27.6 – 1.7. 2022

LIST OF PARTICIPANTS OF SCHOOL

LECTURERS

1. Daniel Beltita (Bucharest) - Quantization and enveloping algebras of Lie groups
2. Sorin Dragomir (Potenza) - Bergman kernels, Fefferman's metric, and quantization of complex manifolds
3. David Fernández (Mexico City) - Supersymmetric quantum mechanics and Painleve equations
4. Mikołaj Rotkiewicz (Warszawa) - On some concepts in the theory of Lie algebroids

PARTICIPANTS

5. Krzysztof Bardadyn (Białystok)
6. Sagnik Biswas (Chennai)
7. Martin Bures (Prague)
8. Luca Campobasso (Łódź)
9. Alonso Contreras-Astorga (Mexico City)
10. Tomasz Czyżycki (Białystok)
11. Alina Dobrogowska (Białystok)
12. Galina Filipuk (Warsaw)
13. Tomasz Goliński (Białystok)
14. Jiří Hrivnák (Prague)
15. Grzegorz Jakimowicz (Białystok)
16. Piotr Kielanowski (Mexico City)
17. Bartosz Kwaśniewski (Białystok)
18. Anatolij Prykarpatski (Kraków)
19. Aneta Sliżewska (Białystok)
20. Marzena Szajewska (Białystok)
21. Bhagwat Thakran (Nagpur)
22. Elwira Wawreniuk (Białystok)

Title: On some concepts in the theory of Lie algebroids

Talk 1:
1. Examples of Lie groupoids and Lie algebroids
2. Lie functor: groupoid - algebroid correspondence

Talk 2:
3. de Rham differential and supergeometry (Vaintrob's theorem)
4. ~~Lie bialgebroids (D. Roytenberg's approach) and~~ double Lie algebroids (T. Voronov's approach)

Talk 3:
5. Higher order analogues of Lie algebroids - comorphism approach to Lie algebroids

Talk 1 : *Examples of Lie groupoids and Lie algebroids*

__Def.__ A groupoid is a small category in which every arrow (morphism) is invertible.

$\Longrightarrow$ A groupoid consist of

- two sets $\mathcal{G}$ (groupoid), M (base);
- two surjective maps $s, t : \mathcal{G} \to M$ (source, target maps);
- the composition map (partial multiplication)

$$m : (g, h) \longmapsto gh \in \mathcal{G} \qquad (s(gh) = s(h), \; t(gh) = t(g))$$

defined for all $g, h \in \mathcal{G}$ s.t. $s(g) = t(h)$

- unit map $u : M \to \mathcal{G}$, $x \mapsto 1_x$;
- inverse map $\mathcal{G} \to \mathcal{G}$, $g \mapsto g^{-1}$

satisfying

- (associativity) $g_1(g_2 g_3) = (g_1 g_2) g_3$ (whenever the composition is possible)
- $1_y g = g$, $g \cdot 1_x = g$ where $x \xrightarrow{g} y$
- $(y \xleftarrow{g} x) \cdot (x \xleftarrow{g^{-1}} y) = (y \xleftarrow{1_y} y)$ and similarly,

$$g^{-1} g = 1_x .$$

__Examples__ (1) group, (1') $\mathcal{G} = \coprod_{j \in J} G_j$, G_j - group

(2) $\mathcal{G} = M \times M$ (pair groupoid)

(2') $\mathcal{G} = M \times G \times M$, G - a group
(trivial groupoid)

$$s(y, g, x) = x, \quad t(y, g, x) = y$$

(3) (general linear groupoid) $\mathcal{G} = GL(E)$, $E \to M$ (a vector bundle)
VB, in short

$$GL(E) = \{ \text{linear isomorphism } E_x \xrightarrow{\xi} E_y \mid x, y \in M \}$$

$$M \simeq \{ id_{E_x} \mid x \in M \}.$$
(base of $\mathcal{G}$)

Notation. $\mathcal{G} \overset{s,t}{\rightrightarrows} M$ (Lie groupoid), $\mathcal{G}_x = s^{-1}(x)$, $\mathcal{G}^y = t^{-1}(y)$,

$\mathcal{G}_x^y = s^{-1}(x) \cap t^{-1}(y)$, $x, y \in M$.

$\mathcal{G}_x^x$ is a group

Motivations (symmetry):

A. Weinstein, ,,Groupoids: Unifying Internal and External Symmetry", 1996, Notices of the AMS

(explains groupoid symmetries)

History:

R. Brown, From groups to groupoids: A brief survey, Bull. London Math. Soc. 19 (1987), 113–134.

groupoid = 'a group with many objects (identities)'

- 1920 – generalization of composition of quadratic forms (Gauss (NT), Brandt (1926)) → notion of groupoid
- 1960's J.P. Pradines: groupoid-algebroid theory (~ Lie theory)
- 1980's A. Weinstein: applications of groupoids and algebroids in Poisson geometry

Interesting approach: A groupoid can be defined in the same way as group, the difference is that the structure map (group multiplication) is assumed to be a relation,

see S. Zakrzewski, Quantum and classical pseudogroups, I and II, Commun. Math. Phys. 134 (1990).

Def. A Lie groupoid is a groupoid whose set of arrows ($\mathcal{G}$) and objects (M) are smooth manifolds, s, t are submersions ($\Rightarrow$ fibers $s^{-1}(x)$ are submanifolds), structure maps, m, u, are smooth (thm $\Rightarrow$ the inverse map is a diffeomorphism).

Examples. Lie groups, (2), (3) (M – manifold)

Exercise. Show that

$$\mathcal{G}_2 = \{(h, g) : s(h) = t(g)\}$$

is a submanifold of $\mathcal{G} \times \mathcal{G}$.

M. Crainic, R.L. Fernandes (2001)
Integration of Lie algebroids

- A. Weinstein, P. Libermann: initiated 'Lie algebroids in geometric mechanics (Lagrangian – hamiltonian formalism, E-L equations on Lie algebroids)
- K., J. Grabowski: geometric mechanics on Lie algebroids

2006 – search for geometric structures fundamental from the point of view of mechanics / variational calculus

MORE EXAMPLES

(4) (Fundamental groupoid of a manifold)

$\Pi(M) = \{\text{paths in } M\}/\sim$ (homotopy classes of paths in M with fixed ends).

Ex. Find $\Pi(S^1)$

(5) (Fundamental groupoid of a foliation)

$\mathcal{F}$ - foliation of M (smooth)

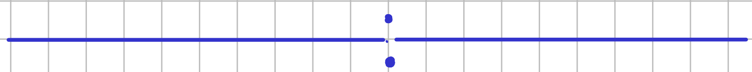
$\Pi(\mathcal{F}) = \{ [g] \mid g : [0,1] \to M, \underbrace{g \in \mathcal{F}} \}$

means g lies in a single leaf of $\mathcal{F}$.

Ex. $\mathcal{F}$ - foliation of $\mathbb{R}^3 \setminus \{(0,0,0)\}$ by horizontal planes.

Show that $\Pi(\mathcal{F})$ is non-Hausdorff

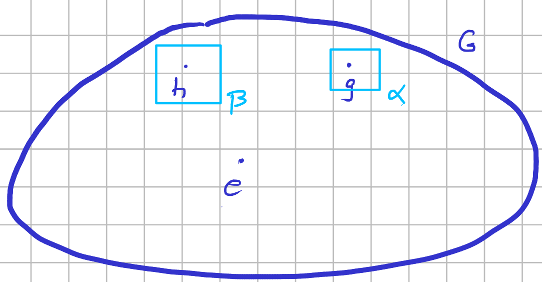
(Examples of non-Hausdorff manifolds :



$\mathbb{R} \sqcup \mathbb{R} /\sim$

$\sim$ identifies two copies of $\mathbb{R}\setminus\{0\}$

Example (5) $\mathcal{G} = T^*G$, G - a Lie group, $\mathfrak{g} := Lie(G) \simeq T_e G$ (Lie algebra)



$s(\alpha) = \alpha \circ TL_g \big|_{\mathfrak{g}}$, $t(\alpha) = \alpha \circ TR_g \big|_{\mathfrak{g}}$

$\{\alpha \in T^*_g G\}$ $\quad \in T_e^* G$

Assume $s(\beta) = t(\alpha)$, $\alpha \cdot TR_g = \beta \cdot TL_h$

Then

$$\beta \cdot \alpha \overset{df}{:=} \beta \circ TR_{g^{-1}} = \alpha \circ TL_{h^{-1}}$$

$\beta \cdot \alpha \in T_{hg} G$

$$\Rightarrow \quad \begin{array}{ccc} T^*G & \longrightarrow & G \\ \downarrow\downarrow & & \\ \mathfrak{g}^* & & \end{array}$$

$Lie(T^*\mathcal{G}) = T^*A(\mathcal{G})^*$

(symplectic groupoid)

The last example can be generalized to $\begin{array}{ccc} T^*\mathcal{G} & \longrightarrow & \mathcal{G} \\ \downarrow\downarrow & & \\ A(\mathcal{G})^* & & \end{array}$

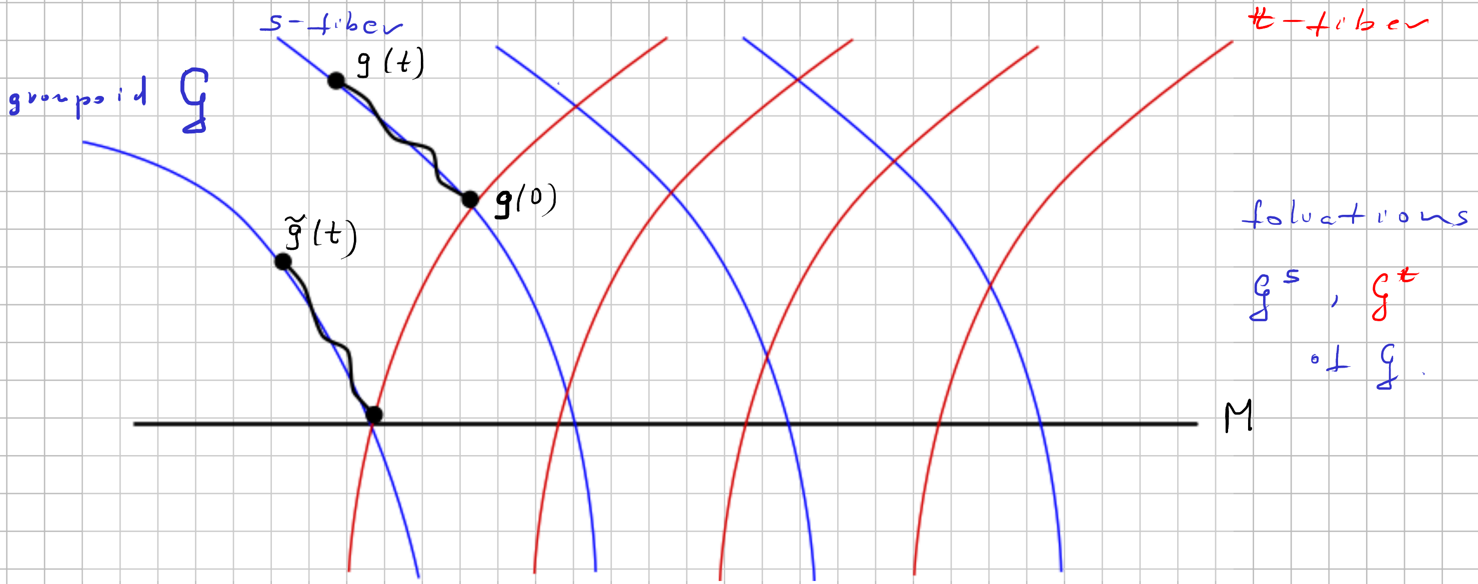
and links theory of groupoids/algebroids with Poisson geometry.

Ex 6. (action groupoid) $G \times M \to M$ - action of a Lie group on a mnfld M

$\mathcal{G} = G \times M$, $s(g,x) = x$,
$t(g,x) = g.x$

Ex. 7 (gauge groupoid) Principal bundle $\begin{matrix} P \\ \downarrow \\ M \end{matrix} \circlearrowleft G \Rightarrow \mathcal{G} = \dfrac{P \times P}{G}$

Lie functor What is the infinitesimal objects that correspond to Lie groupoids?



Reduction of the tangent bundle $T\mathcal{G}$

If $g : [0,1] \to \mathcal{G}$ is a path in $\mathcal{G}$ we write $\underbrace{g \in \mathcal{G}^s}_{\text{path in } \mathcal{G}^s} \Leftrightarrow s(g(t)) = s(g(0))$ for any $t \in [0,1]$

A path g in $\mathcal{G}^s$ can be translated to a path $\tilde{g} \in \mathcal{G}^s$ s.t. $\tilde{g}(0) \in M$

$\Rightarrow$ Reduction map

(fiber-wise isomorphism)
$$\begin{tikzcd} T\mathcal{G}^s \arrow[r, "R"] \arrow[d] & T_M \mathcal{G}^s =: Lie(\mathcal{G}) \arrow[d] \\ \mathcal{G} \arrow[r, "s"] & M \end{tikzcd}$$

$$(T\mathcal{G}^s = \{ [g] = \tfrac{d}{dt}\big|_{t=0} g(t) : g \in \mathcal{G}^s \} \subset T\mathcal{G})$$

- $Lie(\mathcal{G}) \to M$ is a VB,
- (anchor map) $\sharp := Tt\big|_{Lie(\mathcal{G})} : Lie(\mathcal{G}) \to TM$

$\therefore$ $\mathfrak{X}_{RIVF}(\mathcal{G}) \simeq \Gamma(Lie(\mathcal{G}))$ is equipped with Lie bracket

RIVF (right-invariant vector fields) induced from the

Lie bracket of vector fields on $\mathcal{G}$ ($X, Y \in \mathfrak{X}_{RIVF}(\mathcal{G}) \Rightarrow [X, Y]$ the same)

Def. A Lie algebroid is a vector bundle $A \to M$

together with a VB-morphism $\# : A \to TM$ (anchor map)

and a Lie bracket $[\,,\,]_A$ on $\Gamma(A)$ satisfying

$$[\alpha, f\beta]_A = f[\alpha, \beta] + \#(\alpha)(f)\cdot\beta$$

(Leibniz rule) $\quad \forall \alpha, \beta \in \Gamma(A),\ f \in C^\infty(M)$.

Lie bracket: $\begin{cases} \text{skew-symmetric} \\ (\text{Jacobi identity}) \quad [[\alpha,\beta],\gamma] = [\alpha,[\beta,\gamma]] - [\beta,[\alpha,\gamma]]. \end{cases}$

Examples

(1) $A = TM$ (tangent algebroid), (2) $\mathfrak{g}$ – Lie algebra

Exercise TM is the Lie algebroid of the pair groupoid $M \times M$ and the fundamental groupoid.

Ex. Prove that $\# : \Gamma(A) \to \mathfrak{X}(M)$ is a Lie algebra homomorphism.

Ex. $E \to M$ a VB. What is $Lie(GL(E)) = ?$

Consider a path $g \in GL(E)^{\$}$, $g(0) = id_{E_x}$, i.e.

$$g(\varepsilon) : E_x \longrightarrow E_{g_M(\varepsilon)} \quad \text{for some path } g_M \text{ in } M \ (g(0)=x).$$

It defines a derivation

$$\Delta : \Gamma(E) \to E_x, \quad s \longmapsto \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \underbrace{g(\varepsilon)^{-1} s_{g_M(\varepsilon)}}_{\text{in } E_x}$$

(A derivation at $x \in M$ is a $\mathbb{R}$-linear map $\Delta : \Gamma(E) \to E_x$ s.t.

$$\Delta(f \cdot s) = f\,\Delta(s) + v(f)\cdot s$$

(for some $v \in T_x M$ & $\forall \, f \in C^\infty(M), \ s \in \Gamma(E)$)

Derivations of E at x form a vector space, $\mathbb{D}_x E$.

$\Rightarrow$ $\mathbb{D}E$ a VB. Sections of $\mathbb{D}E$ are identified with maps

$$D : \Gamma(E) \to \Gamma(E) \quad \text{s.t.} \quad D(fs) = f D(s) + D_M(f)\cdot s,$$

($\forall f \in C^\infty(M)$, $s \in \Gamma(E)$ and some $D_M \in \mathfrak{X}(M)$)

Such maps D are called derivative endomorphisms or covariant differential operators.)

The VB $\mathbb{D}E$ carries an algebroid structure, $\mathbb{D}E = Lie(GL(E))$.

Ex Show that $\mathbb{D}E \simeq \mathbb{D}E^*$

Hint: Find a map $\mathbb{D}E \longrightarrow \mathbb{D}E^*$, $\Delta \mapsto \Delta^*$ and show that $(\Delta^*)^* = \Delta$ (under $E^{**} = E$)

Ex. Show that $\mathbb{D}E \simeq \mathfrak{X}_0(E)$

$$E : (x^a, y^i), \quad X = \underset{0}{X^a}(x)\,\underset{0}{\partial_{x^a}} + \underset{0}{X^i_j}(x)\,\underset{1}{y^j}\,\underset{-1}{\partial_{y^i}}$$

THM. (Linear Poisson structure)

Let $(A \to M, [\,,\,], \#)$ be a Lie algebroid, then

A^* (dual VB) is a Poisson manifold, which is linear. It is

uniquely determined by

$$\{s_1, s_2\} = [s_1, s_2], \quad \{s, f\} = (\# s)(f)$$

$(A^{**} \simeq A, \; C^\infty_{Lin}(A^*) \simeq \Gamma(A) \ni s, \; f \in C^\infty(M))$

Conversely, any linear Poisson structure on a VB $E \to M$

gives rise to a Lie algebroid structure on $E^* \to M$.

$\square$

In local coordinates: $A: (x^a, y^i)$, $A^*: (x^a, e_i)$, $\{e_i, x^a\} = Q^a_i$, $\{e_i, e_j\} = Q^k_{ij} e_k$

$\Rightarrow$ Poisson tensor $\Lambda_{A^*} = Q^a_i \frac{\partial}{\partial e_i} \wedge \frac{\partial}{\partial x^a} + Q^k_{ij} e_k \frac{\partial}{\partial e_i} \wedge \frac{\partial}{\partial e_j}$.

(† A function $f \in C^\infty(E)$ has weight w if $f(te) = t^w \cdot f(e)$

$C^\infty(E) \supset \bigoplus_{w=0}^{\infty} C^\infty_w(E)$, $C^\infty_w(E) = \Gamma(Sym^w(E^*))$.

Poisson bracket on E is linear if $\{C^\infty_{w_1}(E), C^\infty_{w_2}(E)\} \subseteq C^\infty_{w_1+w_2-1}$.

(equiv. $\{\Gamma(E^*), \Gamma(E^*)\} \subset \Gamma(E^*)$))

Cotangent algebroid $A = T^*P$, where P - Poisson mfld.

(early 80's, a number of different people, K-S)

For $f, g \in C^\infty(M)$ define $\# : T^*P \to TP$, $\# = \Lambda$

$[df, dg] := d\{f, g\}$ Λ - Poisson tensor,

(It can be extended to all 1-forms, not-necessarily exact)

(Link integration of algebroids with symplectic reduction of Poisson manifold)

Summarize: Lie algebroids give rise to Poisson manifolds (of special kind),

but also Poisson mflds form a special class of Lie algebroids.