Geometric aspects of the modular theory of operator algebras

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Local nets in Algebraic Quantum Field Theory (AQFT)

In AQFT one studies nets of von Neumann algebras $(\mathcal{M}(\mathcal{O}))_{\mathcal{O}\subseteq M}$ in $B(\mathcal{H})$, $\mathcal{H}$ a cplx Hilbert space, $\mathcal{M}(\mathcal{O})$ models observables measurable in the "laboratory" $\mathcal{O} \subseteq M$ (an open subset of the space-time manifold M).

Axioms:

- Isotony: $\mathcal{O}_1 \subseteq \mathcal{O}_2$ implies $\mathcal{M}(\mathcal{O}_1) \subseteq \mathcal{M}(\mathcal{O}_2)$
- Locality: $\mathcal{O}_1 \subseteq \mathcal{O}_2'$ implies $\mathcal{M}(\mathcal{O}_1) \subseteq \mathcal{M}(\mathcal{O}_2)'$ [$\mathcal{O}' =$ causal compl.]
- Reeh–Schlieder property: There exists a unit vector $\Omega \in \mathcal{H}$ that is cyclic for each $\mathcal{M}(\mathcal{O})$.
- Covariance: There is a symmetry group G of M and a unitary representation $U\colon G \to \mathrm{U}(\mathcal{H})$ such that, for $g \in G$
  $U_g\mathcal{M}(\mathcal{O})U_g^{-1} = \mathcal{M}(g\mathcal{O})$
- Bisognano–Wichmann property: Ω is separating for some $\mathcal{M}(W)$, $W \subseteq M$ "wedge region" with modular group of $(\mathcal{M}(W), \Omega)$ in G.
- Invariance of vacuum: $U(g)\Omega = \Omega$ for $g \in G$.

Problem: For $\dim G < \infty$, determine structural implications for G and describe wedge regions $W \subseteq M$.

Modular groups of von Neumann algebras

$\mathcal{H}$ a complex Hilbert space, $B(\mathcal{H})$ bounded operators on $\mathcal{H}$

Commutant of $\mathcal{S} \subseteq B(\mathcal{H})$: $\mathcal{S}' = \{a \in B(\mathcal{H}) \colon (\forall s \in \mathcal{S}) as = sa\}$

von Neumann algebra: $\mathcal{M} \subseteq B(\mathcal{H})$ a $*$-subalgebra with $\mathcal{M} = \mathcal{M}''$.

For a von Neumann algebra $\mathcal{M} \subseteq B(\mathcal{H})$, a vector $\Omega \in \mathcal{H}$ is called

- cyclic if $\overline{\mathcal{M}\Omega} = \mathcal{H}$.
- separating if $M \in \mathcal{M}, M\Omega = 0$ implies $M = 0$.

Theorem (Tomita 1967, Takesaki 1970)

Any cyclic and separating vector $\Omega \in \mathcal{H}$ for the von Neumann algebra $\mathcal{M}$ determines a conjugation J (=antilinear isometry) and a positive selfadjoint operator $\Delta > 0$ such that

$$J\Delta J = \Delta^{-1}, \quad J\mathcal{M}J = \mathcal{M}' \quad \text{and} \quad \Delta^{it}\mathcal{M}\Delta^{-it} = \mathcal{M} \quad \text{for} \quad t \in \mathbb{R}$$

(modular automorphism group)

From Ω to (Δ, J): The operator $S(M\Omega) = M^*\Omega$, $M \in \mathcal{M}$, is densely defined. Its closure has the polar decomposition $\overline{S} = J\Delta^{1/2}$.

Examples of wedge regions

Minkowski spacetime: $M = \mathbb{R}^{1,d-1}$,
$G = \mathbb{R}^d \rtimes \mathrm{SO}_{1,d-1}(\mathbb{R})^{\uparrow}$ (Poincaré group),
Wedge region: $W_R = \{(x_0, \mathbf{x}) \in \mathbb{R}^{1,d-1} \colon x_0 > |x_1|\}$ (Rindler wedge)
Modular group is implemented by the Lorentz boosts

$$\gamma(t) = e^{th}, \quad h(x_0, \mathbf{x}) = (x_1, x_0, 0, \ldots, 0).$$

Conformal compactif. of Minkowski space: $M = (\mathbb{S}^1 \times \mathbb{S}^{d-1})/\{\pm\}$,
$G = \mathrm{SO}_{2,d}(\mathbb{R})_e$ (conformal group).
Wedge regions: G-translates $W = g.W_R$ (double cones, future or past light cone) and $\gamma_W(t) = g\gamma(t)g^{-1}$.

CFT: $M = \mathbb{S}^1$, $G = \text{Möb} \cong \mathrm{PGL}_2(\mathbb{R})$, $W \subseteq \mathbb{S}^1$ is an open non-dense interval and the modular group is conjugate to $\mathrm{PSO}_{1,1}(\mathbb{R})$ in G.

de Sitter space: $M = \mathrm{dS}^d = \{x \in \mathbb{R}^{1,d} \colon x_0^2 - \mathbf{x}^2 = -1\}$,
$G = \mathrm{SO}_{1,d}(\mathbb{R})^{\uparrow}$ (Lorentz group),
Wedge region: $W = W_R \cap \mathrm{dS}^d$
Modular group = Lorentz boosts (same as for Minkowski space $\mathbb{R}^{1,d}$).

Def.: A element h of a Lie algebra $\mathfrak{g}$ is called an Euler element if $\mathrm{ad}\, h$ is diagonalizable with eigenvalues $\subseteq \{-1, 0, 1\}$.

Theorem (Euler Element Theorem, Morinelli, N.,'22)

Let $U\colon G \to \mathrm{U}(\mathcal{H})$ be a unitary rep., $\dim G < \infty$ *and* $\ker(U)$ *discrete.*

(a) *Ω is a unit vector fixed by $U(G)$.*

(b) *$\mathcal{M} \subseteq B(\mathcal{H})$ a von Neumann algebra with Ω cylic and separating.*

(c) *$\mathcal{N} \subseteq \mathcal{M}$ a von Neumann algebra for which Ω is cyclic and $\{g \in G\colon U_g \mathcal{N} U_g^{-1} \subseteq \mathcal{M}\}$ is an e-neighborhood in G.*

(d) *For the modular operator Δ of the pair $(\mathcal{M}, \Omega)$, there exists an element $h \in \mathfrak{g}$ such that $U(\exp(th)) = \Delta^{-it/2\pi}$ for $t \in \mathbb{R}$.*

Then h is an Euler element and the modular conjugation J of $(\mathcal{M}, \Omega)$ satisfies

$$JU(\exp x)J = U(\exp \tau_h x) \quad \text{for the involution} \quad \tau_h = e^{\pi i\, \mathrm{ad}\, h}.$$

Important application to nets of local algebras:

$\mathcal{M} = \mathcal{M}(W)$, W wedge region, and $\mathcal{N} = \mathcal{M}(\mathcal{O})$ with $\overline{\mathcal{O}} \subseteq W$ compact.

Euler elements in simple real Lie algebras: (N., Morinelli, CMP, 2021).

$\mathfrak{g} \supseteq \mathfrak{a}$ maximal ad-diagonalizable, $\Sigma(\mathfrak{g}, \mathfrak{a}) \subseteq \mathfrak{a}^*$ restricted root system

$$\{\alpha_1, \ldots, \alpha_n\} \subseteq \Sigma(\mathfrak{g}, \mathfrak{a}) \quad \text{(simple restricted roots; basis of } \mathfrak{a}^*)$$

$h_j \in \mathfrak{a}$ with $\alpha_i(h_j) = \delta_{ij}$ (dual basis). Euler elements h are conjugate to some h_j and red dots mark nodes for which h_j is Euler.

$A_n, \; n \geq 1$:

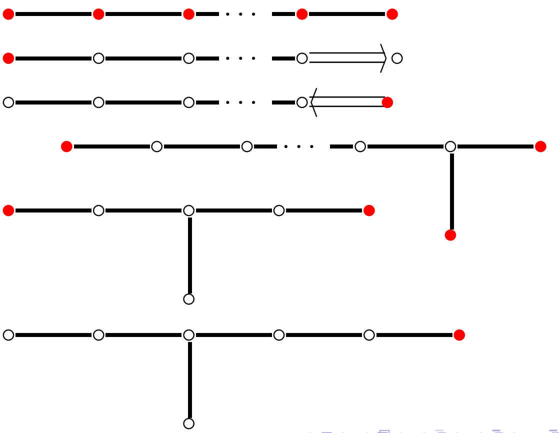
$B_n, \; n \geq 2,$:

$C_n, \; n \geq 3,$:

$D_n, \; n \geq 4,$:

E_6 :

E_7 :



Inclusions of von Neumann algebras

- $\mathcal{M} \subseteq B(\mathcal{H})$ a von Neumann algebra
- $\Omega \in \mathcal{H}$ cyclic separating unit vector for $\mathcal{M}$ with modular data (Δ, J).
- A unitary representation $U\colon G \to \mathrm{U}(\mathcal{H})$ fixing Ω.
- An Euler element $h \in \mathfrak{g}$ such that $\Delta^{-it/2\pi} = U(\exp th)$ for $t \in \mathbb{R}$.
- $JU(\exp x)J = U(\exp \tau_h x)$ for the involution $\tau_h = e^{\pi i \,\mathrm{ad}\, h}$.

Then inclusions of von Neumann algebras among the $(U_g \mathcal{M} U_g^{-1})_{g\in G}$ are encoded in the endomorphism semigroup

$$S_{\mathcal{M}} = \{g \in G\colon U_g \mathcal{M} U_g^{-1} \subseteq \mathcal{M}\}$$
$$\supseteq G_{\mathcal{M}} = \{g \in G\colon U_g \mathcal{M} U_g^{-1} = \mathcal{M}\} = S_{\mathcal{M}} \cap S_{\mathcal{M}}^{-1}.$$

To describe this semigroup, we need the positive cone of U

$$C_U := \{x \in \mathfrak{g}\colon -i \cdot \partial U(x) \geq 0\}, \qquad \partial U(x) = \frac{d}{dt}\Big|_{t=0} U(\exp tx),$$

which is a closed, convex, $\mathrm{Ad}(G)$-invariant cone in $\mathfrak{g}$.

The Euler element h defines a 3-grading

$$\mathfrak{g} = \mathfrak{g}_1(h) \oplus \mathfrak{g}_0(h) \oplus \mathfrak{g}_{-1}(h) \quad \text{with} \quad \mathfrak{g}_\lambda(h) = \ker(\mathrm{ad}\, h - \lambda \mathbf{1}).$$

The endomorphism semigroups can be determined:

Theorem (Structure Theorem for $S_{\mathcal{M}}$), N. 2020, 2021)

If $\ker(U)$ *is discrete and* $C_\pm = \pm C_U \cap \mathfrak{g}_{\pm 1}(h)$, *then*

$$S_{\mathcal{M}} = G_{\mathcal{M}} \exp(C_+ + C_-) = \exp(C_+) G_{\mathcal{M}} \exp(C_-) \quad \textit{and} \quad \mathbf{L}(G_{\mathcal{M}}) = \mathfrak{g}_0(h).$$

Inclusions imply spectral conditions:
$S_{\mathcal{M}} \neq G_{\mathcal{M}} \Rightarrow C_U \neq \{0\}$.

Conclusion: Under suitable non-degeneracy assumptions

- “modular groups” in finite-dim. Lie groups are gen. by Euler elts
- inclusions can be determined by the positive cone of the rep.

Problem: Construct such nets of operator algebras!
On which homogeneous spaces do they exist?

Causal symmetric spaces

- $M = G/H$ symmetric space: $H \subseteq G^\tau$ open subgr., τ involution on G
- $(\mathfrak{g}, \tau)$ symmetric Lie algebra, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}, \quad \mathfrak{h} = \mathfrak{g}^\tau, \quad \mathfrak{q} = \mathfrak{g}^{-\tau}$
- $(\mathfrak{g}, \tau, C)$ causal if $C \subseteq \mathfrak{q}$ is $\mathrm{Ad}(H)$-invariant pointed generating closed convex cone. Then

$$C \subseteq \mathfrak{q} \cong T_{eH}(G/H) \quad \text{and} \quad C_{gH} := g.C \subseteq T_{gH}(G/H)$$

  defines a G-invariant cone field (causal structure) on G/H.
- $(\mathfrak{g}, \tau, C)$ non-compactly causal (ncc) if C is hyperbolic ($x \in C^\circ \Rightarrow \mathrm{ad}\, x$ diagonalizable).
- $(\mathfrak{g}, \tau, C)$ compactly causal (cc) if C is elliptic ($x \in C^\circ \Rightarrow \mathrm{ad}\, x$ elliptic).

Duality: $(\mathfrak{g}, \tau, C)$ is cc $\Longleftrightarrow (\mathfrak{g}^c = \mathfrak{h} + i\mathfrak{q}, \tau^c, iC)$ is ncc

Lorentzian exs.: de Sitter space: $\mathrm{dS}^d \cong \mathrm{SO}_{1,d}(\mathbb{R})/\mathrm{SO}_{1,d-1}(\mathbb{R})$ is ncc
Anti-de Sitter space: $\mathrm{AdS}^d \cong \mathrm{SO}_{2,d-1}(\mathbb{R})/\mathrm{SO}_{1,d-1}(\mathbb{R})$ is cc

From Euler elements to causal symmetric spaces

Let $\mathfrak{g}$ be simple and $h \in \mathfrak{g}$ an Euler element.
There exists a Cartan involution θ with $\theta(h) = -h$
$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ corresp. Cartan decomposition, $\mathfrak{k} = \mathfrak{g}^\theta, \mathfrak{p} = \mathfrak{g}^{-\theta}$

Theorem

For $\tau := \theta e^{\pi i \operatorname{ad} h}$ the triple $(\mathfrak{g}, \tau, C)$ is non-compactly causal for $C := \operatorname{cone}(e^{\operatorname{ad} \mathfrak{h}} h) \subseteq \mathfrak{q}$.

In the context of the Theorem we call h a causal Euler element for $(\mathfrak{g}, \tau)$.

Theorem (Classification Theorem; Ólafsson, 1980s)

The above construction leads to a bijection from the set of G-orbits of Euler elements in $\mathfrak{g}$ to isoclasses of irreducible ncc symm. Lie algebras $(\mathfrak{g}, \tau, C)$, where $C \subseteq \mathfrak{q}$ is a minimal $\operatorname{Ad}(H)$-invariant cone.

Consequence: Classification of irreducible causal symmetric spaces by duality and Euler elements. $\Rightarrow$ Natural causal manifolds from Euler elements.

General scheme to obtain nets of algebras

- M a causal G-manifold,
- $(C_m)_{m\in M}$, $C_m \subseteq T_m(M)$ a G-invariant causal structure on M
- $U\colon G \to \mathrm{U}(\mathcal{H})$ unitary representation

$\mathcal{O}$	$\underset{open}{\subseteq}$	$M = G/H$	causal manifold
$\downarrow \mathtt{V}$			net of closed real subspaces (first quantization)
$\mathtt{V}(\mathcal{O})$	$\subseteq$	$\mathcal{H}$	1-particle space with G-rep U
	$\downarrow \Gamma$		a second quantization functor
$\mathcal{M}(\mathcal{O}) = \mathcal{R}(\mathtt{V}(\mathcal{O}))$	$\subseteq$	$B(\Gamma(\mathcal{H}))$	net of von Neumann algebras (local observables)

Ex: Bosonic second quantization: $\Gamma(\mathcal{H}) = \mathcal{F}_+(\mathcal{H})$ Fock space,
$\mathcal{R}(\mathtt{V}) = W(\mathtt{V})''$, where $W(v) \in B(\Gamma(\mathcal{H}))$ are the Weyl operators.

Def. A closed real subspace $\mathtt{V} \subseteq \mathcal{H}$ is called **standard** if

$$\overline{\mathtt{V} + i\mathtt{V}} = \mathcal{H} \quad (\mathtt{V} \text{ cyclic}) \qquad \text{and} \qquad \mathtt{V} \cap i\mathtt{V} = \{0\} \quad (\mathtt{V} \text{ separating}).$$

Then $S(v + iw) := v - iw$ on $\mathtt{V} + i\mathtt{V}$ is a closed operator on $\mathcal{H}$, polar decomposition $S = J_{\mathtt{V}}\Delta_{\mathtt{V}}^{1/2}$ defines a conjugation $J_{\mathtt{V}}$ and a positive operator $\Delta_{\mathtt{V}}$ with $J_{\mathtt{V}}\Delta_{\mathtt{V}}J_{\mathtt{V}} = \Delta_{\mathtt{V}}^{-1}$ and $\mathtt{V} = \mathrm{Fix}(J_{\mathtt{V}}\Delta_{\mathtt{V}}^{1/2})$.

Axioms for a net of real subspaces:

• G Lie group, • M causal G-manifold, • $U \colon G \to \mathrm{U}(\mathcal{H})$ unitary rep.

A net of real subspaces on M is a family $\mathtt{V}(\mathcal{O}) \subseteq \mathcal{H}$ of closed real subspaces, $\mathcal{O} \subseteq M$ open, such that

- Isotony: $\mathcal{O}_1 \subseteq \mathcal{O}_2$ implies $\mathtt{V}(\mathcal{O}_1) \subseteq \mathtt{V}(\mathcal{O}_2)$
- Locality: $\mathcal{O}_1 \subseteq \mathcal{O}_2'$ implies $\mathtt{V}(\mathcal{O}_1) \subseteq \mathtt{V}(\mathcal{O}_2)' := \mathtt{V}(\mathcal{O}_2)^{\perp_\omega}, \omega = \mathrm{Im}\langle \cdot, \cdot \rangle$
- Reeh–Schlieder property: $\mathtt{V}(\mathcal{O})$ is cyclic if $\mathcal{O} \neq \emptyset$.
- Covariance: $U_g\mathtt{V}(\mathcal{O}) = \mathtt{V}(g\mathcal{O})$ for $g \in G$.
- Bisognano–Wichmann prop.: $\mathtt{V}(W)$ is standard for a "wedge domain" $W \subseteq M$ and $\Delta^{-it/2\pi} = U(\exp th)$ for an Euler element $h \in \mathrm{L}(G)$

Fact: Second quantization translates $\mathtt{V}(\mathcal{O})$ into algebras $\mathcal{R}(\mathtt{V}(\mathcal{O}))$ preserving all five axioms and $U(G)$ fixes Ω.

Nets of real subspaces from distribution vectors

Let $U : G \to \mathrm{U}(\mathcal{H})$ be a unitary representation.
$\mathcal{H}^\infty \subseteq \mathcal{H}$ (smooth vectors), $U^\xi \colon G \to \mathcal{H}, g \mapsto U(g)\xi$ smooth.
$\mathcal{H}^\infty$ carries a natural Fréchet topology, which defines a "rigging"

$$\mathcal{H}^\infty \hookrightarrow \mathcal{H} \xrightarrow{\eta} \mathcal{H}^{-\infty}, \quad \eta(\xi) = |\xi\rangle$$

$\mathcal{H}^{-\infty}$ (distribution vectors) = continuous antilinear functionals on $\mathcal{H}^\infty$.
Any test function $\varphi \in C_c^\infty(G, \mathbb{C})$ defines a smearing operator

$$U(\varphi) \colon \mathcal{H}^{-\infty} \to \mathcal{H}, \quad U(\varphi)\eta = \int_G \varphi(g) U(g)\eta \, dg \in \mathcal{H}.$$

Let $\mathtt{E} \subseteq \mathcal{H}^{-\infty}$ be a real subspace, $\mathcal{O} \subseteq M := G/H$ a homogeneous space and $q_M \colon G \to G/H, g \mapsto gH$. Then

$$\mathsf{H}_{\mathtt{E}}^M(\mathcal{O}) := \overline{\mathrm{span}}\{U(\varphi)\mathtt{E}, \varphi \in C_c^\infty(G, \mathbb{R}), \mathrm{supp}(\varphi) \subseteq q_M^{-1}(\mathcal{O})\}$$

defines a covariant isotone net of closed real subspaces on M.

Wedge domains in causal homogeneous spaces

Let $M = G/H$ be a causal homogeneous space,
$(C_m)_{m\in M}$ the invariant cone field (causal structure).
For an Euler element $h \in \mathfrak{g}$, we define the modular flow on M by

$$\alpha_t(m) = \exp(th).m \quad \text{and} \quad X_h^M(m) := \frac{d}{dt}\Big|_{t=0} \alpha_t(m)$$

is called the modular vector field.

Def.: The open subset

$$W := W_M^+(h) := \{m \in M \colon X_h^M(m) \in C_m^\circ\}$$

is called the positivity domain of the modular flow of h on M.

Ex. (a) For $G = \mathrm{Aff}(\mathbb{R})$, $M = \mathbb{R}$ and $\alpha_t(x) = e^t x$: $W_M^+(h) = (0, \infty)$.

(b) For Lorentz boost h on $M = \mathbb{R}^{1,d-1}$: $W_M^+(h) = W_R$ (Rindler wedge).

(c) Same as (b) in conformal completion of $M = \mathbb{R}^{1,d-1}$.

Existence of nets of real subspaces

Group case (N.,Ólafsson, '21; Daniel Oeh '21)

- Spectral condition: $\mathfrak{g} = \mathbb{R}h + (C_U - C_U)$, C_U pos cone of U.
- $M = G$ left action, $C_g = g.C_U$ (causal structure)
- $\alpha_t(g) = \exp(th)g\exp(-th)$ modular flow
- $W = G^h \exp(C_+^\circ + C_-^\circ)$ (semigroup), $C_\pm = \pm C_U \cap \mathfrak{g}_{\pm 1}(h)$
- Isotony, Reeh–Schlieder, Covariance, Bisognano–Wichmann
- **Generalizations:** Nets on Cayley type spaces G/G^h and simple Jordan space times (coverings of conformal completions of Jordan algebras)

Compactly causal symmetric spaces (N.,Ólafsson, '22)

- Spectral condition: $\mathfrak{g} = C_U - C_U$, C_U pos cone of U, $\mathfrak{g}$ semisimple.
- $M = G/H$, $H \subseteq G^\tau$ open, $\tau(h) = h$, $C = C_U \cap \mathfrak{q}$ defines causal structure
- $\alpha_t(gH) = \exp(th)gH$ modular flow
- $W = G^h \exp(C_+^\circ + C_-^\circ)H$, $C_\pm = \pm C_U \cap \mathfrak{g}_{\pm 1}(h)$
- Isotony, Reeh–Schlieder, Covariance, Bisognano–Wichmann
- $\mathtt{E} = \mathbb{R}\eta$ fixed by H and J.
- **Includes:** Group case and Anti-de Sitter space AdS^d.

An approach for ncc spaces (with Jan Frahm and G. Ó.)

- G a connected simple Lie group
- $h \in \mathfrak{g}$ an Euler element, $\theta(h) = -h$, θ Cartan involution
- $(\mathfrak{g}, \tau, C)$ simple ncc, where $\tau = \tau_h\theta$, $\tau_h := e^{\pi i \operatorname{ad} h}$
- $H \subseteq G^\tau$ an open θ-invariant subgroup with $\mathrm{Ad}(H)C = C$
- $M = G/H$ corresponding ncc symmetric space.
- $(U, \mathcal{H})$ an irreducible unitary representation of G.
- J a conjugation on $\mathcal{H}$ with $JU(g)J = U(\tau_h(g))$ for $g \in G$.

Conjecture 1: For any K-finite vector $v \in \mathcal{H}$, the limit

$$\beta(v) := \lim_{t \to \pi/2} e^{it \cdot \partial U(h)} v \quad \text{exists in} \quad \mathcal{H}^{-\infty}$$

and define a $(\mathfrak{g}_{\mathbb{C}}, H \cap K)$-equivariant linear map $\beta \colon \mathcal{H}^{[K]} \to (\mathcal{H}^{-\infty})^{[H]}$ to the space of H-finite distribution vectors.

Consider a finite-dimensional real K-invariant subspace $\mathtt{E}_K \subseteq \mathcal{H}$ of J-fixed vectors and put

$$\mathtt{E}_H := \beta(\mathtt{E}_K) \subseteq \mathcal{H}^{-\infty}.$$

Conjecture 2: Assume that $U(G)\mathtt{E}_K$ spans a dense subspace of $\mathcal{H}$. Then the following assertions hold:

- Reeh–Schlieder Theorem: If $\emptyset \neq \mathcal{O} \subseteq G$, then $\mathsf{H}^G_{\mathtt{E}_H}(\mathcal{O}) = \overline{\mathrm{span}}\{U(C^\infty_c(G,\mathbb{R}))\mathtt{E}_H\}$ is total in $\mathcal{H}$.
- Bisognano–Wichmann property:
  If $W = W^+_M(h) \subseteq G/H$ is the wedge region, then $\mathsf{H}^{G/H}_{\mathtt{E}_H}(W) = \mathsf{H}^G_{\mathtt{E}_H}(q_M^{-1}(W))$ is standard with modular operator $\Delta = e^{2\pi i \cdot \partial U(h)}$.
- **Example:** de Sitter space dS^d (work of Bros/Moschella in '90s)
- For de Sitter space and H-spherical representations one can use reflection positivity on the sphere (N., 'Olafsson, '20)
- Need realizations of unitary representations in bundle-valued distributions which are boundary values of holomorphic functions on suitable complex manifolds. If $M = G/H$ is ncc, then $M \subseteq \partial\Xi$, Ξ complex crown of G/K
  (Krötz, Gindikin, Stanton, 'Olafsson; 2002-2005).

Further observations:

- For compactly causal spaces M there is no natural order ($\exists$ closed causal curves), but the wedge space $\mathcal{W} := \{gW : g \in G\}$ has an order structure (Ex: Anti-de Sitter space).
- For non-compactly causal spaces M there is a global order on M, but the wedge space $\mathcal{W} := \{gW : g \in G\}$ has no order structure (Ex: de Sitter space).
- Duality $M \leftrightarrow M^c$ between compactly causal and non-compactly causal symmetric spaces with Lie algebra $(\mathfrak{g}, \tau)$
  - $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q} \leftrightarrow \mathfrak{g}^c = \mathfrak{h} \oplus i\mathfrak{q}, \quad \mathfrak{h} = \mathrm{Fix}(\tau), \mathfrak{q} = \mathrm{Fix}(-\tau)$
  - de Sitter $\leftrightarrow$ anti-de Sitter,
  - group type spaces $M = G \leftrightarrow M^c = G_{\mathbb{C}}/G$.
- Geometric characterization of wedge domains (for cc and ncc symmetric spaces on which the modular flow has fixed points): Wedge domains $W = W_M^+(h)$ can also be characterized by KMS-like conditions or by polar decompositions of domains (N., Ólafsson, '21,'22).

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