Differentiable and Kähler geometry in the coadjoint orbit of a nuclear operator

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Definitions

H separable complex Hilbert space

$B(H)$ linear bounded operators on H

$B_1(H)$ *nuclear operators* = compact & singular values in $\ell_1(\mathbb{N})$

$U(\mathcal{H})$ unitary operators; Lie group with Lie algebra

$$Lie(U(H)) = B(H)_{ah} = \{V^* = -V : V \in B(\mathcal{H})\}.$$

$U_1(H)$ *nuclear unitary group* =

$$\{u \in U(H) : u - 1 \in B_1(H)\} = \{e^z : z \in B_1(H)_{ah}\}$$

$\mathcal{O}(\rho)$ The *unitary coadjoint orbit of* $\rho \in B_1(H)_{ah}$

$$\mathcal{O}(\rho) = \{u\rho u^* : \ u \in U_1(H)\} \subset B_1(H)_{ah}.$$

Part 1 (joint work with Daniel Beltiță)

MANIFOLD STRUCTURE OF $\mathcal{O}(\rho)$ AND ITS CLOSURE $\mathcal{O}_\rho$

$\rho^* = -\rho$ $\qquad \rho = i \sum_{j=0}^{\infty} \lambda_j \, p_j \qquad$ real $\lambda_j \neq \lambda_k$, infinitely many

p_0 kernel projection (possibly ∞-dimensional)

p_j eigen-projections ; finite rank for $j \neq 0$

$\mathcal{O}(\rho) = \{u\rho u^* : u \in B_1(H)\} \subset B_1(H)$ is NOT closed
(closed if and only if the spectrum of ρ is finite).

Unitary orbit

Let P_x = orthogonal range projection of $x \in B(H)$, then

$\overline{\mathcal{O}(\rho)}^{\|\cdot\|_\infty} = \overline{\mathcal{O}(\rho)}^{\|\cdot\|_1} = \{v\rho v^* : v^*v = P_\rho\} =: \mathcal{O}_\rho$ (Bel-Lar-22)

the orbit with partial isometries with initial space P_ρ.

$\ker \pi = K$ Action map $\pi : U_1(H) \to B_1(H)_{ah}$, $\pi : u \mapsto u\rho u^*$, with stabilizer $K \subset U_1(H)$ subgroup of unitaries commuting with ρ.

Theorem K Banach-Lie subgroup; $U_1(H)/K$ has a Banach-manifold structure s.t. that $q : U_1(H) \to U_1(H)/K$ is a submersion.

$j : U_1(H)/K \to \mathcal{O}(\rho)$, $j : uK \mapsto u\rho u^*$ is a continuous bijection.

Unitary orbit

Remark

Via $j : uK \mapsto u\rho u^*$, the orbit $\mathcal{O}(\rho) \simeq U_1(H)/K$ has a manifold structure s.t. $i : \mathcal{O}(\rho) \hookrightarrow B_1(H)$ is a *weak immersion*: i_* the diferential of the inclusion map i, is injective but its range is not closed in $B_1(H)$.

Bel-L-22

The following are equivalent:

1. the range of i_* is closed
2. $\mathcal{O}(\rho)$ is closed in $B_1(H)$
3. $\mathcal{O}(\rho) \subset B_1(H)$ is an embedded manifold
4. the action map π has smooth strong cross-sections
5. ρ has finite spectrum.

Groupoid orbit

Recall $\mathcal{O}_\rho = \overline{\mathcal{O}(\rho)} = \{x = v\rho v^* : v^*v = P_\rho,\ vv^* = P_x\}$,

$\mathcal{V}_\rho$ $= $ partial isometries of $B(H)$ s.t. $v^*v = P_\rho$,

K_ρ $= \{v \in \mathcal{V}_\rho : v\rho v^* = \rho\}$

Theorem $\mathcal{V}_\rho$ is a submanifold of $B(H)$ and K_ρ a Lie subgroup

(Andruchow-Corach-Mbekhta-05, Bel-Goliński-Jakimowicz-Pelletier-19)

and $\mathcal{O}_\rho \simeq \mathcal{V}_\rho / K_\rho$ has a Banach manifold structure such that the inclusion maps $\mathcal{O}(\rho) \xhookrightarrow{i_1} \mathcal{O}_\rho \xhookrightarrow{i_2} B(H)$ are weak immersions

Bel-L $\mathcal{O}_\rho$ is closed in $B(H) \Leftrightarrow 0$ is the only accum. point of $\sigma(\rho)$.

Moreover In that case, if the spectrum is infinite, the manifold topology of $\mathcal{O}_\rho$ is strictly finer than the norm topology of $B(H)$.

Part 2 (joint work with Tomasz Goliński)

Kähler geometry in the unitary orbit $\mathcal{O}(\rho)$

- $\rho = i \sum_{j=0}^{\infty} \lambda_j \, p_j$, infinitely many $\lambda_j \neq \lambda_k$
- $B(H)_{ah} = D_\rho \oplus C_\rho$ diagonal / co-diagonal decomposition with respect to ρ: if $x_{jk} = p_j x p_k$ then

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots \\ -x_{12}^* & x_{22} & x_{23} & \cdots \\ -x_{13}^* & -x_{23}^* & x_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} x_{11} & 0 & 0 & \cdots \\ 0 & x_{22} & 0 & \cdots \\ 0 & 0 & x_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} + \begin{pmatrix} 0 & x_{12} & x_{13} & \cdots \\ -x_{12}^* & 0 & x_{23} & \cdots \\ -x_{13}^* & -x_{23}^* & 0 & \cdots \\ \vdots & \vdots & \vdots & 0 \end{pmatrix}$$

- $E : B_1(H) \to B_1(H)$ the map $E : x \mapsto \sum_{j=0}^{\infty} p_j x p_j$ projection onto D_ρ with $\ker E = D_\rho$.

Geometry of the orbit $\mathcal{O}(\rho)$

- $\mathcal{O}(\rho) = \{u\rho u^* : u \in B_1(H)\} \subset B_1(H)$ is NOT closed
- $U_1(H)/K \simeq \mathcal{O}(\rho)$, $\mathcal{O}(\rho) \hookrightarrow B_1(H)$ only a weak immersion
- $Lie(K) = D_\rho \cap U_1(H)$ diagonal unitary operators,
- $T_\rho \mathcal{O}(\rho) \simeq C_\rho \cap B_1(H)_{ah}$ co-diagonal operators
- $TU_1(H) \simeq U_1(H) \times B_1(H)_{ah}$ the tangent bundle of the group
- $T\,\mathcal{O}(\rho) \simeq U_1(H) \times_K C_\rho \cap B_1(H)_{ah}$

where $k \cdot (u, z) = (uk^{-1}, kzk^*)$

Kähler geometry of $\mathcal{O}(\rho)$

$$T_\rho \mathcal{O}(\rho) \simeq C_\rho \cap B_1(H)_{ah}, \quad T\,\mathcal{O}(\rho) \simeq U_1(H) \times_K C_\rho \cap B_1(H)_{ah}$$

Kähler structure: $\boxed{\omega(v, Jw) = g(v, w)}$

ω KKS-symplectic form $\omega_x(v,w) = Tr(x[v,w])$, $x \in \mathcal{O}(\rho)$

J Complex structure $J(v) = i \sum_{j=1}^{\infty} \sum_{k=0}^{j-1} p_k v p_j - p_j v p_k$

g Riemannian metric $G(v) = \sum_{j,k \geq 0} |\lambda_j - \lambda_k| \, p_j v p_k$

$$g(v,w) = -Tr(G(v)w)$$

Levi-Civita connection and geodesics in $(\mathcal{O}(\rho), g)$

$X = ux$, $Y = uy$ with $x, y \in T\,\mathcal{O}(\rho) = C_\rho \cap B_1(H)_{ah}$
left-invariant vector fields then (Goliński-L-22)

∇

$$u^*\nabla_X Y(u) = 1/2\left((1-E)[x,y] + G^{-1}([G(x),y] + [G(y),x]\right)$$

geodesics

γ geodesic $\iff \gamma = \Gamma_t\,\rho\,\Gamma_t^*$ with $\Gamma_t \subset U_1(H)$ such that if $c_t = \Gamma_t^{-1}\Gamma_t^*$, then $c_t \subset C_\rho$ and

$$G(c_t') = [G(c_t), c_t]$$

- $G(c_t)$ has constant eigenvalues, and the eigenspaces of its non-zero eigenvalues have constant dimension.
- $G(c_t) = v_t\,G(c_0)\,v_t^*$ with $v_t^* v_t \in P_{G(c_0)}$ a path in $\mathcal{V}_{G(c_0)}$.

Thank You very much for your kind attention!

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