Reflection Positivity on the Sphere

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Structure of the talk

In this presentation I will give a brief overview of reflection positivity in geometry and representation theory. This is a part of a joint project with K-H. Neeb, including work with J. Frahm, V. Morinelli and B. Ørsted.

In more details:

- Short history of reflection positivity.
- Reflection positive Hilbert spaces.
- Reflection positive representations.
- Cartan duality
- Dissecting manifolds.
- The basic example: The sphere.

A brief history of reflection positivity

- Reflection Positivity originated as one of the Osterwalder-Schrader axioms of constructive quantum field theory (1973/75).

- Very simplified version of two of the Wightman, or Gårding-Wightman axioms, around 1964:

- Hilbert space: States are elements of an Hilbert space $\mathcal{H}$ carrying an unitary positive-energy representation π of the Poincaré group ($n = 4$): $\mathrm{P}_n = \mathrm{O}_{1,n-1}(\mathbb{R})^{\uparrow} \ltimes \mathbb{R}^n$.

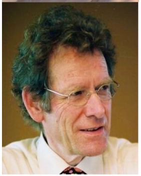
- Representation Theory: P_n acts on $\mathbb{R}^n$ by $(A, \mathsf{x}) \cdot \mathsf{v} = A\mathsf{x} + \mathsf{v}$. Hence it acts on function by

$$\lambda_{(A,\mathsf{x})} f(\mathsf{v}) = f(A^{-1}(\mathsf{v} - \mathsf{x})).$$

Constructive QFT

- Motivated by the fact that

> Eucledian geometry/quantum mechanics
>
> is simpler than
>
> Lorentzian geometry/quantum field theory



Osterwalder-Schrader in 1973/1975
formulated equivalent axioms two of which are:

Two of Osterwalder-Schrader Axioms

- Euclidean covariance:
  $P_{n+1} = O^{\uparrow}_{1.n} \ltimes \mathbb{R}^{1,n}$-covariance is replaced by $E_{n+1} = SO_{n+1} \ltimes \mathbb{R}^{n+1}$-covariance.
  The representation theory of E_n is much simpler.
- Reflection positivity: The tool to transform the euclidean fields to the relativistic fields via time reflection, analytic continuation in the time variable and then restrict to purely imaginary time, Wick rotation moving to imaginary time.

Change of metrics: The Geometry

- On the space time manifold we have a time reflection

$$\tau((x_0, \mathrm{x})) = (-x_0, \mathrm{x})$$

Then multiply time by i transfer the Euclidean metric form

$$\langle (x_0, \mathrm{x}), (y_0, \mathrm{y}) \rangle_{\mathrm{E}} = x_0 y_0 + x_1 y_1 + \cdots + x_n y_n$$

into the Lorentzian form:

$$\langle (x_0, \mathrm{x}), (y_0, \mathrm{y}) \rangle_{\mathrm{L}} = -x_0 y_0 + \langle \mathrm{x}, \mathrm{y} \rangle = [x, y].$$

Action on functions, distributions and transformations

- The time reflection acts on
  - Functions: $\varphi \mapsto \varphi \circ \tau = \varphi^\tau$
  - Distributions (conjugate linear maps $\eta : C_c^\infty(M) \to \mathbb{C}$)

$$\eta^\tau(\varphi) = \eta(\varphi^\tau).$$

  - Isometries: $f^\tau(x) = f(\tau(x))$.
  - In the flat case this leads to a duality of symmetry groups (c-duality, talk by Neeb) $\mathrm{E}_{n+1} \leftrightarrow \mathrm{P}_{n+1}$. both groups acting by $(a, v)(x) = ax + v$.

Question: What about representations of the isometry groups?

Basic Concepts I: Reflection Positive Hilbert Spaces

- Based on the assumption that states are elements of an Hilbert space one define:

Definition

A reflection positive Hilbert space is a triple $(\mathcal{E}, \mathcal{E}_+, \theta)$ s.t.

- (Reflection) $\mathcal{E}$ a Hilbert space and $\theta : \mathcal{E} \to \mathcal{E}$ is unitary linear involution: $\theta^2 = \mathrm{id}$
- (Positivity) $\mathcal{E}_+$ a subspace s.t. the Hermitian form

$$\langle v, w \rangle_\theta = \langle \theta v, w \rangle$$

is non-negative on $\mathcal{E}_+$, $\langle \theta v, v \rangle \geq 0$ for all $v \in \mathcal{E}_+$.

Basic Concepts II: the O-S map/functor

- $\mathcal{N} := \{u \in \mathcal{E}_+ \mid \|u\|_\theta = 0\} = \{u \in \mathcal{E}_+ \mid (\forall w \in \mathcal{E}_+)\ \langle u, w\rangle_\theta = 0\}$.
- $\hat{\mathcal{E}} =$ the completion of $\mathcal{E}_+/\mathcal{N}$ in the norm $\|\cdot\|_\theta$, a Hilbert space.
- $q : \mathcal{E}_+ \to \hat{\mathcal{E}}$ or $v \mapsto \hat{v}$, the quotient map $v \mapsto v + \mathcal{N}$
- $T : \mathcal{D} \to \mathcal{E}_+$, $\mathcal{D} \subseteq \mathcal{E}_+$ a (possibly unbounded) linear operator with $T(\mathcal{D} \cap \mathcal{N}) \subset \mathcal{N}$, then

$$\hat{T} : \hat{\mathcal{E}} \to \hat{\mathcal{E}} \quad \hat{T}\hat{v} := \widehat{(Tv)}$$

denotes the corresponding operator defined on $\hat{\mathcal{D}}$.

• Work by several peopleJ. Dimock, J. Fröhlich, Osterwalder, Seiler, Glimm, Jaffe, Jorgensen, Ritter, Klein, Landau, Nelson, Schrader, ...

c-duality I

Theorem

If X is a connected Riemannian manifold with involution τ_X with isolated fixed points then $G = \mathrm{Iso}\,(X)_e$ is a finite dimensional Lie group with involution $\tau_G(g) = \tau_X g \tau_X$.

- τ_G defines an involution on $\mathfrak{g} = \mathrm{Lie}\, G$ and we have

$$\mathfrak{g} = \underbrace{\mathfrak{h}}_{+1 \text{ eigenspace}} \oplus \underbrace{\mathfrak{q}}_{-1 \text{ eigenspace}}$$

- From now on we simply write τ. We will also choose $H \subset G$ closed such that

$$G_e^\tau \subset H \subset G \quad \text{(symmetric subgroup)}.$$

- We let $\mathfrak{g}^c = \mathfrak{h} \oplus i\mathfrak{q}$ (Lie algebra) and G^c the simply connected Lie group with Lie algebra $\mathfrak{g}^c$ and note that τ defines an involution on G^c.

Unitary Representations

- Starting with a
  - Reflection positive Hilbert space $(\mathcal{E}, \mathcal{E}_+, \theta)$
  - A triple (G, π, τ) where π is a unitary representation of G on $\mathcal{E}$ and $\tau : G \to G$ is an involution

Definition

The representation π is reflection positive if $\pi(H)\mathcal{E}_+ \subset \mathcal{E}_+$ and we have the compatibility condition

$$\theta\pi(g)\theta = \pi(\tau(g)). \tag{1}$$

Transfer of unitarity

- Assuming that the derived representation π^∞ defines a densely defined operator on $\mathcal{E}_+$ then (1) implies that the derived representation on $\hat{\mathcal{E}}$ satisfies

$$\widehat{\pi}^\infty(X) \text{ is } \begin{cases} \text{skew symmetric} & \widehat{\pi}^\infty(X)^* = -\widehat{\pi}^\infty(X),\ X \in \mathfrak{h} \\ \text{symmetric} & \widehat{\pi}^\infty(X)^* = \widehat{\pi}^\infty(X),\ X \in \mathfrak{q} \end{cases}$$

- Thus $\pi^c(X + iY) = \pi^\infty(X) + i\pi^\infty(Y), \quad X \in \mathfrak{h}, Y \in \mathfrak{q}$, defines a representation of $\mathfrak{g}^c$ in the space of slew symmetric operators on $\widehat{\mathcal{E}}$.

Question: When does this define a unitary representation $\widehat{\pi}$ of G^c?

Partial answers: Lüscher-Mack 1975, R. Schrader 1986, Jorgensen-Ó. 1998/2000, Merigon-Neeb-Ó 2015

Dissecting manifolds

Definition

The pair (X, τ), where X is a manifold and $\tau : X \to X$ an involution, is dissecting if $X \setminus X^{\tau}$ is not connected.

The idea behind the definition is that (see more in a moment) we can think of τ as time reflection leading to a decomposition of the manifold into:

$$X = X_+ \dot{\cup} X_0 \dot{\cup} X_-, \quad X_0 = X^{\tau}, \quad \tau(X_+) = X_-.$$

and

$$T_x X = \underbrace{T_x^-(X)}_{\substack{-1 \text{ eigenspace, time} \\ \text{one-dimensional}}} \oplus \underbrace{T_x^+(X)}_{+1 \text{ eigenspace, space}}$$

Fun facts, several authors

Theorem

Let X *be a connected smooth manifold.*

(i) *If* g *is a complete Riemannian metric on* X *and* τ *is a dissecting isometry, then* τ *is an involution.*

(ii) *If* $\tau \in \mathrm{Diff}(M)$ *is an involution, then there exists a* τ*-invariant complete Riemannian metric on* X*.*

(iii) *If* $\tau \in \mathrm{Diff}(\mathsf{X})$ *is a dissecting involution, then* $\mathsf{X} \setminus \mathsf{X}^\tau$ *has two connected components* $\mathsf{X}_\pm$ *with* $\tau(\mathsf{X}_\pm) = \mathsf{X}_\mp$ *and each component of* $\mathsf{X}_0 = \mathsf{X}^\tau$ *is of codimension one.*

(iv) *If* $\tau \in \mathrm{Diff}(\mathsf{X})$ *is a reflection and* X *is simply-connected, then* τ *is dissecting and its fixed point set* X^τ *is a connected orientable hypersurface.*

Classification, X symmetric space $\simeq G/H$. Neeb-Ó, 2019

Theorem

Up to coverings, every connected irreducible symmetric space with a dissecting involutive automorphism is a connected component of a quadric

$$Q := \{x \in \mathbb{R}^{p+q} \mid \beta_{p,q}(x,x) = 1\},$$

where

$$\beta_{p,q}(x,y) = \sum_{j=1}^{p} x_j y_j - \sum_{j=p+1}^{p+q} x_j y_j.$$

Here $G = \mathrm{SO}_{p,q}(\mathbb{R})_0$ *and* $H_0 \cong \mathrm{SO}_{p-1,q}(\mathbb{R})_0$ *or* $H_0 \cong \mathrm{SO}_{p,q-1}$. *Up to conjugation, the dissecting involution is given by*

$$\sigma(x_1, \mathrm{x}) = (-x_1, \mathrm{x}) \text{ or } \sigma(\mathrm{x}, x_n) = (\mathrm{x}, -x_n)$$

Jaffe-Ritter-Anderson/Dimock construction I

- Let (X, τ) be a dissecting Riemannian manifold. Δ the Laplacian.
- For $m > 0$ let

$$C = (m^2 - \Delta)^{-1}$$

bounded on $L^2(\mathrm{X})$ and positive.

- (Sobolev) inner product

$$\langle \varphi, \psi \rangle_m = \langle C\varphi, \psi \rangle_{L^2}$$

- Completion: The Sobolev space $H_{-1}(\mathrm{X})$.
- C commutes with all isometries on X, in particular

$$(C\varphi) \circ \tau = C(\varphi \circ \tau)$$

leading to a unitary rep. of $G = \mathrm{Iso}\,(\mathrm{X})$.

Jaffe-Ritter-Anderson/Dimock

Theorem

Assume that (X, τ) *is dissecting.*

a) Let $\mathcal{E} = H_{-1}(\mathsf{X})$ *be the completion of* $L^2(\mathsf{X})$ *w.r.t.* $\|\cdot\|_m$ *and let* $\mathcal{E}_+$ *be the closed subspace generated by* $L^2(\mathsf{X}_+)$*. Define* $\theta : \mathcal{E} \to \mathcal{E}$ *by* $\theta\varphi = \varphi \circ \tau$*. Then* $(\mathcal{E}, \mathcal{E}_+, \theta)$ *is reflection positive.*
b) Let $G = Iso(\mathsf{X})$ *and define* $\tau : G \to G$ *by* $\tau(g) = \tau g \tau$*. Define a unitary representation of* G *on* $\mathcal{E}$ *by* $\pi_m(g)\varphi = \varphi \circ g^{-1}$*. Then* π_m *is reflection positive.*
c)[Dimock, Zero time realization] *The space* $\hat{\mathcal{E}}$ *is isomorphic to the space of elements in* $H_{-1}(\mathsf{X})$ *supported on* $\mathsf{X}_0 = \mathsf{X}^\tau$*.*

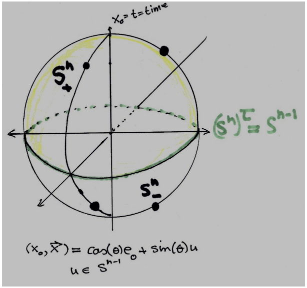
The sphere S^n

- The only compact, dissecting symmetric space is

$$\mathrm{S}^n = \mathrm{SO}_{n+1}/\mathrm{SO}_n = \mathrm{O}_{n+1}/\mathrm{O}_n$$

with

$$\tau(x_0, \mathrm{x}) = (-x_0, \mathrm{x}),$$
$$\tau(g) = \mathrm{I}_{1,n} g \mathrm{I}_{1,n}$$
$$\mathrm{S}^n_0 = \mathrm{S}^{n\tau} = \{(0, \mathrm{x}) \mid \mathrm{x} \in \mathrm{S}^{n-1}\}$$
$$\mathrm{S}^n_\pm = \{(x_0, \mathrm{x}) \mid \pm x_0 > 0\}.$$



The distribution Ψ

- Positive definite distribution defined by

$$\Phi(\varphi \otimes \psi) = \langle \overline{\varphi}, \psi \rangle_m = \int_{\mathrm{S}^n \times \mathrm{S}^n} C \bar{\varphi}(x) \bar{\psi}(y) d\sigma(x) d\sigma(y)$$

where σ is the unique rotational invariant probability measure on S^n.

- Satisfies the equation

$$(m^2 - \Delta)_x \Phi = (m^2 - \Delta)\Phi = \delta(x, y)$$

- $\Rightarrow$ On $\mathrm{S}^n \times \mathrm{S}^n \setminus \mathrm{S}^n$ given by a G-invariant analytic positive definite function $\phi_m(x, y) = \phi_y(x)$.

- Twisting by τ in the first variable we obtain the twisted inner product on $\mathcal{E}_+$ resulting in the function $\psi(x, y) = \phi(\tau x, y)$. Let $\psi(x) := \psi(x, -e_0)$.

The function ψ

Theorem (Neeb-Ó, 2020)

We have:

- *The function* ψ *is analytic on* $\mathrm{S}^n \setminus \{-e_0\}$*. and satisfies the differential equation*

$$\Delta\psi = m^2\psi \quad \text{on } \mathrm{S}^n \setminus \{-e_0\} \tag{2}$$

  and is invariant under rotation around the e_0*-axis, i.e.* $\psi(gx) = \psi(x)$ *for all* $g \in \mathrm{O}_n$.

- $\psi(\cos te_0 + \sin tu) = \gamma_2 F_1(\rho + \lambda, \rho - \lambda, \frac{n}{2}; \sin^2(t/2))$ *with a known constant* γ, $\rho = (n-1)/2$, $\lambda = \sqrt{\rho^2 - m^2}$.

The crown

- We can take $\mathrm{O}_{1,n}^{\uparrow}$ as G^c and try to understand the representation $\widehat{\pi}$. For that one define (see also Bros and co-authors):

$$\Xi = G^c \mathrm{S}_+^n \subset \mathrm{S}_{\mathbb{C}}^n = \mathrm{SO}_{n+1}(\mathbb{C})/\mathrm{SO}_n(\mathbb{C})$$

- As we are working inside $\mathrm{S}_{\mathbb{C}}^n$ and $\mathrm{O}_{n+1,\mathbb{C}}$ one has to trace where to put the is. For that let $V = \mathbb{R}e_0 + i\sum_{j=1}^n \mathbb{R}e_j$ and note that $\langle (v_0, \mathrm{v}), (v_0, \mathrm{v})\rangle_E = v_0^2 - \|\mathrm{v}\|^2 = \langle v, v\rangle_L$.

Important description of Ξ

Theorem (N-Ó, 2019)

We have the following description of Ξ.

- *Let* $C_+ \subset V$ *be the open light cone*

$$C_+ = \{(v_0, \mathrm{v}) \in V \mid v_0 > 0\langle v, v\rangle > 0\}$$

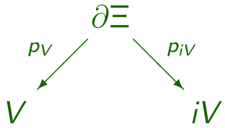
  and $T_{C_+} = V + iC_+$, *Then* $\Xi = \mathrm{S}^n_{+\mathbb{C}} \cap T_{C_+}$.
- $\Xi = \{v \in V_{\mathbb{C}} \mid [z, z]_V \not\in (-\infty, -1]\}$.
- Ξ *is a complex manifold isomorphic to the Lie ball* $\mathrm{SO}_{2,n}/\mathrm{S}(\mathrm{O}_2 \times \mathrm{O}_n)$ *(see the work of Stanton-Krötz).*
- *The kernel* Ψ *has a holomorphic extension to a holomorphic positive definite kernel on* $\Xi \times \overline{\Xi}$ *given by*

$$\psi(z, w) = \gamma_2 F_1(\rho + \lambda, \rho - \lambda, n/2; \frac{1 - [z, \bar{w}]}{2})$$

  $\bar{w}$ *conjugation wrt* V.

Important geometric facts about Ξ, I

Consider the double fibration:



Both p_V and p_{iV} are G^c-equivariant maps. Hence the projection of a G^c-invariant subset is again G^c-invariant in V, resp., iV.

Theorem

The boundary of Ξ in $\mathrm{S}^n_{\mathbb{C}}$ consists of two G^c-orbits:

- *The deSitter space*

$$\mathrm{dS}^n := i\{v \in V \; [v,v]_V = -1\} = \mathrm{S}^n_{\mathbb{C}} \cap iV = \mathrm{SO}_{1,n}/\mathrm{SO}_{1,n-1}.$$

- *The light like vectors*

$$\mathbb{L}^n_+ := \{v = (v_0, i\mathrm{v}) \in V \; [v,v]_V = 0, v_0 > 0\} = G^c.\xi^0 \simeq G^c/P.$$

More geometric facts and limits

Theorem

Let $\mathrm{H}^n = \{[v,v]_V = 1, v_0 > 0\} = G^c e_0$*. The space* H^n *is a Riemannian symmetric space and* $\mathrm{H}^n \subset \Xi$*.*

The crown can be used to transfer information between S^n_+, H^n, $\mathbb{L}^n_+$ and dS^n via analytic continuation, restriction and boundary value.

Theorem

Let $y \in \mathrm{dS}^n$*. The limit process*

$$\lim_{\Xi \ni v \to y} \int_{\mathrm{dS}^n} \overline{\varphi(x)} \psi(x,v)\, d\mu(x), \quad \varphi \in C_c^\infty(\mathrm{dS}^n)$$

defines a distribution with singular support $\{(y,-y) \mid y \in \mathrm{dS}^n\}$*.*

See Frahm-Neeb-Ó, Iswarya-Ó (both in preparation) and Gindikin-Krötz-Ó for details

The representation π_m

- The positive definite kernel ψ defines a Hilbert space $\mathcal{H}_m \subset \mathcal{O}(\Xi \times \overline{\Xi})$ and unitary representation π_m defined by the GNS construction:
  - The space $\{\sum_{\text{finite}} \psi(\cdot, x_j) \mid x_i \in \Xi\}$ is dense and
    $$\langle \psi(\cdot, x), \psi(\cdot y)\rangle = \psi(y, x).$$
  - $\pi_m(g)\psi(x, y) = \psi(g^{-1}x, y) = \psi(x, gy)$.

Identification of the representation $\widehat{\pi}$

Theorem (Neeb-Ó 2019)

The following holds true:

- *The representation π_m is irreducible with a $K = \mathrm{SO}_n$-invariant vector $\psi(\cdot, e_0)$.*
- $\widehat{\pi} \simeq \pi_m$.
- *If π is an irreducible unitary representation of G^c with a non-zero K-fixed vector then $\pi \simeq \pi_m$ for some m.*
- *The representation π_m extends to a anti-unitary representation of $G^c \ltimes \{\mathrm{id}, \tau\}$ by*

$$\pi_m(\tau)F(x.y) = \overline{F(\overline{\tau(x)}, \overline{\tau(y)})}.$$