Quadratic algebras and spectrum of superintegrable systems

Fazlul Hoque
Faculty of Nuclear Sciences and Physical Engineering
Czech Technical University in Prague

XXXIX Workshop on Geometric Methods in Physics

University of Bialystok, Poland

Joint Work: F. Correa, I. Marquette and Y-Z. Zhang

June 19-25, 2022

Superintegrable systems

- In a classical mechanics, an n-dimensional Hamiltonian system with Hamiltonian

$$H = \frac{1}{2} g_{ik} p_i p_j + V(\vec{x}, \vec{p}), \quad X_a = f_a(\vec{x}, \vec{p}), a = 1, \dots, n-1,$$

  is called completely integrable (Liouville integrable) if it allows n integrals of motion (including H) that are well-defined functions on phase space, are in involution

$$\{H, X_a\}_p = 0, \quad \{X_a, X_b\}_p = 0, a, b = 1, \dots, n-1$$

  and are functionally independent
- The system is superintegrable if it is integrable and allows additional integrals of motion $Y_b(\vec{x}, \vec{p})$, $\{H, Y_b\}_p = 0, b = n, n+1, \dots, n+k,\ k = 1, \dots, n-1$ that are also well-defined functions on phase space and the integrals $\{H, X_1, \dots, X_{n-1}, Y_n, \dots, Y_{n+k}\}$ are functionally independent
- It is maximally superintegrable if the set contains $2n-1$ functions and minimally superintegrable if it contains $n+1$ such integrals.
- The same definitions apply in quantum mechanics but $\{H, X_a, Y_b\}$ are well-defined quantum mechanical operators, assumed to form an algebraically independent set
- The best known examples of (maximally) superintegrable systems are the Kepler-Coulomb $V(\vec{x}) = \frac{\alpha}{r}$ (Fock-1935, Bargmann 1936) and the harmonic oscillator $V(\vec{x}) = \alpha r^2$ (Jauch, Hill 1940, Moshinsky, Smirnov 1966)
- Miller, Post, Winternitz 2013, J.Phys.A: Math.Theor. 46, 423001 (review paper)

$$H = -\frac{1}{2}\sum_{i=1}^{N}\partial_i^2 + b\sum_{i=1}^{N}x_i^2 + \sum_{i=1}^{N}\frac{a_i}{x_i^2}, \tag{1}$$

where all masses are equal and we set $\hbar = m_i = 1$, $\partial_i = \partial/\partial x_i$.

- The SW system on 2- and 3-dimensional Euclidean space are best examples of maximally superintegrable systems
  - Winternitz P, Smorodinsky Y A,Uhlir Mand Fris I 1967, Sov. J. Nucl. Phys. 4 44
  - Makarov A A, Smorodinsky J A, Valiev K and Winternitz P 1967, Nuov Cim. A 52 1061
  - Evans N W 1990 Superintegrability in classical mechanics Phys. Rev. A 41 5666
- how to apply R-matrix approach to the Rosochatius model, which is the generalization of the SW system, studied
  - Gagnon L, Harnad J and Winternitz P 1985, J. Math. Phys. 26 7
- The Rosochatius model and its various applications (e.g. to Myers–Perry black holes and resonant space-times) were studied
  - Ivanov E, Nersessian A and Shmavonyan H 2019, Phys. Rev. D 99 085007
  - Galajinsky A, Nersessian A and Saghatelian A 2013, J. High Energy Phys. JHEP06(2013)002
  - Evnin O, Demirchian H and Nersessian A 2018, Phys. Rev. D 97 025014

The N-dimensional Smorodinsky-Winternitz system

- The quadratic algebras and algebraic derivations of spectra for 2D models were presented
  - Daskaloyannis C 2001, J. Math. Phys. 42 1100
- A similar approach was studied for many other 2D models (e.g. SW-i,ii,iii, and others)
  - Post S 2011, SIGMA 7 036
- The *N*-dimensional analogs of the SW system have been formulated
  - Evans N W 1990 Super-integrability of the Winternitz system Phys. Lett. A 147 483
  - Evans N W 1991 Group theory of the Smorodinsky–Winternitz system J. Math. Phys. 32 3369
- The symmetry algebra of the classical SW system with a magnetic field was constructed
  - Shmavonyan H 2019 C^N-Smorodinsky-Winternitz system in a constant magnetic field Phys. Lett. A 383 1223
- The supersymmetric extensions of the SW system in the complex Euclidean space $\mathbb{C}^N$ were investigated
  - Ivanov E, Nersessian A and Shmavonyan H 2019, Phys. Rev. D 99 085007
- Other types of algebraic construction of the SW system based on dynamical potential algebra and dynamical symmetries have been obtained
  - Quesne C 2011 Revisiting the symmetries of the quantum Smorodinsky-Winternitz system in D-dimensions SIGMA 7 035
  - Kerimov G A 2012, J. Phys. A: Math. Theor. 45 185201

The N-dimensional Smorodinsky-Winternitz system

Our purpose is to-

- present an algebraic derivation of the spectrum of the N-dimensional SW system based on the complete symmetry algebra
- obtain various subalgebraic structures of the symmetry algebra which consist of distinct quadratic algebras $Q(3)$ and their Casimirs
- show how these substructures enable us to algebraically determine the spectrum of the SW system
- Separation of variables of the N-dimensional SW system

Separation of Variables

The Schrödinger equation H $\Psi = E\Psi$

- The corresponding Schrödinger equation and Hamilton-Jacobi equation of the SW systems allow separation of variables in various coordinate systems
- In quantum mechanics, the separation of variables in Cartesian coordinates of the $H\Psi = E\Psi$ is done[1] via

$$\Psi = \prod_{i=1}^{N} \psi_{n_i} \ , \quad \psi_{n_i} = N_{n_i} e^{-\sqrt{\frac{b}{2}} x_i^2} x_i^{\frac{1}{2} \pm \nu_i} L_{n_i}^{\pm \nu_i}(\sqrt{2b} x_i^2),$$

in terms of the associated Laguerre polynomials $L_n^a(x)$ and $\nu_i = \frac{1}{2}\sqrt{1+8a_i}$, $i = 1, ..., N$.

- The corresponding spectrum and the degeneracies of each level are

$$E = \sqrt{2b} \sum_{i=1}^{N} (2n_i \pm \nu_i + 1), \qquad \deg(n) = \binom{N+n-1}{N-1}, \tag{2}$$

where $n = \sum_{i=1}^{N} n_i$.

[1] Evans 1991, J. Math. Phys. 32 3369

Hyperspherical coordinates

- The system H is also separable in N-dimensional hyperspherical coordinates, and the Hamitonian operator H reduces to

$$
\begin{aligned}
H = & -\frac{1}{2}\left[\frac{\partial^2}{\partial r^2}+\frac{N-1}{r}\frac{\partial}{\partial r}-2br^2\right]-\frac{1}{2r^2}\left\{\left(\frac{\partial^2}{\partial\theta_1^2}+(N-2)\cot\theta_1\frac{\partial}{\partial\theta_1}-\frac{2a_1}{\cos^2\theta_1}\right.\right.\\
& +\frac{1}{\sin^2\theta_1}\left(\frac{\partial^2}{\partial\theta_2^2}+(N-3)\cot\theta_2\frac{\partial}{\partial\theta_2}-\frac{2a_2}{\cos^2\theta_2}\right.\\
& +\frac{1}{\sin^2\theta_2}\left(\frac{\partial^2}{\partial\theta_3^2}+(N-4)\cot\theta_3\frac{\partial}{\partial\theta_3}-\frac{2a_3}{\cos^2\theta_3}\right.\\
& \quad \cdots\\
& \quad \cdots\\
& +\frac{1}{\sin^2\theta_{N-3}}\left(\frac{\partial^2}{\partial\theta_{N-2}^2}+\cot\theta_{N-2}\frac{\partial}{\partial\theta_{N-2}}-\frac{2a_{N-2}}{\cos^2\theta_{N-2}}\right.\\
& \left.\left.\left.\left.\left.\left.+\frac{1}{\sin^2\theta_{N-2}}\left(\frac{\partial^2}{\partial\theta_{N-1}^2}-\frac{2a_{N-1}}{\cos^2\theta_{N-1}}-\frac{2a_N}{\sin^2\theta_{N-1}}\right)\right)\cdots\right)\right)\right)\right)\right\}.
\end{aligned}
$$

- The ansatz

$$\Psi = \psi(r) \prod_{l=1}^{N-1} \psi(\theta_l)$$

in the Schrödinger equation $H\Psi = E\Psi$, reduce to

$$-\frac{1}{2}\left(\frac{\partial^2}{\partial r^2} + \frac{N-1}{r}\frac{\partial}{\partial r} - 2br^2 + \frac{k_1}{r^2}\right)\psi(r) = E\psi(r), \quad (3)$$

$$\left(\frac{\partial^2}{\partial \theta_\ell^2} + (N-\ell-1)\cot\theta_\ell \frac{\partial}{\partial \theta_\ell} - \frac{2a_\ell}{\cos^2\theta_\ell} + \frac{k_{\ell+1}}{\sin^2\theta_\ell}\right)\psi(\theta_\ell) = -k_\ell \psi(\theta_\ell), \quad (4)$$

$$\left(\frac{\partial^2}{\partial \theta_{N-1}^2} - \frac{2a_{N-1}}{\cos^2\theta_{N-1}} - \frac{2a_N}{\sin^2\theta_{N-1}}\right)\psi(\theta_{N-1}) = -k_{N-1}\psi(\theta_{N-1}), \quad (5)$$

where $\ell = 1, 2, \ldots, N-2$.

- After a long computation, the solutions are as follows

$$\psi(\theta_{N-1}) \propto \cos^{1/2\pm\nu_{N-1}}(\theta_{N-1}) \sin^{1/2\pm\nu_N}(\theta_{N-1}) P_{\tau_{N-1}}^{(\pm\nu_N, \pm\nu_{N-1})}(\cos(2\theta_{N-1})).$$

$$\psi(\theta_l) \propto \cos^{1/2\pm\nu_l}(\theta_l) \sin^{\mu_{l+1}+1-(N-l)/2}(\theta_{N-1}) P_{\tau_l}^{(\mu_{l+1}, \pm\nu_l)}(\cos(2\theta_l)),$$

$$\psi(r) := \psi_{\tau_r}^{2\nu}(r) \propto e^{-\sqrt{\frac{b}{2}}r^2} r^{2\nu - \frac{N-2}{2}} L_{\tau_r}^{2\nu}(\varepsilon r^2),$$

- the energy spectrum of the system

$$E = \sqrt{2b}\left(2\tau_r + 2\sum_{i=1}^{N-1}\tau_i \pm \sum_{i=1}^{N}\nu_i + N\right), \tag{6}$$

where

$$k_l = \left[2\sum_{i=l}^{N-1}\tau_i \pm \sum_{i=l}^{N}\nu_i + (N-l)\right]^2 - \frac{1}{4}(N-l-1)^2, \quad l = 1, \dots, N-3,$$

$$\mu_l = 2\sum_{i=l}^{N-1}\tau_i \pm \sum_{i=l}^{N}\nu_i + \frac{N-l-2}{2}, \quad l = 1, \dots, N-1.$$

$$\tau_r = \frac{E}{2\varepsilon} - \nu - \frac{1}{2}, \qquad 2\nu = 2\sum_{i=1}^{N-1}\tau_i \pm \sum_{i=1}^{N}\nu_i + (N-1), \quad \varepsilon = \sqrt{2b}. \tag{7}$$

Algebraic Derivations

The N-dimensional SW system H is superintegrable. It has the following second order integrals of motion

$$B_i = -\partial_i^2 + 2bx_i^2 + 2\frac{a_i}{x_i^2},$$

$$A_{ij} = -J_{ij}^2 + 2\frac{a_i x_j^2}{x_i^2} + 2\frac{a_j x_i^2}{x_j^2} + \frac{1}{2} \quad (= A_{ji}),$$

where

$$J_{ij} = x_i\partial_j - x_j\partial_i, \quad i, j = 1, 2, \ldots, N.$$

From the definition of the Hamiltonian H, it is clear the integrals B_i satisfy

$$H = \frac{1}{2}\sum_i^N B_i$$

We can easily verify the following commutation relations

$$[H, B_i] = [H, A_{ij}] = [B_i, B_j] = [A_{ij}, B_k] = 0, \quad i, j, k = 1, 2, \ldots, N \quad \text{and} \quad k \neq i, j.$$

We can further define more conserved charges

$$C_{ij} = [B_i, A_{ij}] = [B_j, A_{ij}], \quad D_{ijk} = [A_{ij}, A_{jk}],$$

$$[C_{ij}, H] = 0 = [D_{ijk}, H]$$

Symmetry algebra $\mathcal{SW}(N)$

It can be shown that the above constants of motion of the system H close to satisfy the following quadratic symmetry algebra $\mathcal{SW}(N)$ relations,

$$
\begin{aligned}
&[A_{jk}, D_{ijk}] = 4\{A_{ik}, A_{jk}\} - 4\{A_{jk}, A_{ij}\} + 4(8a_j - 3)A_{ik} - 4(8a_k - 3)A_{ij},\\
&[A_{kl}, D_{ijk}] = 4\{A_{ik}, A_{jl}\} - 4\{A_{jk}, A_{il}\},\\
&[D_{ijk}, D_{jkl}] = 4\{D_{jkl}, A_{ij}\} - 4\{D_{ikl}, A_{jk}\} - 4\{D_{ijk}, A_{jl}\} - 4(8a_j - 3)D_{ikl},\\
&[D_{ijk}, D_{klm}] = 4\{D_{ilm}, A_{jk}\} - 4\{D_{jlm}, A_{ik}\},\\
&[C_{ik}, C_{kl}] = 4\{C_{li}, B_k\},\\
&[B_i, D_{ijk}] = 4\{B_k, A_{ij}\} - 4\{B_j, A_{ik}\},\\
&[B_i, C_{ij}] = -4\{B_i, B_j\} + 32bA_{ij},\\
&[C_{ij}, D_{jkl}] = 4\{C_{il}, A_{jk}\} - 4\{C_{ik}, A_{jl}\},\\
&[C_{ij}, D_{ijk}] = -4\{C_{ik}A_{ij}\} - 4\{C_{jk}, A_{ij}\},\\
&[A_{ij}, C_{ij}] = 4\{A_{ij}, B_j\} - 4\{A_{ij}, B_i\} - 4(8a_j - 3)B_i + 4(8a_i - 3)B_j,\\
&[A_{ij}, C_{ki}] = 4\{A_{kj}, B_i\} - 4\{A_{ik}, B_j\},
\end{aligned}
$$

where $i \neq j \neq k \neq l \neq m$ with $i, j, k, l, m \in \{1, ..., N\}$ covering all non-vanishing commutators.

- The relations involving A_{ij} and D_{lmn} define the Racah algebra $\mathcal{R}(N)$, which has been the subject of attention in last years with connections to many other algebraic structures.
- It is interesting to see $\mathcal{R}(N)$ is embedded in the larger symmetry algebra $\mathcal{SW}(N)$ of the N-dimensional Smorodinsky-Winternitz system.

The quadratic algebra $\mathcal{Q}(3)$

- The structures of the $\mathcal{SW}(N)$ and $\mathcal{R}(N)$ are complicated for $N > 3$ and higher rank
- To algebraically derive the spectrum, we exploit the existence of set of commutating integrals, i.e., different subalgebras involving 3 generators which has similarity with the quadratic algebra $\mathcal{Q}(3)$ introduced in context of two-dimensional systems[2]
- The algebraic approach involves identifying N substructures $\mathcal{Q}_i(3)$, each involving 3 generators $\{E_i, F_i, G_i\}$ for any *fixed* $i = 1, \ldots N$ and satisfy the general commutation relations

$$\begin{aligned} [E_i, F_i] &= G_i, \\ [E_i, G_i] &= \alpha_i A_i^2 + \gamma_i\{E_i, G_i\} + \delta_i E_i + \epsilon_i F_i + \zeta_i, \\ [F_i, G_i] &= a_i E_i^2 - \gamma_i F_i^2 - \alpha_i\{E_i, F_i\} + d_i E_i - \delta_i F_i + z_i, \end{aligned} \tag{8}$$

- The structure constants for each of the substructures, α_i, γ_i, δ_i, ϵ_i, ζ_i, a_i, d_i, z_i, are constants or more generally polynomials of central elements of the i-th substructure
- Each substructure has a cubic Casimir invariant as,

$$\begin{aligned} K_i = {} & G_i^2 - \alpha_i\{E_i^2, F_i\} - \gamma_i\{E_i, F_i^2\} + (\alpha_i\gamma_i - \delta_i)\{E_i, F_i\} + (\gamma_i^2 - \epsilon_i)F_i^2 \\ & + (\gamma_i\delta_i - 2\zeta_i)F_i + \frac{2a_i}{3}E_i^3 + \left(d_i + \frac{a_i\gamma_i}{3} + \alpha_i^2\right)E_i^2 + \left(\frac{a_i\epsilon_i}{3} + \alpha_i\delta_i + 2z_i\right)E_i. \end{aligned} \tag{9}$$

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[2] Daskaloyannis 2001, J. Math. Phys.42 1100

Deformed oscillator algebra

- The quadratic algebra $\mathcal{Q}_i(3)$ (8) for any fixed i value can be realized in terms of the deformed oscillator algebra[3],

$$[\aleph_i, b_i^\dagger] = b_i^\dagger, \quad [\aleph_i, b_i] = -b_i, \quad b_i b_i^\dagger = \Phi(\aleph_i + 1), \quad b_i^\dagger b_i = \Phi(\aleph_i), \tag{10}$$

- the function $\Phi(x)$ is real valued function satisfying

$$\Phi(0) = 0, \quad \Phi(x) > 0, \quad \forall x > 0$$

- The structure function is given by,

$$\begin{aligned}
\Phi_i(n_i) = & \frac{1}{4}\left[-\frac{K_i'}{\epsilon_i} - \frac{z_i}{\sqrt{\epsilon_i}} - \frac{\delta_i}{\sqrt{\epsilon_i}}\frac{\zeta_i}{\epsilon_i} + \left(\frac{\zeta_i}{\epsilon_i}\right)^2\right] \\
& - \frac{1}{12}\left[3d_i - a_i\sqrt{\epsilon_i} - 3\alpha_i\frac{\delta_i}{\sqrt{\epsilon_i}} + 3\frac{\delta_i^2}{\epsilon_i} - 6\frac{z_i}{\sqrt{\epsilon_i}} + 6\alpha_i\frac{\zeta_i}{\epsilon_i} - 6\frac{\delta_i}{\sqrt{\epsilon_i}}\frac{\zeta_i}{\epsilon_i}\right](n_i + u_i) \\
& + \frac{1}{4}\left[\alpha_i^2 + d_i - a_i\sqrt{\epsilon_i} - 3\alpha_i\frac{\delta_i}{\sqrt{\epsilon_i}} + \frac{\delta_i^2}{\epsilon_i} + 2\alpha_i\frac{\zeta_i}{\epsilon_i}\right](n_i + u_i)^2 \\
& - \frac{1}{6}\left[3\alpha_i^2 - a_i\sqrt{\epsilon_i} - 3\alpha_i\frac{\delta_i}{\sqrt{\epsilon_i}}\right](n_i + u_i)^3 + \frac{1}{4}\alpha^2(n_i + u_i)^4
\end{aligned} \tag{11}$$

for $\gamma_i = 0$, $\epsilon_i \neq 0$,

[3] Daskaloyannis 2001, J. Math. Phys.42 1100

Deformed oscillator algebra

- and by

$$\Phi_i(n_i) = \gamma_i^8(3\alpha_i^2+4a_i\gamma_i)[2(n_i+u_i)-3]^2[2(n_i+u_i)-1]^4[2(n_i+u_i)+1]^2 - 3072\gamma_i^6K_i[2(n_i+u_i)-1]^2$$
$$- 48\gamma_i^6(\alpha_i^2\epsilon_i-\alpha_i\gamma_i\delta_i+a_i\gamma_i\epsilon_i-\gamma_i^2d_i)[2(n_i+u_i)-1]^4[2(n_i+u_i)+1]^2[2(n_i+u_i)-3]$$
$$+ 32\gamma_i^4\left(3\alpha_i^2\epsilon_i^2+4\alpha_i\gamma_i^2\zeta_i-6\alpha_i\gamma_i\delta_i\epsilon_i+2a_i\gamma_i\epsilon_i^2+2\gamma_i^2\delta_i^2-4\gamma_i^2d_i\epsilon_i+8\gamma_i^3z_i\right)\times \qquad (12)$$
$$[2(n_i+u_i)-1]^2[12(n_i+u_i)^2-12(n_i+u_i)-1] + 768(\alpha_i\epsilon_i^2+4\gamma_i^2\zeta_i-2\gamma_i\delta_i\epsilon_i)^2$$
$$- 256\gamma_i^2[2(n_i+u_i)-1]^2(3\alpha_i^2\epsilon_i^3+4\alpha_i\gamma_i^4\zeta_i+12\alpha_i\gamma_i^2\zeta_i\epsilon_i-9\alpha_i\gamma_i\delta_i\epsilon_i^2+a_i\gamma_i\epsilon_i^3+2\gamma_i^4\delta_i^2$$
$$- 12\gamma_i^3\delta_i\zeta_i+6\gamma_i^2\delta_i^2\epsilon_i+2\gamma_i^4d_i\epsilon_i-3\gamma_i^2d_i\epsilon_i^2-4\gamma_i^5z_i+12\gamma_i^3z_i\epsilon_i)$$

for $\gamma_i \neq 0$.

- The construction of the deformed oscillator algebra rely on the integrals E_i being realized only in terms of the number operators $\mathcal{N}_i$ associated with n_i and provide constraints for the eigenvalues of the operator E_i (Daskaloyannis 2001, J. Math. Phys.42 1100)

$$e(E_i) = E_i(q_i) = \frac{\gamma_i}{2}\left((q_i+u_i)^2 - \frac{\epsilon_i}{\gamma_i^2} - \frac{1}{4}\right), \qquad \gamma_i \neq 0; \qquad (13)$$
$$e(E_i) = E_i(q_i) = \sqrt{\epsilon_i}(q_i+u_i), \qquad \gamma_i = 0, \quad \epsilon_i \neq 0. \qquad (14)$$

- Here denote the eigenvalues of the generators E_i in terms of q_i
- Other constraints on the structure functions $\Phi_i(n_i, u_i, H)$ of each substructures take the form of $\Phi_i(0, u_i, H) = 0$ and $\Phi_i(p_i+1, u_i, H) = 0$ where $q_i = 0, 1, ..., p_i$.

The algebra $\mathcal{Q}(3)$ for $N = 3$ case

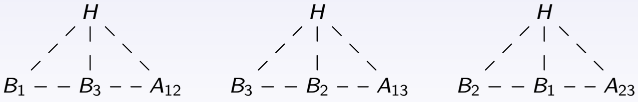
- To motivate our general discussions, we examine the distinct subalgebra structures of $\mathcal{SW}(3)$.
- The Hamiltonian system H for $N = 3$ reads,

$$H = -\frac{1}{2}(\partial_1^2 + \partial_2^2 + \partial_3^2) + b(x_1^2 + x_2^2 + x_3^2) + \frac{a_1}{x_1^2} + \frac{a_2}{x_2^2} + \frac{a_3}{x_3^2}.$$

- The corresponding second order integrals of motion are B_1, B_2, B_3 and A_{12}, A_{13}, A_{23} and satisfy

$$[H, B_i] = 0, \quad [H, A_{ij}] = 0, \quad [B_i, B_j] = 0, \quad i, j = 1, 2, 3;$$
$$[A_{23}, B_1] = 0, \quad [A_{13}, B_2] = 0, \quad [A_{12}, B_3] = 0.$$

- For more convenience, the diagrams below represent the above relations.



The algebra $\mathcal{Q}(3)$ for $N = 3$ case

- We also have the following four linearly independent commutators of the second order integrals,

$$
\begin{aligned}
C_{12} &= [B_1, A_{12}] = -[A_{12}, B_2], \\
C_{23} &= [B_2, A_{23}] = -[A_{23}, B_3] \\
C_{31} &= [B_3, A_{31}] = -[A_{31}, B_1], \\
D_{123} &= [A_{12}, A_{31}] = [A_{13}, A_{23}] = [A_{23}, A_{12}]
\end{aligned}
$$

- The above diagram shows that there are three possible subalgebras generated by three generators

$$
\begin{aligned}
\{E_1, F_1, C_1\} &\equiv \{A_{12}, B_1, C_{12}\} \\
\{E_2, F_2, C_2\} &\equiv \{A_{23}, B_2, C_{23}\} \\
\{E_3, F_3, C_3\} &\equiv \{A_{31}, B_3, C_{31}\}
\end{aligned}
$$

- Each set satisfies the commutation relations (8) of the associate substructure with appropriate structure constants

The general N case

- We now generalize the above results to the general N case and consider subalgebra structures generated by $\{B_i, B_j, A_{ij}; H, B_k,\ k = 1, 2, \ldots, N,\ k \neq i, j\}$ for any fixed $i, j = 1, 2, \ldots, N$.
- By direct computations, we get the following quadratic subalgebra structure, denoted by $\mathcal{Q}_{ij}(3)$ for any fixed $i, j = 1, 2, \ldots, N$,

$$
\begin{aligned}
&[B_i, A_{ij}] = C_{ij},\\
&[B_i, C_{ij}] = 8B_i^2 - 8(2H - \sum_{k\neq i,j} B_k)B_i + 32bA_{ij},\\
&[A_{ij}, C_{ij}] = -8\{B_i, A_{ij}\} + 8(2H - \sum_{k\neq i,j} B_k)(A_{ij} + \frac{1}{2}[8a_i - 3]) - 8(4a_i + 4a_j - 3)B_i.
\end{aligned}
$$

- The corresponding Casimir operator takes the form,

$$
\begin{aligned}
K_{ij} = {}& C_{ij}^2 - 8\{B_i^2, A_{ij}\} + 8(2H - \sum_{k\neq i,j} B_k)\{B_i, A_{ij}\} - 8(4a_i + 4a_j - 11)B_i^2\\
&+ 8(8a_i - 11)(2H - \sum_{k\neq i,j} B_k)B_i - 32bA_{ij}^2
\end{aligned}
$$

- The Casimir operator can also be written in terms of only the central elements H and all $B_k, k \neq i, j$ as

$$
K'_{ij} = 4(8a_i - 3)\left(2H - \textstyle\sum_{k\neq i,j} B_k\right)^2 - 8b(8a_i - 3)(8a_j - 3)
$$

Deformed oscillators realization

- In order to obtain the energy spectrum of the system H from the above subalgebra, we construct its realization in terms of the deformed oscillator algebra (Daskaloyannis 2001, J. Math. Phys.42 1100), the structure functions,

$$\Phi(n_{ij};u_{ij},H)=\frac{1}{1024b^2}\left[4(n_{ij}+u_{ij})-2-2\nu_i\right]\left[4(n_{ij}+u_{ij})-2+2\nu_i\right]$$
$$\left[8b(n_{ij}+u_{ij})-4b+4b\nu_j+\sqrt{2b}(\sum_{k\neq i,j}B_k-2H)\right]\left[8b(n_{ij}+u_{ij})-4b-4b\nu_j+\sqrt{2b}(\sum_{k\neq i,j}B_k-2H)\right]$$

- The values of parameter u_{ij} and the eigenvalues of the operators $\sum_{k\neq i,j}B_k$ are determined by requiring that the corresponding representation of the deformed oscillator algebra is finite dimensional, i.e.,

$$\Phi(p_{ij}+1;u_{ij},E)=0,\quad \Phi(0;u_{ij},E)=0,\quad \Phi(n_{ij})>0,\quad \forall\quad n_{ij}>0,$$

  where p_{ij} are positive integer.
- These constraints give

$$u_{ij}=\frac{1}{2}+\frac{\varepsilon_i\nu_i}{2},\qquad \sum_{k\neq i,j}B_k=2H-2\sqrt{2b}(p_{ij}+1+\varepsilon_i\nu_i+\varepsilon_j\nu_j),$$
$$\Phi(n_{ij})=n_{ij}(n_{ij}+\varepsilon_i\nu_i)(n_{ij}-p_{ij\,-}1)(n_{ij}+\varepsilon_j\nu_j-p_{ij}-1),$$

  where $\varepsilon_i=\pm1,\varepsilon_j=\pm1$.
- The spectrum of H could be determined in terms of the p_{ij}, $i,j\in\{1,\dots,N\}$ from selected subsets of N substructures.
- This can be seen alternatively guided by the form of the spectrum in Cartesian coordinates, and relation among the H and the B_i operators.

Energy spectrum

- We now use the constraints for the spectrum of B_i (Daskaloyannis 2001, J. Math. Phys.42 1100) which is given by

$$e(B_i(x)) = 2\sqrt{2b}(2q_i + \varepsilon_i\nu_i + 1),$$

and by virtue of

$$H = \frac{1}{2}\sum_i^N B_i,$$

- the energy spectrum of system H is

$$E = \sum_i^N \sqrt{2b}(2q_i + \varepsilon_i\nu_i + 1)$$

- this can be seen in the form of the spectrum of the Cartesian coordinates
- the above results are obtained based on the algebraic manipulation only without using explicitly the corresponding Schrödinger equation

Algebraic derivation based on Racah algebra $\mathcal{R}(N)$

- The Racah R(N) subalgebra can also be used to derive the spectrum of the form obtained in the hyperspherical coordinates
- The Racah subalgebra is related to the separation of variables in hyperspherical coordinates via the relation

$$\sum_i \frac{\partial^2}{\partial x_i^2} = \frac{\partial^2}{\partial r^2} + \frac{N-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\sum_{i<j} J_{ij}^2.$$

- Define the new operator Z associated with the separation of variables in hyperspherical coordinates,

$$Z = \sum_{i<j} A_{ij} = -\sum_{i<j} J_{ij}^2 + 2r^2 \sum_i \frac{a_i}{x_i^2} - 2\sum_i a_i + \frac{N(N-1)}{4},$$

such that H acquires the form

$$H = -\frac{1}{2}\left[\frac{\partial^2}{\partial r^2} + \frac{N-1}{r}\frac{\partial}{\partial r} - 2br^2 - \frac{4Z + 8\sum_i a_i - N(N-1)}{4r^2}\right] \tag{15}$$

- Comparing the above equation (15) with the radial equation (3) and using spectrum equation (6), it leads to the spectrum of Z,

$$
\begin{aligned}
e(Z) &= k_1 - 2\sum_i a_i + \frac{N(N-1)}{4} \\
&= \left[2\sum_{i=1}^{N-1}\tau_i \pm \sum_{i=1}^{N}\nu_i + N - 1\right]^2 - \frac{1}{4}(N-2)^2 - 2\sum_i a_i + \frac{N(N-1)}{4} \\
&= \nu^2 + \frac{1}{4}(3N-4) - 2\sum_i a_i,
\end{aligned}
$$

where ν is given by (7), which allows to rewrite the eigenvalues (6) as

$$E = \sqrt{2b}(2\tau_r + 2\nu + 1)$$

The $su(1,1)$ algebra and spectrum

- The Hamiltonian written in terms of the radial variable r and Z, where Z can be seen as a Casimir of the Racah $\mathcal{R}(N)$ algebra, has similarities with the N-dimensional radial oscillator.
- This suggests the existence of ladder differential operators in the radial variable r,

$$\mathcal{D}^{\pm} = H \pm \sqrt{2b}r\frac{\partial}{\partial r} - 2br^2 \pm \sqrt{\frac{b}{2}}N,$$

whose action on the wave functions is given by

$$\mathcal{D}^{+}\psi^{2\nu}_{\tau_r} = 2\sqrt{2b}(\tau_r+1)\psi^{2\nu}_{\tau_r+1}, \quad \mathcal{D}^{-}\psi^{2\nu}_{\tau_r} = 2\sqrt{2b}(\tau_r+2\nu)\psi^{2\nu}_{\tau_r-1}$$

- Then

$$\mathcal{D}^{+}\mathcal{D}^{-}\psi^{2\nu}_{\tau_r} = 8b\tau_r(\tau_r+2\nu)\psi^{2\nu}_{\tau_r}, \qquad \mathcal{D}^{-}\mathcal{D}^{+}\psi^{2\nu}_{\tau_r} = 8b(\tau_r+1)(\tau_r+2\nu+1)\psi^{2\nu}_{\tau_r}$$

- The differential operators $\mathcal{D}^{\pm}$ satisfy the following $su(1,1)$ algebra relations,

$$[\mathcal{D}^{+}, H] = -2\sqrt{2b}\mathcal{D}^{+}, \quad [\mathcal{D}^{-}, H] = 2\sqrt{2b}\mathcal{D}^{-}, \quad [\mathcal{D}^{-}, \mathcal{D}^{+}] = 4\sqrt{2b}H.$$

- This means that the spectrum of the N-dimensional SW system H can be obtained in same way as that for rotationally invariant systems with the Racah algebra $\mathcal{R}(N)$ playing the same role as the angular momentum algebra, to show this,

The new integrals and $R(3)$ subalgebras

- we define new integrals,

$$Z_l = \sum_{1\leq i<k\leq l+1} A_{ik}, \quad 1 \leq l \leq N-2, \qquad Y_p = \sum_{p\leq i<k\leq N} A_{ik}, \quad 1 \leq p \leq N-1.$$

- We examine the subalgebra structures generated by Y_i, Z_{i-1} and the central elements Y_1, Y_{i+1}, Z_{i-2} with $2 \leq i \leq N-1$, $Z_0 = 4a_1 + \frac{1}{2}$ and $Y_N = 0$.
- After a long computation we find that these elements obey the following quadratic algebra relations,

$$\begin{aligned}
&[Z_{i-1}, Y_i] = C_i, \\
&[Z_{i-1}, C_i] = 8Z_{i-1}^2 + 8\{Z_{i-1}, Y_i\} - (8Y_1 + 8Y_{i+1} + 8Z_{i-2} - 32a_i + 12)Z_{i-1} \\
&\qquad +4(8\sum_{j=1}^{i} a_i - 3i)Y_i - 4(8a_i - 3)Y_1 - 4(8\sum_{j=1}^{i-1} a_j - 3(i-1))Y_{i+1} \\
&\qquad +8Y_1Z_{i-2} - 8Y_{i+1}Z_{i-2}, \\
&[Y_i, C_i] = -8Y_i^2 - 8\{Z_{i-1}, Y_i\} - 4(8\sum_{j=i}^{N} a_j - 3(N-i+1))Z_{i-1} \\
&\qquad +(8Y_1 + 8Y_{i+1} + 8Z_{i-2} - 32a_i + 12)Y_i + 4(8a_i - 3)Y_1 \\
&\qquad +4(8\sum_{j=i+1}^{N} a_j - 3(N-i))Z_{i-2} - 8Y_1Y_{i+1} + 8Y_{i+1}Z_{i-2}.
\end{aligned} \tag{16}$$

- It follows that $\{Y_i, Z_{i-1}, C_i; Y_1, Y_{i+1}, Z_{i-2}, 2 \leq i \leq N-1, Y_N \equiv 0, Z_0 = 4a_1 + \frac{1}{2}\}$ form the subalgebra $\mathcal{R}(3)$ for fixed i.

Casimir operator

- This quadratic subalgebra can be fitted into the general form (8) with

$$\alpha_i = 8, \quad \gamma_i = 8, \quad \delta_i = -8(Y_1+Y_{i+1}+Z_{i-2}-4a_i+3/2), \quad \epsilon_i = 4(8\sum_{j=1}^{i} a_j - 3i),$$

$$\zeta_i = -4(8a_i-3)Y_1 - 4[8\sum_{j=1}^{i-1} a_j - 3(i-1)]Y_{i+1} + 8Y_1 Z_{i-2} - 8Y_{i+1}Z_{i-2}, \quad a_i = 0,$$

$$d_i = -4[8\sum_{j=i}^{N} a_j - 3(N-i+1)],$$

$$z_i = 4(8a_i-3)Y_1 + 4[8\sum_{j=i+1}^{N} a_j - 3(N-i)]Z_{i-2} - 8Y_1 Y_{i+1} + 8Y_{i+1}Z_{i-2}.$$

- The corresponding Casimir operator involving only the central elements Y_1, Y_{i+1}, Z_{i-2} takes the form,

$$\begin{aligned} K_i' &= 4(8a_i-3)Y_1^2 - 64Y_1 Y_{i+1} + 4[8\sum_{j=1}^{i-1} a_j - 3(i-1)]Y_{i+1}^2 + 32(8a_i-3)Y_1 \\ &\quad -4(8a_i-3)[8\sum_{j=1}^{i-1} a_j - 3(i-1)]Y_{i+1} + 16Z_{i-2}^2 Y_{i+1} - 64Z_{i-2}Y_1 - 16(8a_i-3)Z_{i-2}Y_{i+1} \\ &\quad -4(8a_i-3)[8\sum_{j=i+1}^{N} a_j - 3(N-i)]Z_{i-2} + 4[8\sum_{j=i+1}^{N} a_j - 3(N-i)]Z_{i-2}^2 \\ &\quad -16Z_{i-2}Y_1 Y_{i+1} + 16Z_{i-2}Y_{i+1}^2 - (8a_i-3)[8\sum_{j=1}^{i-1} a_j - 3(i-1)][8\sum_{j=i+1}^{N} a_j - 3(N-i)]. \end{aligned}$$

The realizations and spectrum

- The subalgebra algebra (16) can be realized in terms of the deformed oscillator algebra with structure function given by

$$\begin{aligned}\phi(n_i,u_i) \quad = \quad & [n_i+u_i-\frac{1}{4}(2-y_1-y_{i+1})][n_i+u_i-\frac{1}{4}(2-y_1+y_{i+1})]\\ & [n_i+u_i-\frac{1}{4}(2+y_1-y_{i+1})][n_i+u_i-\frac{1}{4}(2+y_1+y_{i+1})]\\ & [n_i+u_i-\frac{1}{4}(2+z_{i-2}+2\nu_i)][n_i+u_i-\frac{1}{4}(2+z_{i-2}-2\nu_i)]\\ & [n_i+u_i-\frac{1}{4}(2-z_{i-2}-2\nu_i)][n_i+u_i-\frac{1}{4}(2-z_{i-2}+2\nu_i)],\end{aligned}$$

where $y_1, y_{i+1}, z_{i-2}, \nu_i$ satisfy

$$\begin{aligned}Y_1 &= \frac{1}{4}(3N-4-8\sum_{j=1}^{N}a_j+y_1^2),\\ Y_{i+1} &= \frac{1}{4}\left((3N-3i-4)-8\sum_{j=i+1}^{N}a_j+y_{i+1}^2\right),\\ Z_{i-2} &= \frac{1}{4}(3i-7-8\sum_{j=1}^{i-1}a_j+z_{i-2}^2).\end{aligned} \tag{17}$$

The realizations and spectrum

- Imposing the constraints $\phi(0, u_i) = 0$ and $\phi(p_i + 1, u_i) = 0$, where p_i is positive integer, we obtain

$$\begin{aligned} u_i &= \frac{1}{4}(2 + \varepsilon_1 y_1 + \varepsilon_2 y_{i+1}), \quad \text{or} \quad u_i = \frac{1}{4}(2 + \varepsilon_1 z_{i-2} + 2\varepsilon_2 \nu_i), \\ z_{i-2} &= 4\bar{\varepsilon}_1(p_i + 1) + \bar{\varepsilon}_2 y_1 + \bar{\varepsilon}_3 y_{i+1} + 2\bar{\varepsilon}_4 \nu_i, \\ \text{or} \quad y_1 &= 4\bar{\varepsilon}_1(p_i + 1) + \bar{\varepsilon}_2 y_{i+1} + 2\bar{\varepsilon}_3 \nu_i + \bar{\varepsilon}_4 z_{i-2}, \end{aligned} \tag{18}$$

where $\varepsilon_1, \varepsilon_2, \bar{\varepsilon}_1, \bar{\varepsilon}_2, \bar{\varepsilon}_3, \bar{\varepsilon}_4$ take the values ± 1.

- In the following we will take

$$u_i = \frac{1}{4}(2 + z_{i-2} + 2\nu_i)$$

- Using the fact that the integrals Z_{i-1} is diagonal in the number operator in the oscillator realization, we obtain

$$Z_{i-1} = 4(n_i + u_i)^2 + \frac{1}{4}(3i - 8\sum_{j=1}^{i} a_j) - 1,$$

- From the last relation of (17),

$$Z_{i-1} = \frac{1}{4}(3i - 8\sum_{j=1}^{i} a_j) + \frac{1}{4} z_{i-1}^2 - 1$$

- Comparing these two relations, we have

$$z_{i-1} = 4(q_i + u_i)$$

- This gives us the recurrence relation,

$$z_{i-1} = 4q_i + z_{i-2} + 2\nu_i + 2,$$

with the initial condition $z_0 = \nu_1$, from which we get

$$z_{N-2} = 4\sum_{i=1}^{N-1} q_i + 2\sum_{i=1}^{N-1} \nu_i + 2(N-2). \tag{19}$$

- By (18) we can write

$$y_1 = 4\bar{\varepsilon}_1(p_N + 1) + \bar{\varepsilon}_2 y_{N+1} + 2\bar{\varepsilon}_3\nu_N + \bar{\varepsilon}_4 z_{N-2}.$$

- Choosing suitable sign of $\bar{\varepsilon}_i$ and setting $y_{N+1} = 0$, we have

$$y_1 = 4(p_N + 1) + 4\sum_{1}^{N-1} n_i + 2\sum_{1}^{N} \nu_i + 2(N-2).$$

- Substitute into the first equation in (17), we obtain

$$e(Z) = e(Y_1) = \left[2p_N + 2\sum_{i=1}^{N-1} q_i + \sum_{i=1}^{N} \nu_i + N\right]^2 + \frac{1}{4}(3N-4) - 2\sum_{j=1}^{N} a_j.$$

The spectrum of H

- To derive the spectrum of H, we consider the algebra generated by the integrals, $\{Y_1, B_N; H, Z_{N-2}\}$, which close to form the quadratic algebra

$$
\begin{aligned}
&[Y_1, B_N] = D, \\
&[Y_1, D] = 8\{Y_1, B_N\} - 16HY_1 + 4(8\sum_{j=1}^{N} a_j - 3N)B_N + 16HZ_{N-2} - 8(8a_N - 3)H, \qquad (20) \\
&[B_N, D] = -8B_N^2 - 32bY_1 + 16HB_N + 32bZ_{N-2},
\end{aligned}
$$

- Comparing these with the quadratic algebra (8), we have

$$
\begin{aligned}
&\alpha = 0, \quad \gamma = 8, \quad \delta = -16H, \quad \epsilon = 4(8\sum_{j=1}^{N} a_j - 3N), \\
&\zeta = 16HZ_{N-2} - 8(8a_N - 3)H, \quad a = 0, \quad d = -32b, \quad z = 32bZ_{N-2}
\end{aligned}
$$

- The Casimir operator in terms of only the central elements H and Z_{N-2} has the form,

$$
K' = 32bZ_{N-2}^2 + 16(8a_N - 3)H^2 - 32b(8a_N - 3)Z_{N-2} - 8b(8a_N - 3)\left[8\sum_{j=1}^{N-1} a_j - 3(N-1)\right]
$$

The realizations

- The quadratic algebra can be realized in terms of the oscillator algebra with the structure function,

$$\begin{aligned}\phi(n_N, u_N) \quad = \quad & [n_N + u_N - \frac{1}{4}(2 - \sqrt{\frac{2}{b}}H)][n_N + u_N - \frac{1}{4}(2 + \sqrt{\frac{2}{b}}H)] \\ & [n_N + u_N - \frac{1}{4}(2 + z_{N-2} + 2\nu_N)][n_N + u_N - \frac{1}{4}(2 + z_{N-2} - 2\nu_N)] \\ & [n_N + u_N - \frac{1}{4}(2 - z_{N-2} - 2\nu_N)][n_N + u_N - \frac{1}{4}(2 - z_{N-2} + 2\nu_N)],\end{aligned}$$

where z_{N-2} satisfy

$$Z_{N-2} = \frac{1}{4}(3N - 7 - 8\sum_{j=1}^{N-1} a_j + z_{N-2}^2)$$

- Imposing the constraints $\phi(0, u_N) = 0$ and $\phi(p_N + 1, u_N) = 0$ (where p_N is positive integer) to the structure function gives

The spectrum of H

- the solutions

$$u_N = \frac{1}{4}(2 + \varepsilon_1\sqrt{\frac{2}{b}}H), \quad \text{or} \quad u_N = \frac{1}{4}(2 + \varepsilon_1 z_{N-2} + 2\varepsilon_2\nu_N),$$

$$e(H) = \sqrt{\frac{b}{2}}(4(p_N + 1) + \varepsilon_1 z_{N-2} + 2\varepsilon_2\nu_N),$$

$$Y_1 = \frac{\gamma}{2}\left[(n_N + u_N)^2 - \frac{1}{4} - \frac{\epsilon}{\gamma^2}\right], \qquad \epsilon_1, \epsilon_2 = \pm 1.$$

- By means of the recurrence relation (19), we have

$$e(H) = \sqrt{2b}\left[2p_N + 2\sum_{i=1}^{N-1} q_i + \sum_{i=1}^{N} \nu_i + N\right].$$

- This formula coincides with the result from separation of variables in hyperspherical coordinates
- This emphasizes the fact that algebraic derivations of the spectrum for N-dimensional SW systems can be based only on differential operators and their operator algebra.

Summary

- the symmetry algebra of a N-dimensional quantum superintegrable system is in general a quite complicated algebraic structure
- the complete symmetry algebra for the N-dimensional SW system is a higher rank quadratic algebra $\mathcal{SW}(N)$
- the algebra $\mathcal{SW}(N)$ contains the Racah algebra $\mathcal{R}(N)$ as a subalgebra
- two distinct approaches discussed here rely on the construction of different sets of substructures involving three generators (and central elements)
- present their corresponding deformed oscillator algebra and their cubic Casimir operators
- the algebraic derivation is not unique for a superintegrable system
- the higher rank quadratic algebras are useful in deriving the spectrum of a Hamiltonian in quantum mechanics.

– F. Correa, M.F. Hoque, I. Marquette and Y-Z. Zhang, J. Phys. A: Math. Theor. 54 (2021), 395201

Acknowledgement

- Doc. Ing. Libor Snobl, PhD
  Department of Physics, Faculty of Nuclear Sciences and Physical Engineering,
  Czech Technical University in Prague, Czech Republic
- Dean
  Faculty of Nuclear Sciences and Physical Engineering,
  Czech Technical University in Prague, Czech Republic
- Organizer
  XXXIX Workshop on Geometric Methods in Physics
  University of Bialystok, Bialystok, Poland
- The presentation is supported by the grant CZ.02.2.69/0.0/0.0/18_053/0016980,
  -co-financed by the European Union


Thank You