On partial Banach-Lie algebroid structure: some motivations

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2. Problems about generalization of the notion of Poisson manifolds.

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4. Partial Poisson manifold in the Banach setting.

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1. Introduction

In finite dimension it is well known that, for any Poisson manifold, its cotangent bundle can be provided with a natural structure of Lie algebroid. On the other hand the same is true for prolongation of a Lie algebroid.

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For an adaptation for such results in the Banach setting, we propose the notion partial Poisson structure on a Banach manifold M, which is defined by an anchor on a weak subbundle $T^\flat M$ of T^*M. In this context, we obtain only a "partial structure of Banach-Lie algebroid" on $T^\flat M$ and not a Banach-Lie algebroid, even if $T^\flat M = T^*M$.

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For an adaptation for such results in the Banach setting, we propose the notion partial Poisson structure on a Banach manifold M, which is defined by an anchor on a weak subbundle $T^\flat M$ of T^*M. In this context, we obtain only a "partial structure of Banach-Lie algebroid" on $T^\flat M$ and not a Banach-Lie algebroid, even if $T^\flat M = T^*M$.
Although the prolongation of a Banach Lie algebroid is naturally provided with an anchor, when the typical fiber of the Lie algebroid is not finite dimensional, the Lie bracket gives rise only to a "partial Banach-Lie algebroid structure" on this prolongation.

2 : Problems about generalization of the notion of Poisson manifolds.

1. Weak symplectic. On a Banach manifold M, any symplectic form ω on M is such that the morphism $\omega^\flat : X \mapsto \omega(.,X)$ is not surjective in general and so ω is weak symplectic form.

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2. Locality of a bracket. In the finite dimensional setting, we have bump functions and so for any x in M, any germ of function at x can be extended to a global one. If we consider a Banach manifold M which has bump functions the same is true.

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2. Locality of a bracket. In the finite dimensional setting, we have bump functions and so for any x in M, any germ of function at x can be extended to a global one. If we consider a Banach manifold M which has bump functions the same is true. But many interesting geometrical or physical examples of Banach manifolds do not satisfy such assumption. Thus a Poisson Lie bracket defined on global smooth functions can be not defined on local ones. So, it seems natural to assume that a Poisson bracket must be defined on smooth functions over open set in M .

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2 :continuation

3. Dependence on jets of functions. In finite dimension, any Poisson bracket depends on 1-jet of functions. Unfortunately, in Banach setting, there exist Poisson Lie brackets which are localizable which satisfy Leibniz property and Jacobi identity on any $C^\infty(U)$ **but** depends on k-jets with $k > 1$.

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In fact, any skew symmetric bilinear derivation D on $C^\infty(U)$ gives rise to a bracket on $C^\infty(U)$ which satisfies the Leibniz property and depend on k-jets of functions. Since a "Schouten bracket" can be defined for such derivations, if the Schouten bracket of D is zero, the associated bracket is a Poisson bracket which depends on some k-jets with $k \geq 1$.

2 :continuation

4. Module of local sections. In finite dimension, we can define an almost Lie Bracket $[.,.]_P$ on the cotangent bundle, in term of Lie derivative. The Jacobi identity for a Lie Poisson bracket and the fact that the set of sections of $T^*M_{|U}$ is a finite dimensional module generated by $\{df, f \in C^\infty(U)\}$ imply that the bracket $[.,.]_P$ satisfies the Jacobi identity. Unfortunately this no longer true in Banach setting.

3 : Partial Banach-Lie algebroid

Given an anchored bundle $(\mathcal{A}, \pi, M, \rho)$ on M, the classical notion of a Banach-Lie algebroid structure $(\mathcal{A}, \pi, M, \rho, [.,.]_{\mathcal{A}})$ (cf. [1]) is equivalent to the datum of a sheaf of Lie algebras structure on the sheaf of modules

$$\{\Gamma(\mathcal{A}_U),\ U \text{ open in } M\}$$

of sections of $\mathcal{A}_{|U}$ such that, the Lie bracket $[.,.]_U$ on $\Gamma(\mathcal{A}_U)$ and ρ satisfy the following conditions, for any $(\mathfrak{a}, \mathfrak{a}') \in \Gamma(\mathcal{A}_U)^2$, any $f \in C^\infty(U)$ and any open set U in M :

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(i) $[\mathfrak{a}, \mathfrak{a}']_U$ only depends on the 1-jets of $\mathfrak{a}$ and $\mathfrak{a}'$

(ii) $[\mathfrak{a}, f\mathfrak{a}']_U = df(\rho(\mathfrak{a}))\mathfrak{a}' + f[\mathfrak{a}, \mathfrak{a}']_U$ (Leibniz rule).

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(iii) ρ induces a Lie algebra morphism from $\Gamma(\mathcal{A}_U)$ to $\mathfrak{X}(U)$ where $\{\mathfrak{X}(U),\ U \text{ open in } M\}$ is the sheaf of vector fields on M

3 : Continuation

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Thus the following definition seems natural :

Definition 1

Let $(\mathcal{A}, \pi, M, \rho)$ be a Banach anchored bundle. Given a sheaf $\mathfrak{E}_M$ of subalgebras of the sheaf C^∞_M of smooth functions on M, let $\mathfrak{P}_M$ be a sheaf of $\mathfrak{E}_M$-modules of sections of $\mathcal{A}$. Assume that $\mathfrak{P}_M$ can be provided with a structure of Lie algebras sheaf which satisfies, for any open set U in M.

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- **(CPLA 1)** *the Lie bracket $[.,.]_{\mathfrak{P}(U)}$ on $\mathfrak{P}(U)$ only depends on the 1-jets of sections of $\mathfrak{P}(U)$*
- **(CPLA 2)** *for any $(\mathfrak{a},\mathfrak{a}') \in (\mathfrak{P}(U))^2$ and any $f \in \mathfrak{E}(U)$, we have the Leibniz conditions* $[\mathfrak{a}, f\mathfrak{a}']_{\mathfrak{P}(U)} = df(\rho(\mathfrak{a}))\mathfrak{a}' + f[\mathfrak{a},\mathfrak{a}']_{\mathfrak{P}(U)}$

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- **(CPLA 3)** *ρ induces a Lie algebra morphism from $\mathfrak{P}(U)$ to $\mathfrak{X}(U)$, for any open set U in M.*

3 :continuation

In this context, the family $\{[.,.]_{\mathfrak{P}(U)}, U \text{ open set in } M\}$ is called a sheaf brackets, is denoted $[.,.]_{\mathcal{A}}$, and $(\mathcal{A}, \pi, M, \rho, \mathfrak{P}_M, [.,.]_{\mathcal{A}})$ is called **a partial Banach-Lie algebroid**..

A partial Banach Lie algebroid $(\mathcal{A}, \pi, M, \rho, \mathfrak{P}_M, [.,.]_{\mathcal{A}})$ is called **strong partial Lie algebroid** if for any $x \in M$, the stalk

$$\mathfrak{P}_x = \varinjlim \{\mathfrak{P}(U), \quad U \text{ open neighbourhood of } x\}$$

is equal to the fiber $\mathcal{A}_x$ for any $x \in M$.

Note that for a strong partial Lie algebroid, the exterior differential of forms on $\mathcal{A}$ is well defined (cf. [3]).

4 :Partial Banach Lie Poisson structure

Let M be a Banach manifold modelled on a Banach space $\mathbb{M}$. We denote by : $p_M : TM \to M$ its tangent bundle and by $p^*_M : T^*M \to M$ its cotangent bundle

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Definition 2

A vector subbundle $p^\flat : T^\flat M \to M$ of $p_M^* : T^*M \to M$ is called a weak subbundle of $p_M^* : T^*M \to M$ if $p^\flat : T^\flat M \to M$ is a Banach bundle and if the canonical injection $\iota : T^\flat M \to T^*M$ is a Banach vector bundle morphism.

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For any open set U in M we introduce :

Definition 3

Let $\mathfrak{A}(U)$ be the set of smooth functions $f \in C^\infty(U)$ such that each iterated derivative $d^k f(x) \in L^k_{\text{sym}}(T_x M, \mathbb{R})$ ($k \in \mathbb{N}^*$) satisfies :
$\forall x \in U, \forall (u_2, \dots, u_k) \in (T_x M)^{k-1},\ d^k_x f(., u_2, \dots, u_k) \in T^\flat_x M.$

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$\mathfrak{A}(U)$ is sub-algebra of $C^\infty(U)$ (cf. [3])

4 :continuation

Considering the canonical bilinear crossing $< \, , \, >$ between T^*M and TM, we introduce :

Definition 4

A morphism $P : T^\flat M \to TM$ is called skew-symmetric if it satisfies the relation

$< \xi, P(\eta) >= - < \eta, P(\xi) >$

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Given such a morphism P, on $\mathfrak{A}(U)$ we define the bracket :
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Given such a morphism P, on $\mathfrak{A}(U)$ we define the bracket :
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From the definition 1 , the relation (P) defines a skew-symmetric bilinear map $\{.,.\}_P : \mathfrak{A}(U) \times \mathfrak{A}(U) \to \mathfrak{A}(U)$ and satisfies the Leibniz property.

4 :continuation

Definition 5

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In this case, P is called a Poisson anchor.

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In this case, P is called a Poisson anchor.

Note that for a weak symplectic form ω, when $\omega^\flat(TM)$ has a structure of weak subbundle of T^*M or if ω is strong symplectic, the situation $T^\flat M = \omega^\flat(TM)$ and $P = (\omega^\flat)^{-1}$ are particular cases of Definition 5.
When $T^\flat M = T^*M$, the Definition 5 is precisely the definition of Banach-Lie Poisson manifold defined in [4].

4 :continuation

As classically, given a partial Poisson manifold $(T^{\flat}M, p^{\flat}, M, P, \{.,.\}_P)$, any function $f \in \mathfrak{A}(U)$ is called a Hamiltonian and the associated vector field $X_f = P(df)$ is called a Hamiltonian vector field.

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As classically, given a partial Poisson manifold $(T^\flat M, p^\flat, M, P, \{.,.\}_P)$, any function $f \in \mathfrak{A}(U)$ is called a Hamiltonian and the associated vector field $X_f = P(df)$ is called a Hamiltonian vector field. We then have

$\{f,g\}_P = X_f(g) \text{ and also } [X_f, X_g] = X_{\{f,g\}}$

which is equivalent to

$P(d\{f,g\})_P) = [P(df), P(dg)] \quad (PP)$.

4 :continuation

Theorem 1

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Let $(T^\flat M, p^\flat, M, P, \{.,.\}_P)$ be a partial Poisson manifold. We denote by $\mathfrak{P}_M$ the sheaf of $\mathfrak{A}(U)$-modules generated by the set $\{df, f \in \mathfrak{A}(U)\}$. Then we have the following properties :

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1. We can define a sheaf of "almost" Lie brackets $[.,.]_P$ on the sheaf $\mathfrak{P}_M$ by :

$$[\alpha, \beta]_P = L_{P(\alpha)}\beta - L_{P(\beta)}\alpha - d < \alpha, P(\beta) >$$

for any open set U in M and any sections α and β in $\mathfrak{P}(U)$ where L_X is the Lie derivative.

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Moreover, we have $\forall\,(f,g) \in (\mathfrak{A}(U))^2,\ [df, dg]_P = P(d\{f,g\}_P)$

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for any open set U *in* M *and any sections* α *and* β *in* $\mathfrak{P}(U)$ *where* L_X *is the Lie derivative.*

Moreover, we have $\forall\,(f,g) \in (\mathfrak{A}(U))^2,\ [df, dg]_P = P(d\{f,g\}_P)$

2. $(\mathfrak{P}_M, [.,.]_P)$ *is a sheaf of Poisson-Lie algebras. In particular,* $(T^\flat M, p^\flat_M, M, P, \mathfrak{P}_M, [.,.]_P)$ *is a strong partial Banach-Lie algebroid*

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3. If $T^\flat M = T^* M$ and P is an injective morphism, then $(T^* M, M, P, [.,.]_P)$ is a Lie algebroid. This situation occurs in particular for strong symplectic structures.

5 : Prolongation of a Banach-Lie algebroid

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We consider a Banach Lie algebroid $(\mathcal{A}, \pi, M, \rho, [.,.]_{\mathcal{A}})$ with typical fiber $\mathbb{A}$. Let $\mathcal{A}_x := \pi^{-1}(x)$ be the fiber over $x \in M$.

The prolongation $\mathbf{T}\mathcal{A}$ of the anchored Banach bundle $(\mathcal{A}, \pi, M, \rho)$ over $\mathcal{A}$ is the set
$\left\{(x, a, u, X) \;\; u \in \mathcal{A}_x, \;\; X \in T_{(x,a)}\mathcal{A} \; : \; \rho(x, u) = T\pi(X)\right\}.$

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$\left\{(x, a, u, X) \;\; u \in \mathcal{A}_x, \;\; X \in T_{(x,a)}\mathcal{A} \; : \; \rho(x, u) = T\pi(X)\right\}$.
If we set $\mathbf{p}(x, a, u, X) = (x, a)$ we have a Banach vector bundle $\mathbf{p} : \mathbf{T}\mathcal{A} \to \mathcal{A}$ with typical fiber $\mathbb{A} \times \mathbb{A}$ which is also the pull-back of $\pi : \mathcal{A} \to M$ over $\rho : \mathcal{A} \to TM$. We have an anchor $\hat{\rho} : \mathbf{T}\mathcal{A} \to T\mathcal{A}$ given by $\hat{\rho}(x, a, u, X) = X \in T_{(x,a)}\mathcal{A}$.

5 : continuation

Note that the restriction of $\hat{\rho}$ to $\ker \mathbf{p}$ is an isomorphism onto the vertical bundle of the tangent bundle $p_{\mathcal{A}} : T\mathcal{A} \to \mathcal{A}$. Therefore, we can identify these bundles which will be denoted $\mathbf{V}\mathcal{A}$ and so can be identified with $\mathcal{A} \times_M \mathcal{A}$. By the way, each vertical vector field on $\mathcal{A}$ can be considered as a section of $\mathbf{V}\mathcal{A}$.

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Let $\widetilde{\mathcal{A}}$ be the pull-back of $\pi : \mathcal{A} \to M$ over π. Then we have the following commutative diagrams :

$$\begin{array}{ccccc}
\mathbf{T}\mathcal{A} & \xrightarrow{\pi_{\widetilde{\mathcal{A}}}} & \widetilde{\mathcal{A}} & \xrightarrow{\tilde{\pi}} & \mathcal{A} \\
\Big\downarrow{\mathbf{p}} & & \Big\downarrow{\widehat{\pi}} & & \Big\downarrow{\pi} \\
\mathcal{A} & \xrightarrow{\mathrm{Id}} & \mathcal{A} & \xrightarrow{\pi} & M
\end{array}$$

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$$\begin{array}{ccccc}
\mathbf{T}\mathcal{A} & \xrightarrow{\pi_{\widetilde{\mathcal{A}}}} & \widetilde{\mathcal{A}} & \xrightarrow{\tilde{\pi}} & \mathcal{A} \\
\downarrow{\scriptstyle \mathbf{p}} & & \downarrow{\scriptstyle \widehat{\pi}} & & \downarrow{\scriptstyle \pi} \\
\mathcal{A} & \xrightarrow{\mathrm{Id}} & \mathcal{A} & \xrightarrow{\pi} & M
\end{array}$$

Since for local section $\widetilde{\mathfrak{u}}$ of $\widetilde{\mathcal{A}}$ the value $\widetilde{\mathfrak{u}}(x,u)$ belongs to $\widetilde{\mathcal{A}}_{(x,a)}$ which is identified with $\mathcal{A}_x$, it follows that $\rho(\widetilde{\mathfrak{u}}(x,u))$ is well defined and belongs to $T_x\mathcal{A}$.

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Thus each local section $\mathcal{X}$ of $\mathbf{T}\mathcal{A}$ can be identified with a pair $(\widetilde{\mathfrak{u}}, X)$ where $\widetilde{\mathfrak{u}}$ is a section of $\widetilde{\mathcal{A}}$ and X a vector field on $\mathcal{A}$ such that we have $\hat{\rho}(\widetilde{\mathfrak{u}}, X) = X$ where $T\pi(X) = \rho(\widetilde{\mathfrak{u}})$

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As $\mathbf{V}\mathcal{A}$ is also isomorphic to $\mathcal{A} \times_M \mathcal{A}$, each any section $\mathfrak{u}$ of $\mathcal{A}$ is associated a canonical section of $\mathfrak{u}^v$ of $\mathbf{V}\mathcal{A}$ given by $\mathfrak{u}^v(x, a) = (x, a, 0, \mathfrak{u}(x))$.

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In finite dimension, $\mathbf{T}\mathcal{A}$ can be provided with a canonical Lie bracket induced by the Lie bracket on $\mathcal{A}$. **Unfortunately this result is no more true in infinite dimensional Banach setting**. Note that since each vertical section of $\mathbf{T}\mathcal{A}$ is a vertical vector field, the usual Lie bracket of vector field gives rises to a Lie bracket for sections of $\mathbf{V}\mathcal{A}$ and we get a natural structure on Lie algebroid on $\mathbf{V}\mathcal{A}$.

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We can extend the Lie bracket on $\mathbf{V}\mathcal{A}$, **only** to some type of local or global sections, but not for all sections of $\mathbf{T}A$ and so $\mathbf{T}\mathcal{A}$ does not have a Lie algebroid structure.

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Precisely, a section of type $\mathcal{X} = (\mathfrak{u} \circ \hat{\pi}, X)$ where $\mathfrak{u}$ is a section of $\mathcal{A}$, is called **a projectable section** . Since ρ induces a morphism of Lie algebra for modules of local sections, and since $\mathcal{A}$ is a Lie algebroid this implies that, for projectable sections, we can define a Lie bracket by

$$[(\mathfrak{u} \circ \hat{\pi}, X), (\mathfrak{u}' \circ \hat{\pi}, X')]_{\mathbf{T}\mathcal{A}} := ([\mathfrak{u}, \mathfrak{u}']_{\mathcal{A}} \circ \hat{\pi}, [X, X']) \quad (Brak)$$

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We denote by $\mathfrak{P}(\mathbf{T}\mathcal{A}_U)$ the $C^\infty(\mathcal{A}_U)$-module generated by the set of projectable sections defined on $\mathcal{A}_U$. Each module $\mathfrak{P}(\mathbf{T}\mathcal{A}_U)$ has the following properties :

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Lemma

For any open subset U in M, there exists a well defined Lie bracket $[.,.]_{\mathbf{T}\mathcal{A}_U}$ on $\mathfrak{P}(\mathbf{T}\mathcal{A}_U)$ which provides $\mathfrak{P}(\mathbf{T}\mathcal{A}_U)$ with a Lie algebra structure and whose restriction to projectable sections is given by the relation (Brak).

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Finally we have :

Theorem 2

The set of modules $\{\mathfrak{P}(\mathbf{T}\mathcal{A}_U) : U \text{ open set in } M\}$ *defines a sheaf of* $C^\infty(\mathcal{A}_U)$*-module modules denoted* $\mathfrak{P}_\mathcal{A}$ *on* $\mathcal{A}$ *which gives rise to a strong partial Banach-Lie algebroid on the anchored bundle* $(\mathbf{T}\mathcal{A}, \mathbf{p}, \mathcal{A}, \hat{\rho}, [.,.]_{\mathbf{T}\mathcal{A}})$.

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If the Banach bundle $\pi : \mathcal{A} \to M$ *has a finite dimensional fiber then* $(\mathbf{T}\mathcal{A}, \mathbf{p}, \mathcal{A}, \hat{\rho}), [.,.]_{\mathbf{T}\mathcal{A}})$ *is a Banach-Lie algebroid.*

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Thank you for your attention !