Supersymmetric quantum mechanics and Painlevé equations

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Introduction

- Interesting to explore links of SUSY QM with nonlinear differential equations
- Simplest case: connection with Riccati equation
- SUSY partners of the free particle lead to solutions of the KdV equation
- A link between the harmonic oscillator SUSY partners and Painlevé IV (PIV) equation will be found
- A procedure for generating solutions of the PIV (and PV) equation will be available

Introduction

- The first people who realized that there is a connection between second-degree polynomial Heisenberg-algebras, PIV equation and first-order SUSY QM were Veselov and Shabat (1993), Dubov, Eleonsky and Kulagin (1994), Adler (1994)
- This link was further explored in the higher-degree case by Andrianov, Cannata, Ioffe and Nishnianidze (2000), Fernández, Negro and Nieto (2004), Carballo, Fernández, Negro and Nieto (2004), Mateo and Negro (2008), Bermúdez, Fernández, González, Morales-Salgado and Negro (starting from 2010)

First-order SUSY QM

Let us take two Schrödinger type Hamiltonians

$$H_i = -\frac{1}{2}\frac{d^2}{dx^2} + V_i(x), \quad i = 0, 1 \tag{1}$$

which are intertwined as

$$H_1 A_1^+ = A_1^+ H_0$$
$$A_1^\pm = \frac{1}{\sqrt{2}}\left(\mp\frac{d}{dx} + \alpha_1(x)\right)$$

Let us stress that

$$\frac{d}{dx}f = f\frac{d}{dx} + f', \qquad \frac{d^2}{dx^2}f = f\frac{d^2}{dx^2} + 2f'\frac{d}{dx} + f''$$

First-order SUSY QM

Thus

$$\sqrt{2}H_1A_1^+ = \frac{1}{2}\frac{d^3}{dx^3} - \frac{\alpha_1}{2}\frac{d^2}{dx^2} - (V_1 + \alpha_1')\frac{d}{dx} + \alpha_1 V_1 - \frac{\alpha_1''}{2}$$

$$\sqrt{2}A_1^+H_0 = \frac{1}{2}\frac{d^3}{dx^3} - \frac{\alpha_1}{2}\frac{d^2}{dx^2} - V_0\frac{d}{dx} + \alpha_1 V_0 - V_0'$$

which implies that

$$V_1 = V_0 - \alpha_1'$$

$$\alpha_1 V_1 - \frac{\alpha_1''}{2} = \alpha_1 V_0 - V_0'$$

First-order SUSY QM

Substituting V_1 and integrating

$$\alpha_1' + \alpha_1^2 = 2[V_0(x) - \epsilon_1]$$

In terms of $u_1^{(0)}(x)$ such that $\alpha_1(x) = \frac{u_1^{(0)\prime}}{u_1^{(0)}}$:

$$-\frac{1}{2}u_1^{(0)\prime\prime} + V_0 u_1^{(0)} = H_0 u_1^{(0)} = \epsilon_1 u_1^{(0)}$$

The relevant factorizations:

$$H_0 = A_1^- A_1^+ + \epsilon_1$$
$$H_1 = A_1^+ A_1^- + \epsilon_1$$

First-order SUSY QM

Suppose that H_0 is a solvable Hamiltonian such that

$$H_0\psi_n^{(0)} = E_n\psi_n^{(0)}, \quad n = 0, 1, \dots$$

A nodeless mathematical eigenfunction $u_1^{(0)}$ for $\epsilon_1 \leq E_0$ is chosen. Thus, if $A_1^+\psi_n^{(0)} \neq 0$ then $\left\{\psi_n^{(1)} = \frac{A_1^+\psi_n^{(0)}}{\sqrt{E_n - \epsilon_1}}\right\}$ is an orthonormal set of eigenfunctions of H_1 with eigenvalues $\{E_n\}$. This set constitutes a basis if $\nexists$ a normalizable eigenfunction $\psi_{\epsilon_1}^{(1)}$ which is orthogonal to the previous set

First-order SUSY QM

Hence, let us look for $\psi_{\epsilon_1}^{(1)}$ such that

$$(\psi_{\epsilon_1}^{(1)}, \psi_n^{(1)}) \propto (\psi_{\epsilon_1}^{(1)}, A_1^+ \psi_n^{(0)}) = 0 \quad \Rightarrow \quad A_1^- \psi_{\epsilon_1}^{(1)} = 0$$

By solving this first-order differential equation

$$\psi_{\epsilon_1}^{(1)} \propto e^{-\int_0^x \alpha_1(y) dy} = \frac{1}{u_1^{(0)}}$$

Since $H_1 \psi_{\epsilon_1}^{(1)} = \epsilon_1 \psi_{\epsilon_1}^{(1)}$, then $\mathrm{Sp}(H_1)$ depends on either $\psi_{\epsilon_1}^{(1)}$ is normalizable or not. Three different cases arise.

First-order SUSY QM

(i) For $\epsilon_1 = E_0$ and $u_1^{(0)} = \psi_0^{(0)}$, which is nodeless in the domain of V_0, $\alpha_1 = \psi_0^{(0)\prime}/\psi_0^{(0)}$. Thus $V_1 = V_0 - \alpha_1'$ is non-singular, the associated eigenfunctions and eigenvalues of H_1 become

$$\psi_n^{(1)} = \frac{A_1^+ \psi_n^{(0)}}{\sqrt{E_n - E_0}}$$

$$\mathrm{Sp}(H_1) = \{E_n,\ n = 1, 2, \dots\}$$

Note that $E_0 \notin$ Sp(H_1) since $\psi_{\epsilon_1}^{(1)} \propto 1/u_1^{(0)}$ is not normalizable

First-order SUSY QM

(ii) For $\epsilon_1 < E_0$ a nodeless seed solution $u_1^{(0)}$ can be chosen and thus $\alpha_1 = u_1^{(0)\prime}/u_1^{(0)}$ is non-singular. Since in general $u_1^{(0)}$ diverges at the ends of the domain it turns out that the eigenfunctions and eigenvalues of H_1 are given by

$$\psi_{\epsilon_1}^{(1)} \propto \frac{1}{u_1^{(0)}}, \quad \psi_n^{(1)} = \frac{A_1^+ \psi_n^{(0)}}{\sqrt{E_n - \epsilon_1}}$$

$$\mathrm{Sp}(H_1) = \{\epsilon_1, E_n,\ n = 0, 1, 2, \dots\}$$

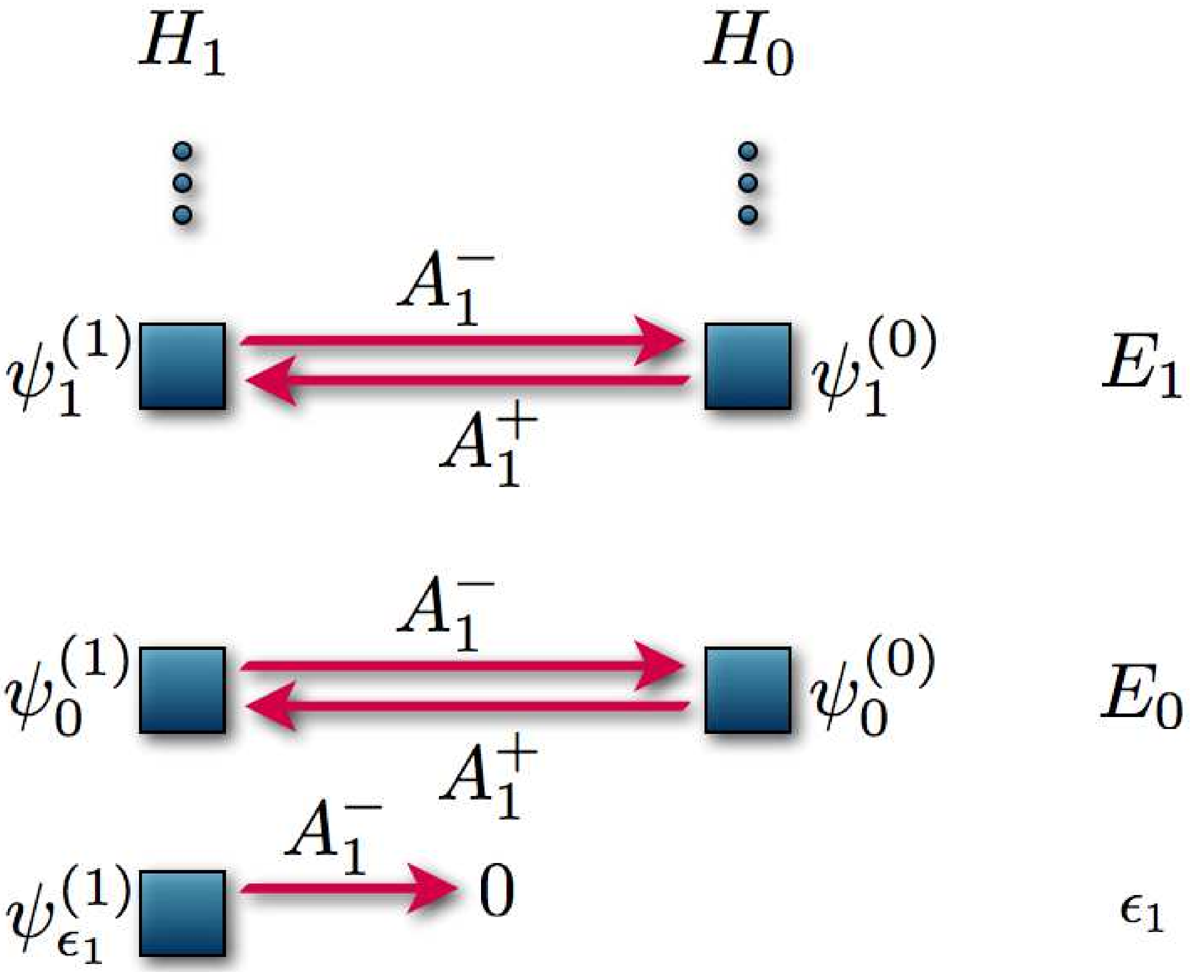
First-order SUSY QM

(iii) For $\epsilon_1 < E_0$ solutions $u_1^{(0)}$ with a node at one end of the problem domain can be found, the transformation induced by $\alpha_1 = {u_1^{(0)}}'/u_1^{(0)}$ is still non-singular. The eigenfunctions and eigenvalues of H_1 become

$$\psi_n^{(1)} = \frac{A_1^+ \psi_n^{(0)}}{\sqrt{E_n - \epsilon_1}}$$

$$\mathrm{Sp}(H_1) = \{E_n,\ n = 0, 1, 2, \ldots\}$$

First-order SUSY QM



First-order SUSY QM

Example: harmonic oscillator potential

$$V_0(x) = \frac{x^2}{2}$$

Eigenfunctions and eigenvalues

$$\psi_n^{(0)}(x) = \sqrt{\frac{1}{2^n n! \sqrt{\pi}}} H_n(x) e^{-\frac{x^2}{2}}, \quad E_n = n + \frac{1}{2}, \; n = 0, 1, \ldots$$

$H_n(x)$ are the Hermite polynomials

First-order SUSY QM

The SE for arbitrary ϵ

$$-\frac{1}{2}u^{(0)''} + \frac{x^2}{2}u^{(0)} = \epsilon u^{(0)}$$

The general solution [$a = (1-2\epsilon)/4$]

$$u^{(0)}(x) = e^{-\frac{x^2}{2}}\left[{}_1F_1\left(a, \tfrac{1}{2}; x^2\right) + 2\nu\frac{\Gamma(a+\frac{1}{2})}{\Gamma(a)}\, x\, {}_1F_1\left(a+\tfrac{1}{2}, \tfrac{3}{2}; x^2\right)\right]$$

where ${}_1F_1\left(a, b; y\right) = \frac{\Gamma(b)}{\Gamma(a)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)}{\Gamma(b+n)}\frac{y^n}{n!}$ is the confluent hypergeometric function, $\Gamma(x)$ is the Gamma function

First-order SUSY QM

(i) 1-SUSY through the ground state: $\epsilon_1 = E_0 = \frac{1}{2}$ and

$$u_1^{(0)} = \psi_0^{(0)} \propto e^{-\frac{x^2}{2}}$$

Thus

$$V_1(x) = \frac{x^2}{2} - \{\log[u_1^{(0)}]\}'' = \frac{x^2}{2} + 1$$

Just the initial potential displaced!

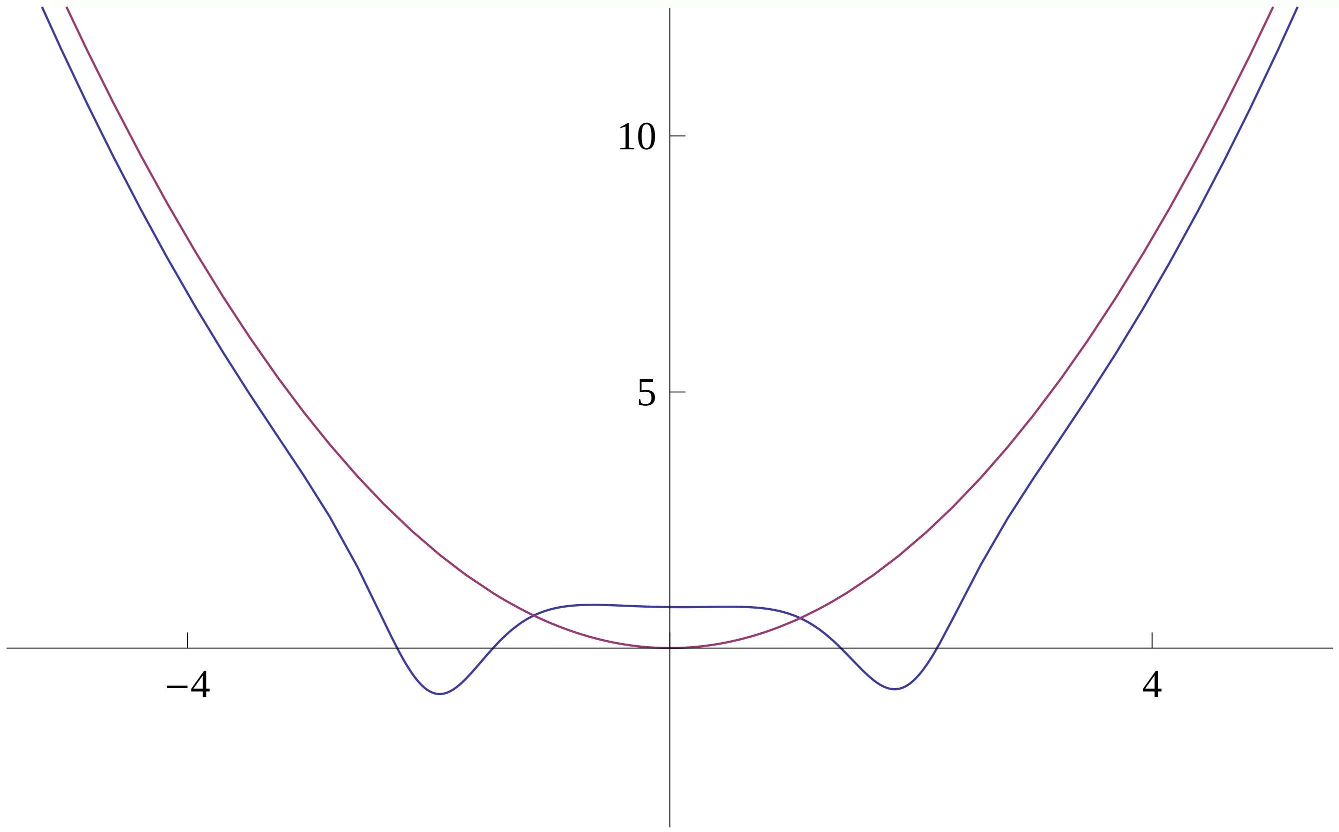
First-order SUSY QM

(ii) 1-SUSY through general $u_1^{(0)}(x)$ with $\epsilon_1 < E_0$ and $|\nu_1| < 1$ it is obtained

$$V_1(x) = \frac{x^2}{2} - \{\log[u_1^{(0)}]\}''$$

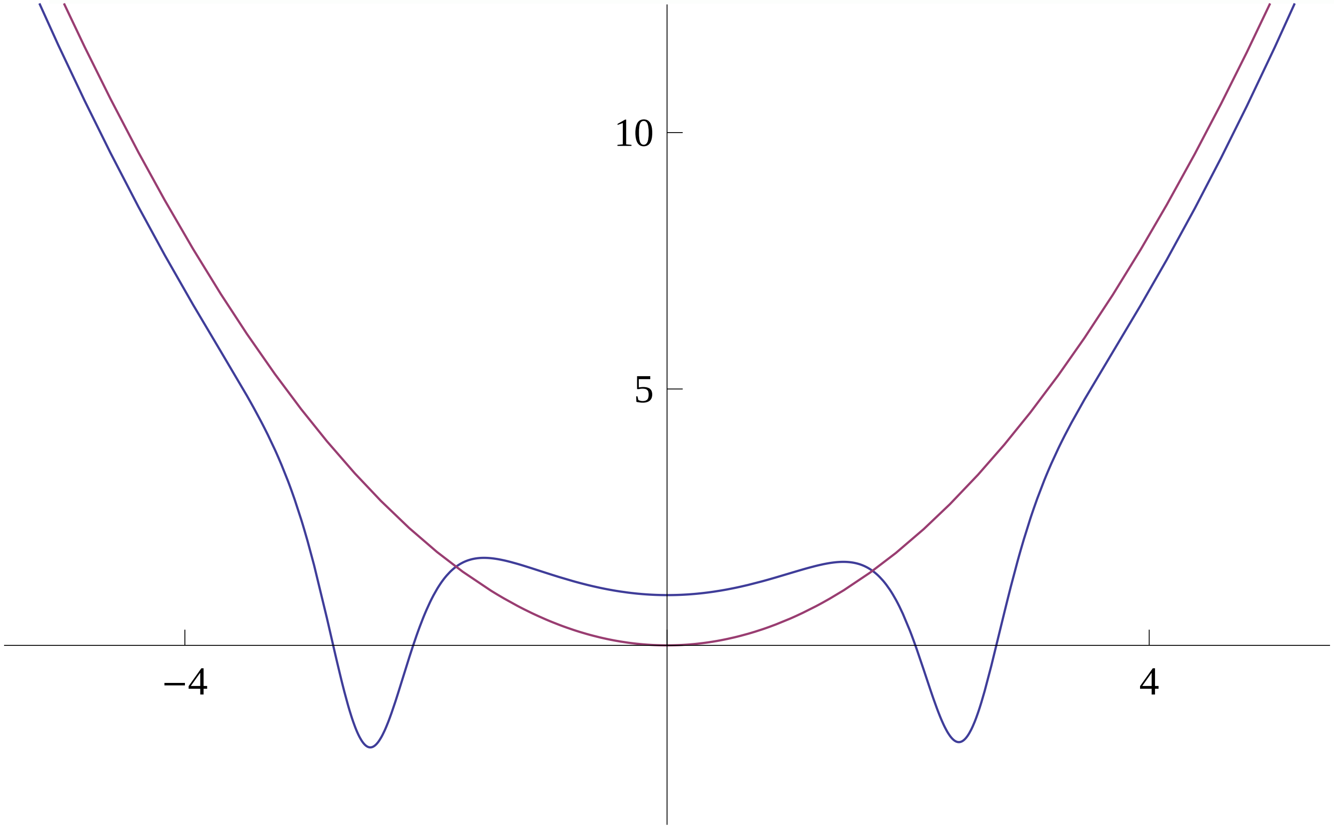
which is essentially different from the initial potential

First-order SUSY QM



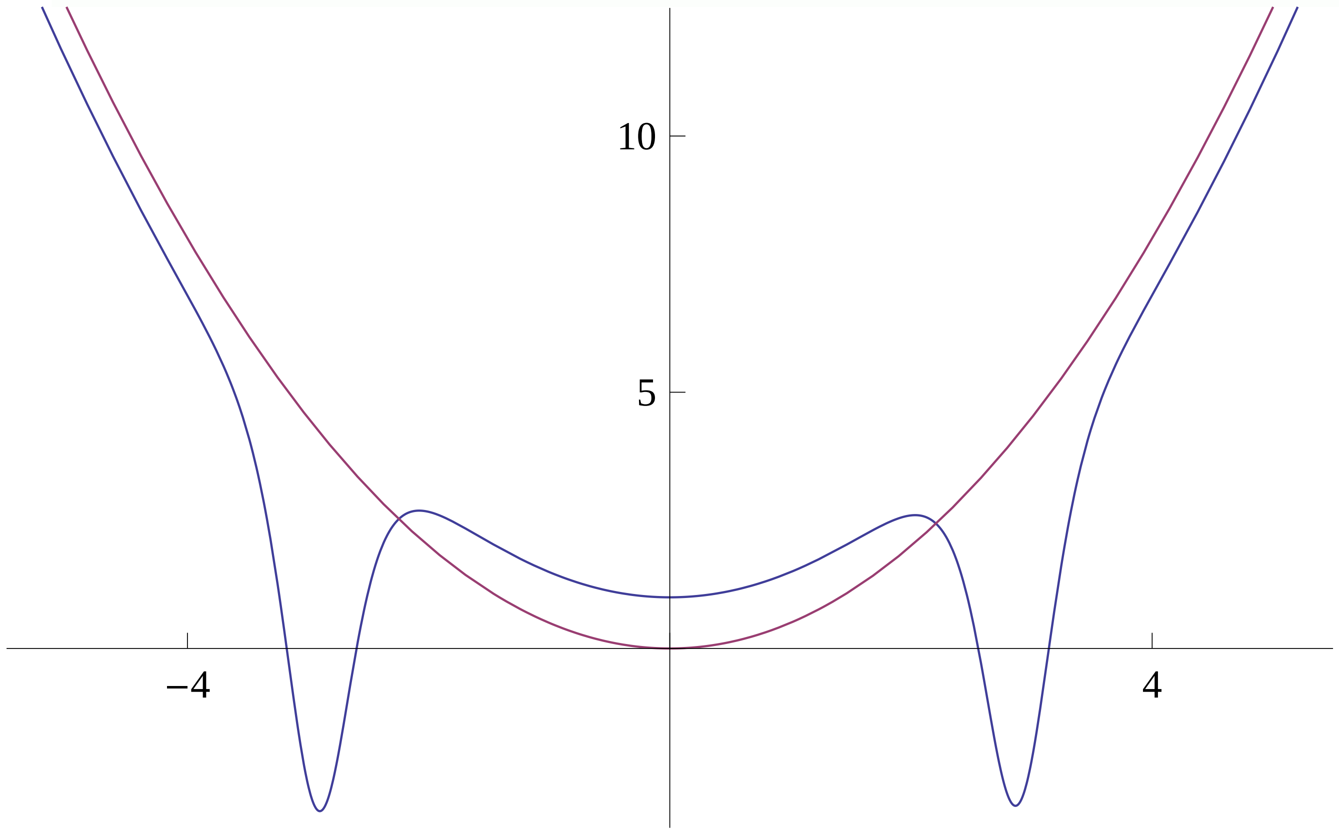
$$\epsilon_1 = \frac{1}{2} - 0.1 \qquad \nu_1 = 0.1$$

First-order SUSY QM



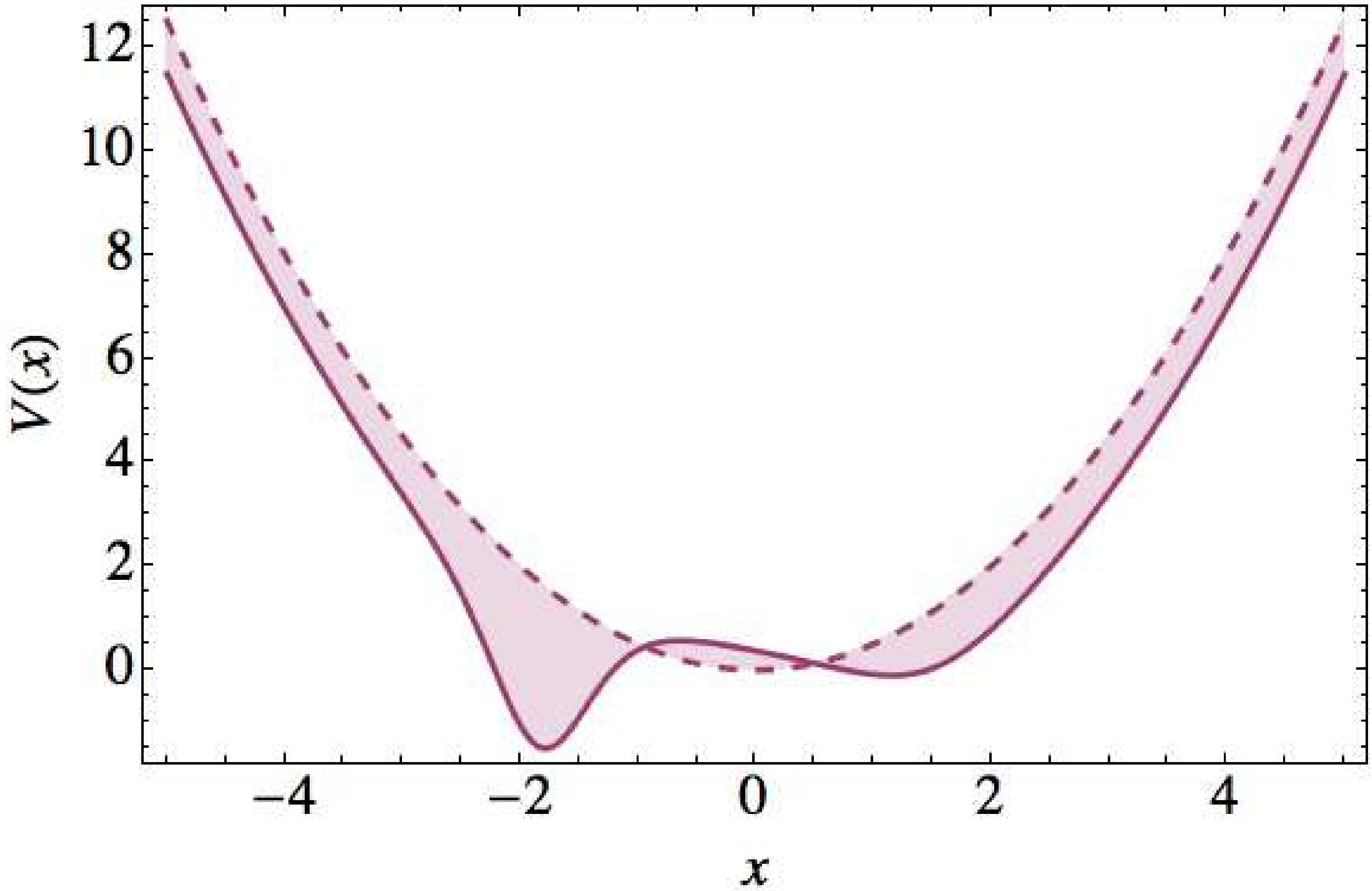
$$\epsilon_1 = \frac{1}{2} - 0.01 \qquad \nu_1 = 0.1$$

First-order SUSY QM



$$\epsilon_1 = \frac{1}{2} - 0.001 \qquad \nu_1 = 0.1$$

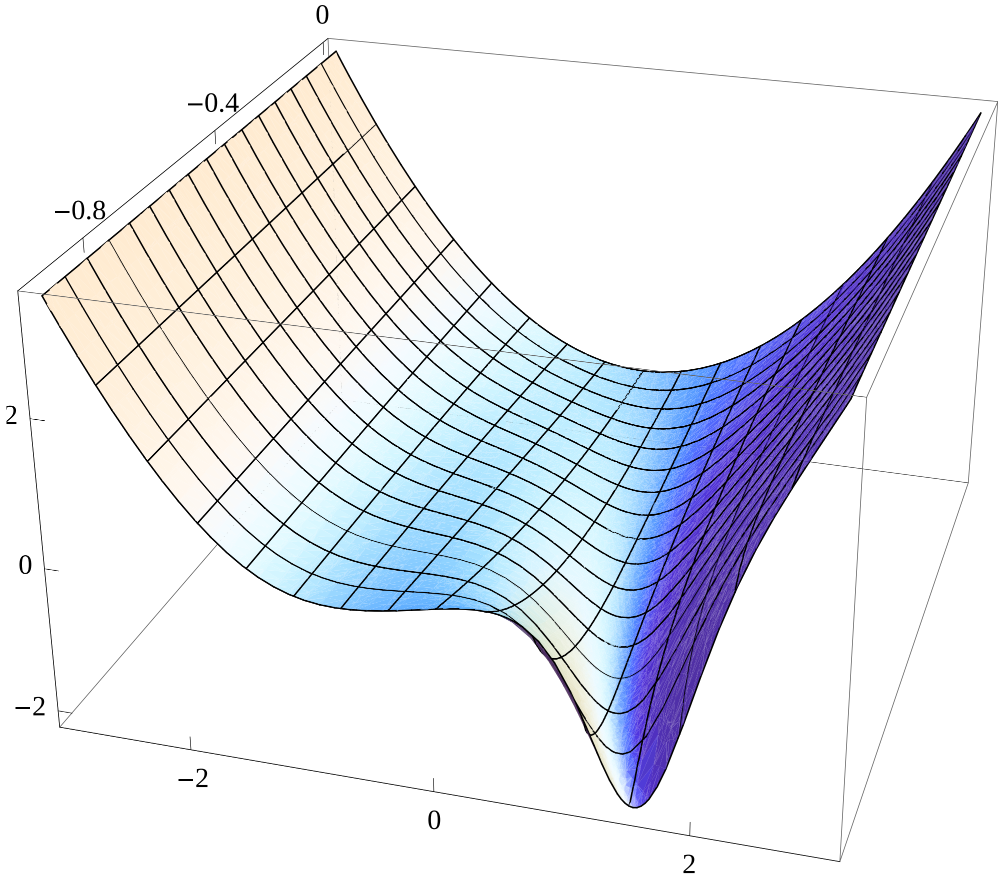
First-order SUSY QM



$$\epsilon_1 = 0, \quad \nu_1 = 0.9$$

First-order SUSY QM

Interesting case: $\epsilon_1 = -\frac{1}{2}$, $u_1^{(0)}(x) = e^{\frac{x^2}{2}}[1 + \nu_1 Erf(x)]$



Abraham-Moses-Mielnik potentials

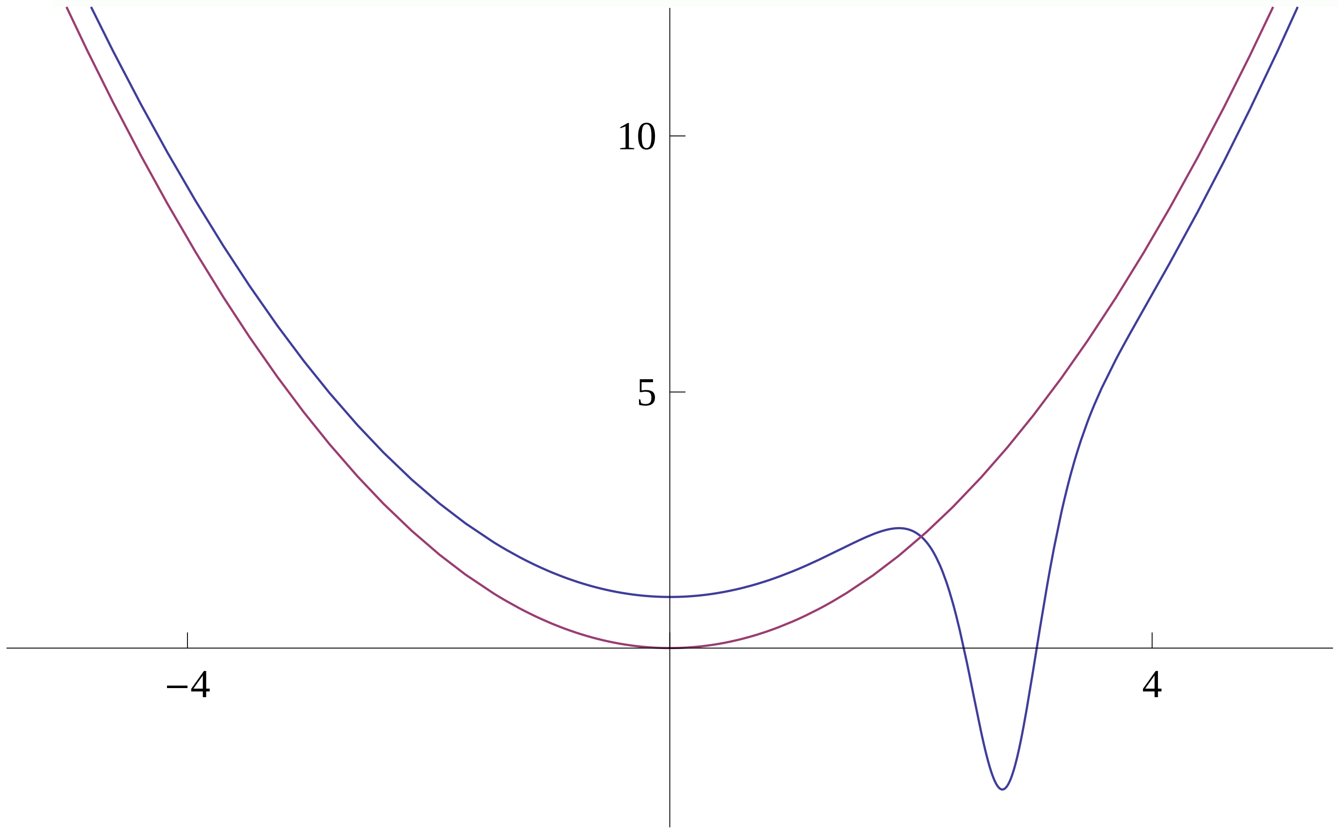
First-order SUSY QM

(iii) 1-SUSY through $u_1^{(0)}(x)$ associated to $\epsilon_1 < E_0$ and $|\nu_1| = 1$

$$V_1(x) = \frac{x^2}{2} - \{\log[u_1^{(0)}]\}''$$

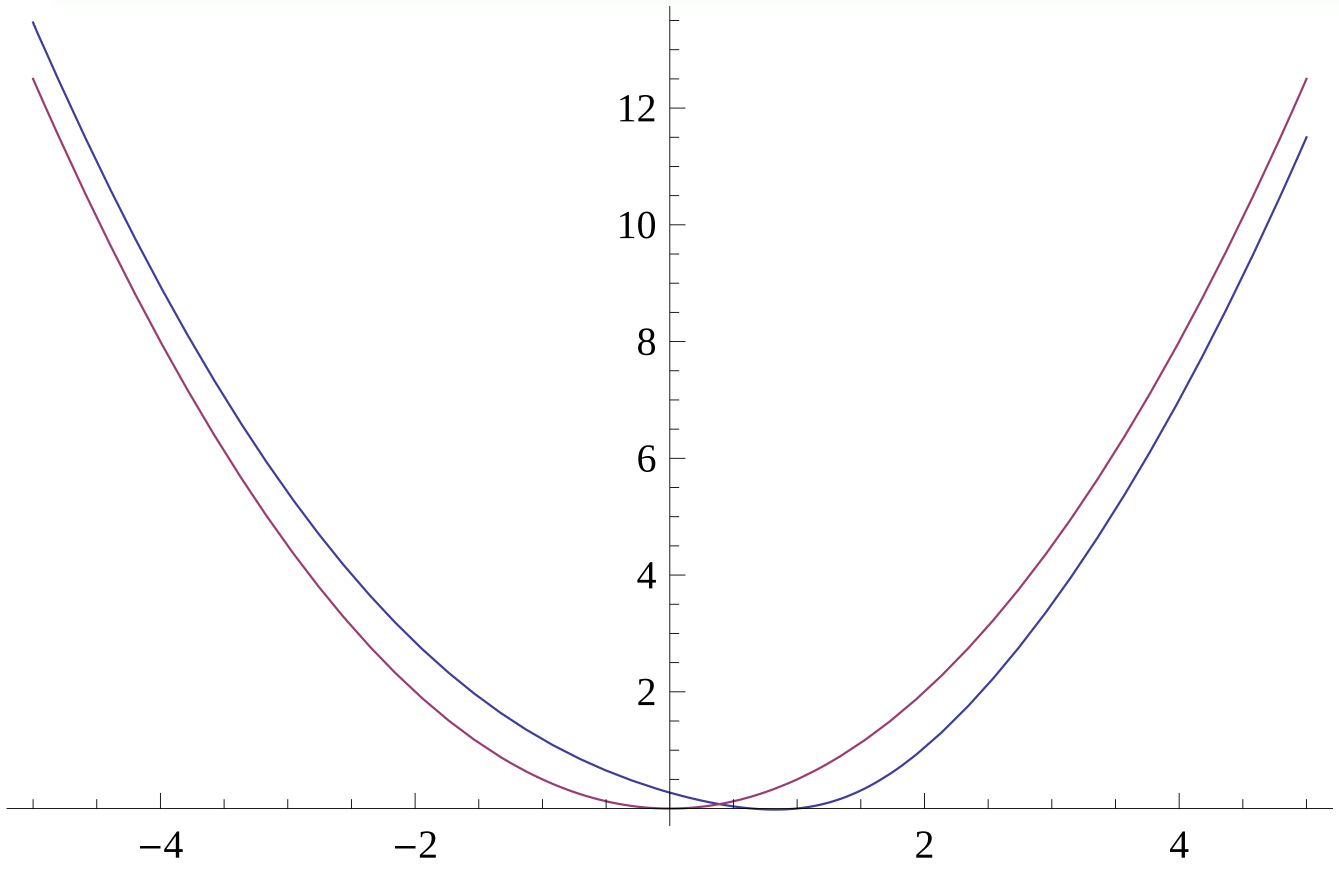
is also different from the initial potential

First-order SUSY QM



$$\epsilon_1 = \frac{1}{2} - 0.001 \qquad \nu_1 = 1$$

First-order SUSY QM



Abraham-Moses-Mielnik with $\epsilon_1 = -\frac{1}{2},\ \nu_1 = 1$

Higher-order SUSY QM

Let us iterate the 1-SUSY procedure taking V_1 and V_2 as the known and new potentials respectively and

$$A_2^+ = \frac{1}{\sqrt{2}}\left(-\frac{d}{dx} + \alpha_2(x,\epsilon_2)\right)$$

where $\alpha_2(x,\epsilon_2) = u_2^{(1)\prime}/u_2^{(1)}$, $\epsilon_2 < \epsilon_1$. It is required that

$$H_2 A_2^+ = A_2^+ H_1 \quad \Rightarrow$$

$$V_2 = V_1 - \alpha_2'(x,\epsilon_2)$$

$$\alpha_2'(x,\epsilon_2) + \alpha_2^2(x,\epsilon_2) = 2[V_1 - \epsilon_2]$$

$$-\tfrac{1}{2}u_2^{(1)\prime\prime} + V_1 u_2^{(1)} = H_1 u_2^{(1)} = \epsilon_2 u_2^{(1)}$$

Higher-order SUSY QM

$$\alpha_2(x,\epsilon_2) = -\alpha_1(x,\epsilon_1) - \frac{2(\epsilon_1-\epsilon_2)}{\alpha_1(x,\epsilon_1)-\alpha_1(x,\epsilon_2)}$$

Finite difference formula!
The new potential

$$\begin{aligned} V_2 &= V_1 - \alpha_2'(x,\epsilon_2) = V_0 + \left[\frac{2(\epsilon_1-\epsilon_2)}{\alpha_1(x,\epsilon_1)-\alpha_1(x,\epsilon_2)}\right]' \\ &= V_0 - [\log W(u_1^{(0)}, u_2^{(0)})]'' \end{aligned}$$

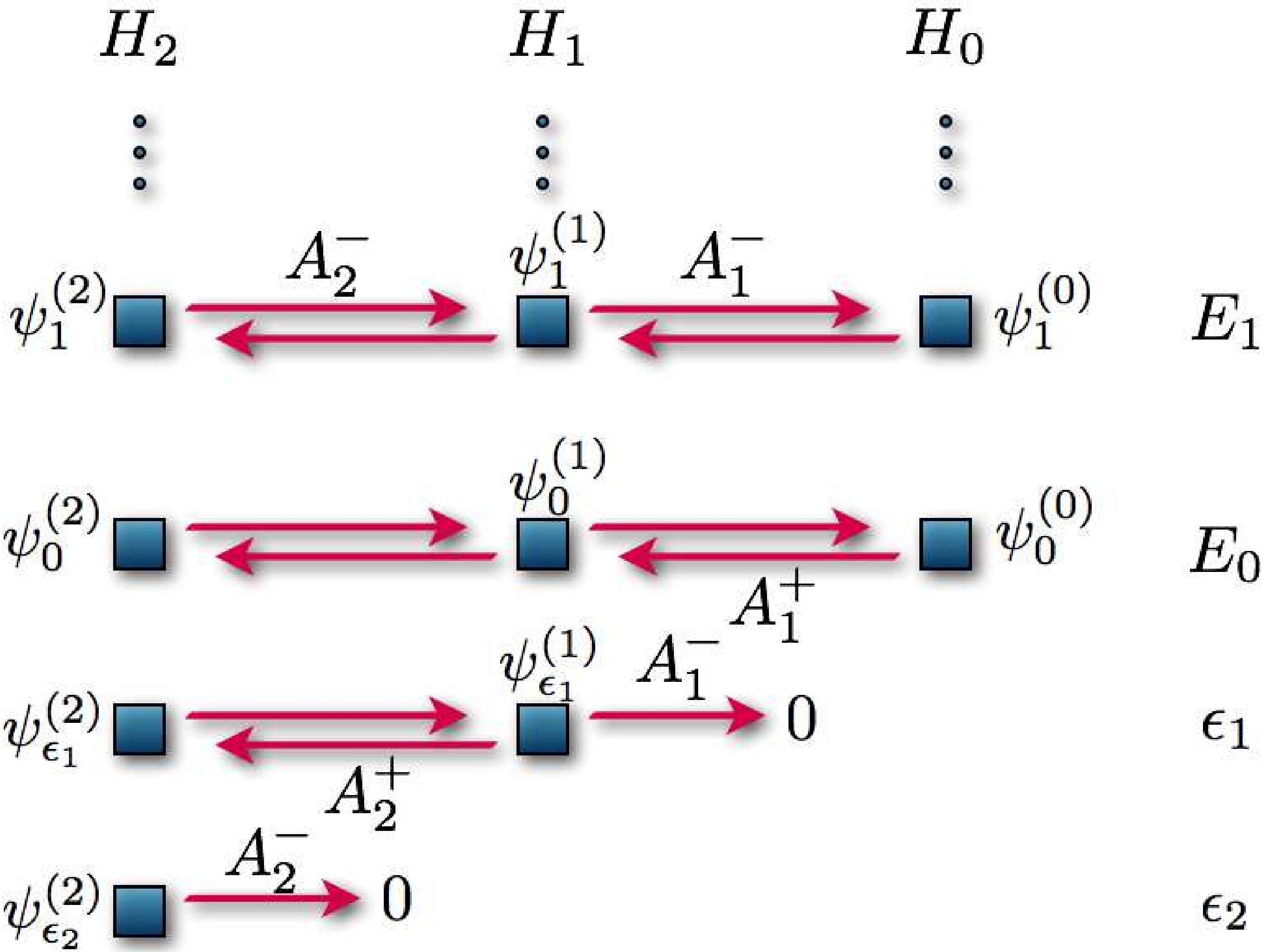
Higher-order SUSY QM

The maximal set of eigenfunctions and eigenvalues of H_2

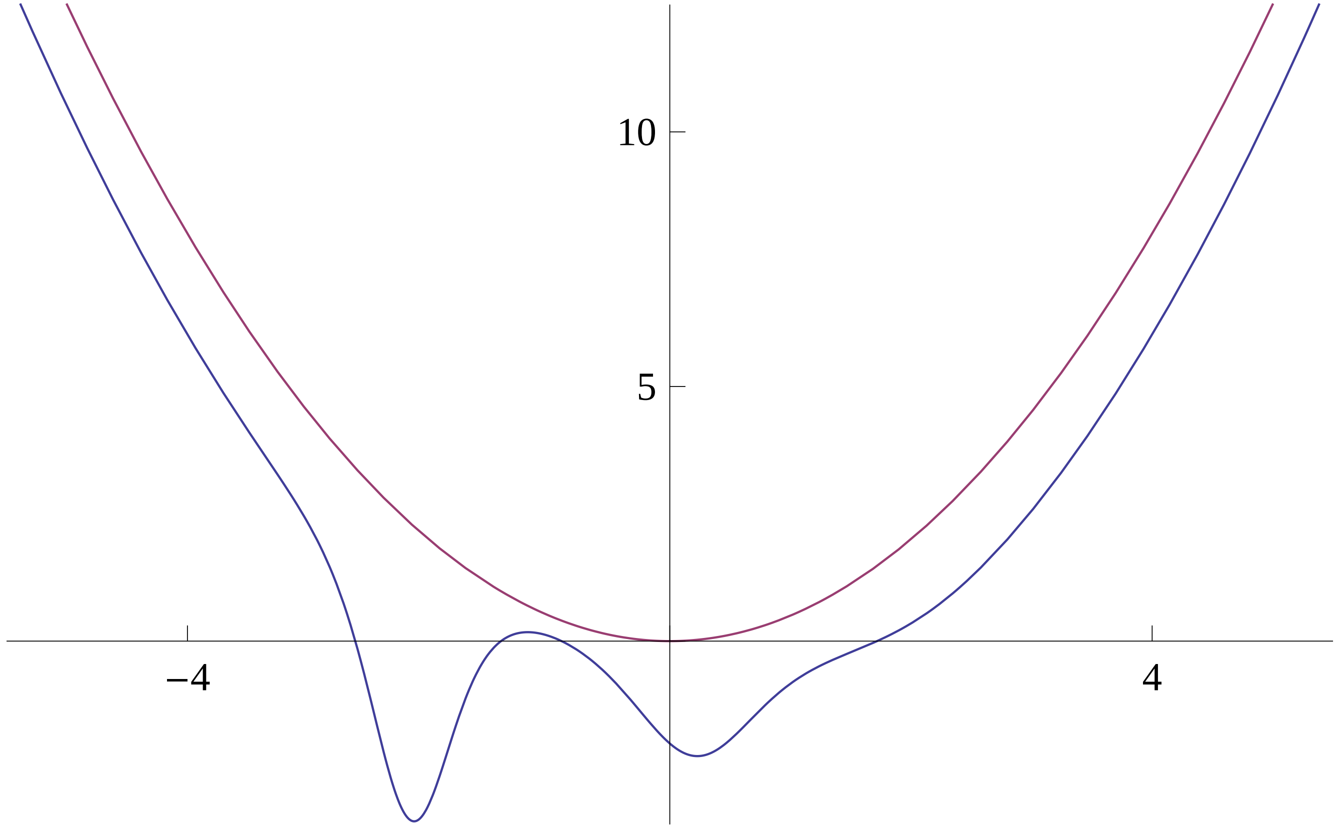
$$
\begin{aligned}
\psi_{\epsilon_2}^{(2)} &\propto \frac{1}{u_2^{(1)}} \propto \frac{u_1^{(0)}}{W(u_1^{(0)}, u_2^{(0)})} \\
\psi_{\epsilon_1}^{(2)} &= \frac{A_2^+ \psi_{\epsilon_1}^{(1)}}{\sqrt{\epsilon_1 - \epsilon_2}} \propto \frac{u_2^{(0)}}{W(u_1^{(0)}, u_2^{(0)})} \\
\psi_n^{(2)} &= \frac{A_2^+ \psi_n^{(1)}}{\sqrt{E_n - \epsilon_2}} = \frac{A_2^+ A_1^+ \psi_n^{(0)}}{\sqrt{(E_n - \epsilon_1)(E_n - \epsilon_2)}}
\end{aligned}
$$

$$\mathrm{Sp}(H_2) = \{\epsilon_2,\ \epsilon_1,\ E_n,\ n = 0, 1, 2, \dots\}$$

Higher-order SUSY QM

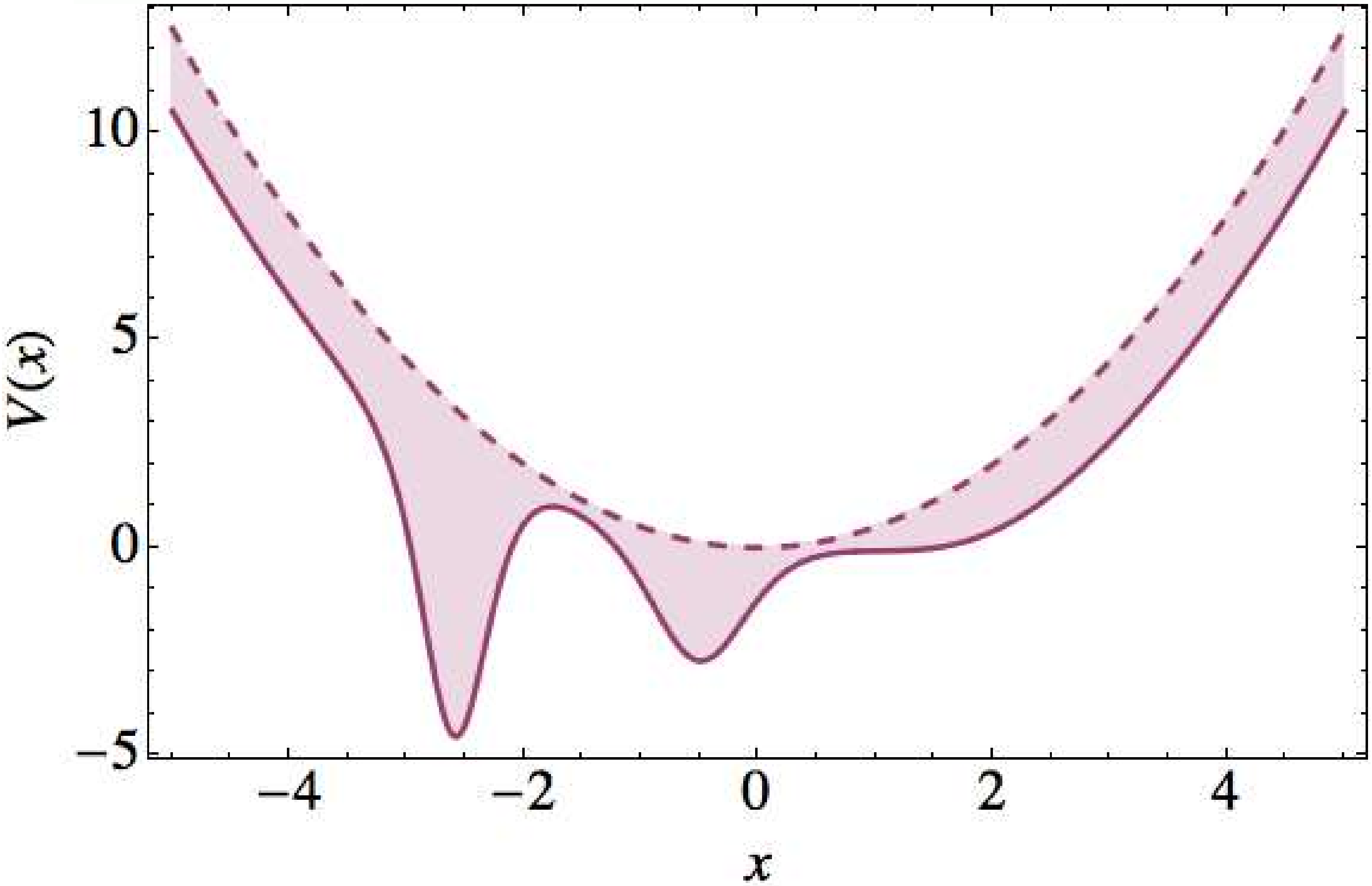


Example: harmonic oscillator



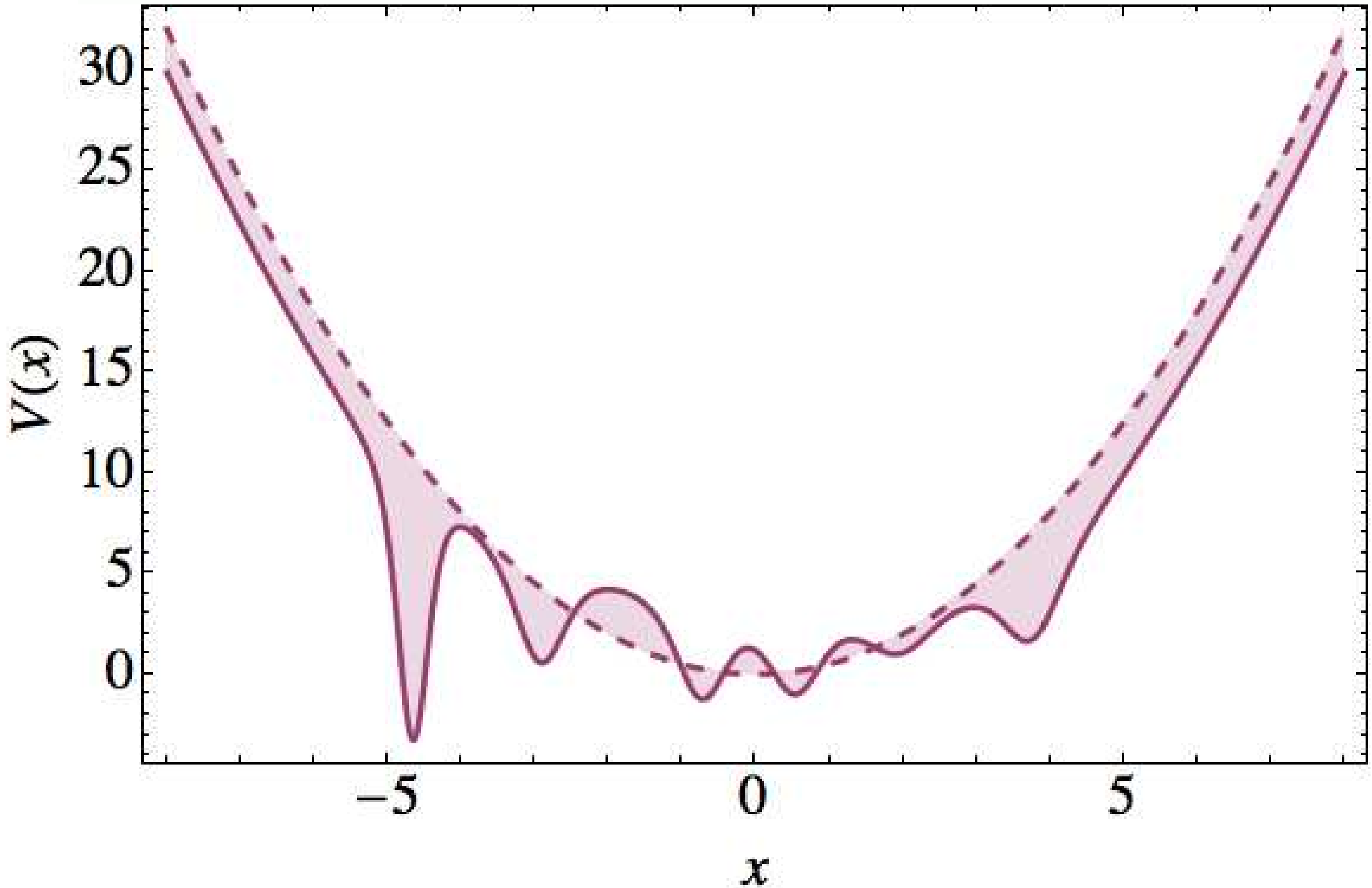
$$(\epsilon_1, \epsilon_2) = (-1, -1.2) \quad (\nu_1, \nu_2) = (0.9, 1.1)$$

Higher-order SUSY QM



$$(\epsilon_1, \epsilon_2) = (-1, -1.2) \quad (\nu_1, \nu_2) = (0.99, 1.01)$$

Higher-order SUSY QM



$$(\epsilon_1, \epsilon_2) = (3.2, 3) \quad (\nu_1, \nu_2) = (1.01, 0.99)$$

Higher-order SUSY QM

Repeating the 1-SUSY procedure k times taking k solutions $\{\alpha_1(x,\epsilon_i), i=1,2,\ldots,k,\ \epsilon_{i+1}<\epsilon_i\}$ it is obtained the Hamiltonian H_k with associated potential:

$$V_k = V_{k-1} - \alpha_k'(x,\epsilon_k) = V_0 - \sum_{i=1}^{k} \alpha_i'(x,\epsilon_i)$$

where

$$\alpha_{i+1}(x,\epsilon_{i+1}) = -\alpha_i(x,\epsilon_i) - \frac{2(\epsilon_i - \epsilon_{i+1})}{\alpha_i(x,\epsilon_i) - \alpha_i(x,\epsilon_{i+1})}$$

Higher-order SUSY QM

The chain of intertwining relations:

$$H_i A_i^+ = A_i^+ H_{i-1}, \quad i = 1, \dots, k$$

The chain of factorizations

$$\begin{aligned} H_0 &= A_1^- A_1^+ + \epsilon_1 \\ H_i &= A_i^+ A_i^- + \epsilon_i = A_{i+1}^- A_{i+1}^+ + \epsilon_{i+1},\ i = 1, \dots, k-1 \\ H_k &= A_k^+ A_k^- + \epsilon_k \end{aligned}$$

The potential V_k is determined by k Riccati solutions $\alpha_1(x, \epsilon_i),\ i = 1, \dots, k$ leading to k factorizations of H_0

$$H_0 = \frac{1}{2}\left[\frac{d}{dx} + \alpha_1(x,\epsilon_i)\right]\left[-\frac{d}{dx} + \alpha_1(x,\epsilon_i)\right] + \epsilon_i, \quad i = 1, \dots, k$$

Higher-order SUSY QM

The maximal set of eigenfunctions and eigenvalues of H_k:

$$\psi_{\epsilon_k}^{(k)} \propto e^{-\int_0^x \alpha_k(y,\epsilon_k)dy}$$

$$\psi_{\epsilon_{k-1}}^{(k)} = \frac{A_k^+ \psi_{\epsilon_{k-1}}^{(k-1)}}{\sqrt{\epsilon_{k-1}-\epsilon_k}}$$

$$\vdots$$

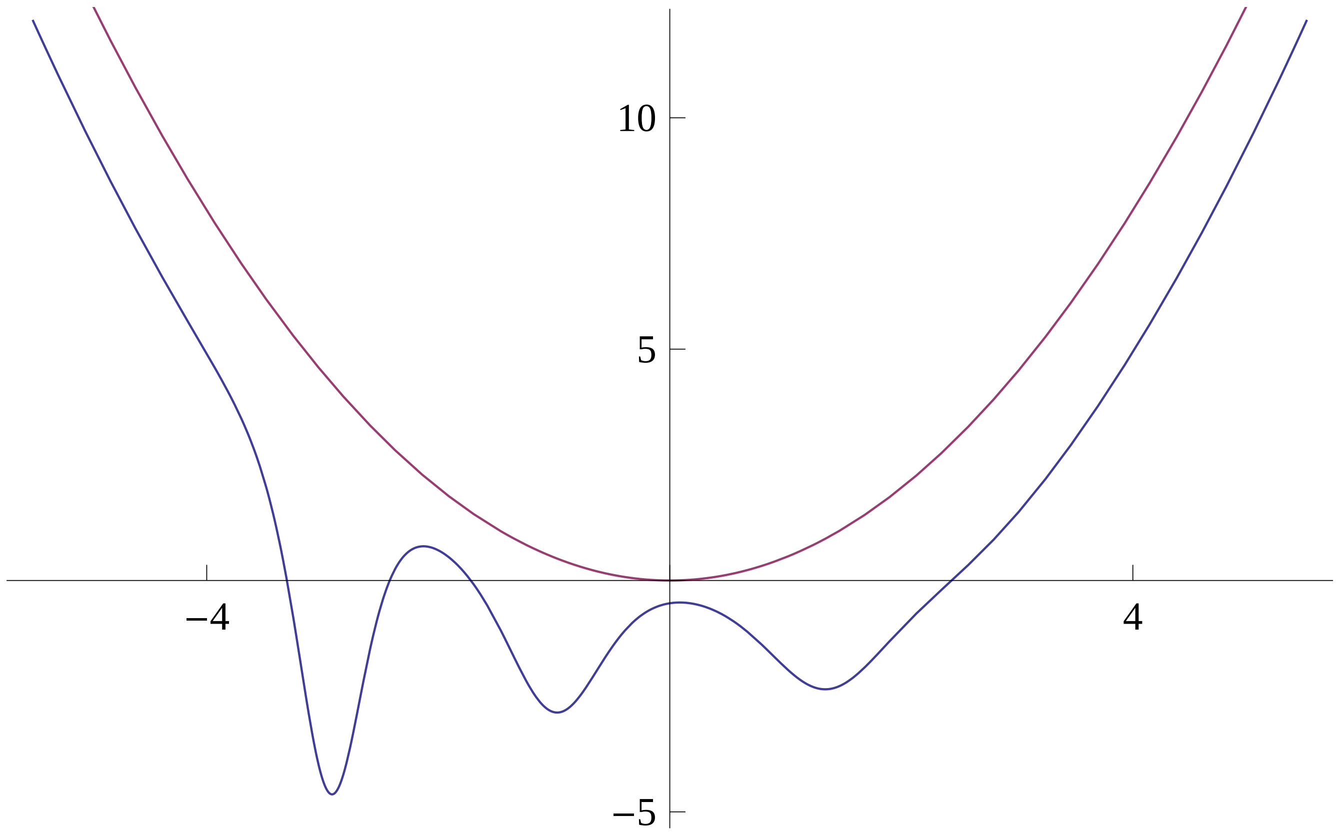
$$\psi_{\epsilon_1}^{(k)} = \frac{A_k^+ \dots A_2^+ \psi_{\epsilon_1}^{(1)}}{\sqrt{(\epsilon_1-\epsilon_2)\dots(\epsilon_1-\epsilon_k)}}$$

$$\psi_n^{(k)} = \frac{A_k^+ \dots A_1^+ \psi_n^{(0)}}{\sqrt{(E_n-\epsilon_1)\dots(E_n-\epsilon_k)}}$$

$$\mathrm{Sp}(H_k) = \{\epsilon_i, E_n, i = k, \dots, 1, n = 0, 1, 2, \dots\}$$

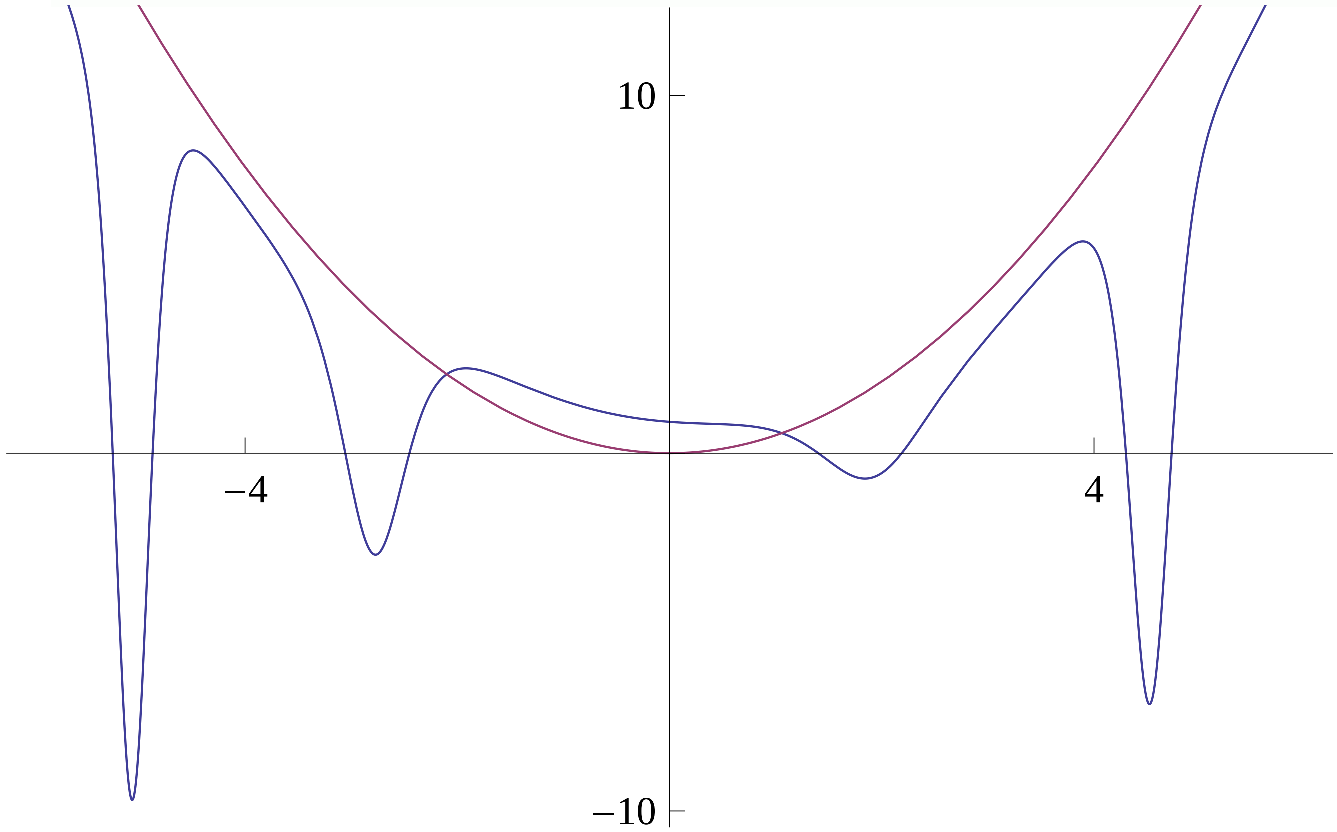
Higher-order SUSY QM

Example: harmonic oscillator with $k = 3$



$$(\epsilon_1, \epsilon_2, \epsilon_3) = (-1, -1.2, -1.4) \quad (\nu_1, \nu_2, \nu_3) = (0.9, 1.1, 0.9)$$

Higher-order SUSY QM



$$(\epsilon_1, \epsilon_2, \epsilon_3) = (\tfrac{1}{2} - 0.01, \tfrac{1}{2} - 0.02, \tfrac{1}{2} - 0.03)$$
$$(\nu_1, \nu_2, \nu_3) = (0.9, 1.1, 0.9)$$

Higher-order SUSY QM

There are intertwining operators of order kth

$$B_k^+ = A_k^+ \dots A_1^+, \quad B_k = A_1 \dots A_k$$
$$H_k B_k^+ = B_k^+ H_0, \quad H_0 B_k = B_k H_k$$

such that

$$\begin{aligned}
B_k^+ B_k &= A_k^+ \dots A_1^+ A_1 \dots A_k = A_k^+ \dots A_2^+ (H_1 - \epsilon_1) A_2 \dots A_k \\
&= A_k^+ \dots A_3^+ (H_2 - \epsilon_1)(H_2 - \epsilon_2) A_3 \dots A_k \\
&= \dots = (H_k - \epsilon_1) \dots (H_k - \epsilon_k) \\
B_k B_k^+ &= (H_0 - \epsilon_1) \dots (H_0 - \epsilon_k) \qquad \Rightarrow \\
B_k^+ \psi_n^{(0)} &= \sqrt{(E_n - \epsilon_1) \dots (E_n - \epsilon_k)}\, \psi_n^{(k)} \\
B_k \psi_n^{(k)} &= \sqrt{(E_n - \epsilon_1) \dots (E_n - \epsilon_k)}\, \psi_n^{(0)}
\end{aligned}$$

Harmonic oscillator SUSY partners

Recall the solution of the SE for $V_0(x) = \frac{x^2}{2}$ and arbitrary ϵ

$$u = e^{-\frac{x^2}{2}}[{}_1F_1(\tfrac{1-2\epsilon}{4}, \tfrac{1}{2}; x^2) + 2x\nu \frac{\Gamma(\frac{3-2\epsilon}{4})}{\Gamma(\frac{1-2\epsilon}{4})}\, {}_1F_1(\tfrac{3-2\epsilon}{4}, \tfrac{3}{2}; x^2)]$$

A k-th order SUSY transformation creating k new levels is performed with the factorization energies ordered as

$$\epsilon_k < \epsilon_{k-1} < \cdots < \epsilon_1 < E_0 = \frac{1}{2}$$

The constants $\nu_i, i = 1, \ldots, k$ in u_i associated to ϵ_i fulfill

$$|\nu_i| < 1 \text{ for odd } i, \quad |\nu_i| > 1 \text{ for even } i$$

Harmonic oscillator SUSY partners

The intertwining relations

$$H_k B_k^+ = B_k^+ H_0$$
$$H_0 B_k = B_k H_k$$

The new potential

$$V_k(x) \quad = \frac{x^2}{2} - \{\ln[W(u_1, \dots, u_k)]\}'', \quad k \geq 1$$

The eigenfunctions of H_k

$$\psi_n^{(k)} = \frac{B_k^+ \psi_n}{[(E_n - \epsilon_1)\dots(E_n - \epsilon_k)]^{1/2}}, \qquad E_n$$

$$\psi_{\epsilon_j}^{(k)} \propto \frac{W(u_1, \dots, u_{j-1}, u_{j+1}, \dots, u_k)}{W(u_1, \dots, u_k)}, \qquad \epsilon_j$$

Harmonic oscillator SUSY partners

The factorizations

$$\begin{aligned} B_k^+ B_k &= (H_k - \epsilon_1) \dots (H_k - \epsilon_k) \\ B_k B_k^+ &= (H_0 - \epsilon_1) \dots (H_0 - \epsilon_k) \end{aligned}$$

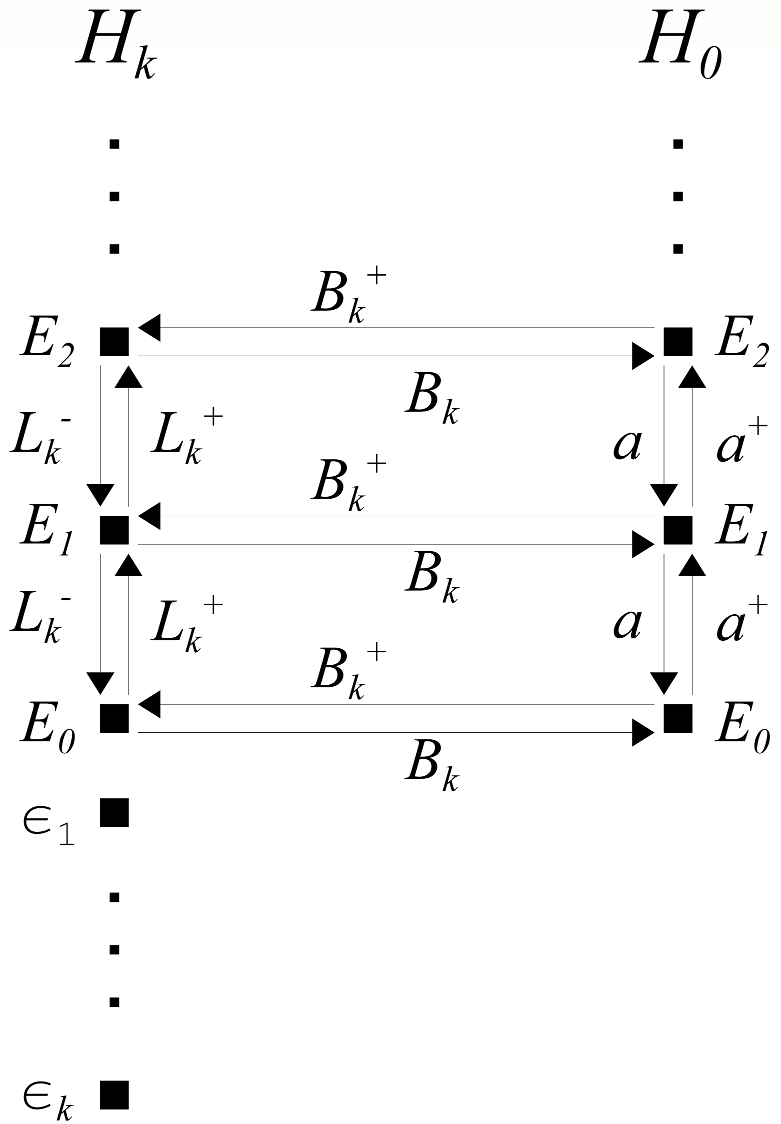
The spectrum of the Hamiltonian H_k

$$\mathrm{Sp}(H_k) = \{\epsilon_j, E_n = n + \tfrac{1}{2}, j = 1, \dots, k, n = 0, 1, \dots\}$$

suggests the following natural ladder operators for H_k

$$\begin{aligned} L_k^- &= B_k^+ a B_k \\ L_k^+ &= B_k^+ a^+ B_k \end{aligned}$$

Harmonic oscillator SUSY partners



Harmonic oscillator SUSY partners

The $(2k+1)$-th order differential ladder operators $L_k^{\pm}$ are such that:

$$[H_k, L_k^-] = -L_k^-$$
$$[H_k, L_k^+] = L_k^+$$

In fact, since $H_0\, a^+ = a^+(H_0 + 1),\ H_0\, a = a(H_0 - 1)$

$$H_k L_k^- = H_k B_k^+ a B_k^- = B_k^+ H_0\, a B_k^-$$
$$= B_k^+ a(H_0 - 1) B_k^- = B_k^+ a B_k^- (H_k - 1)$$
$$= L_k^- (H_k - 1)$$

Harmonic oscillator SUSY partners

The analogue of the number operator $N(H_k) \equiv L_k^+ L_k^-$ is a $(2k+1)$-th degree polynomial in H_k

$$
\begin{aligned}
N(H_k) &\equiv L_k^+ L_k^- = B_k^+ a^+ B_k^- B_k^+ a B_k^- \\
&= B_k^+ a^+ \prod_{i=1}^{k} (H_0 - \epsilon_i)\, a B_k^- \\
&= B_k^+ \left(H_0 - \frac{1}{2} \right) \prod_{i=1}^{k} (H_0 - \epsilon_i - 1)\, B_k^- \\
&= \left(H_k - \frac{1}{2} \right) \prod_{i=1}^{k} (H_k - \epsilon_i - 1)\,(H_k - \epsilon_i)
\end{aligned}
$$

Harmonic oscillator SUSY partners

Conclusion: the operator set $\{L_k^-,\ L_k^+,\ H_k\}$ close a $(2k)$-th degree polynomial Heisenberg algebra since

$$[L_k^-, L_k^+] = N(H_k + 1) - N(H_k)$$

The roots of $N(H_k)$ suggest that Sp(H_k) is composed of $k+1$ ladders: an infinite one departing from $E_0 = 1/2$ and k finites ones (of just one step), starting and ending at $\epsilon_i, i = 1, \dots, k$

The operator L_k^- annihilates the $k+1$ extremal states $\{\psi_0^{(k)}, \psi_{\epsilon_i}^{(k)}, i = 1, \dots, k\}$

The operator L_k^+ annihilates the k extremal states $\{\psi_{\epsilon_i}^{(k)}, i = 1, \dots, k\}$

Harmonic oscillator SUSY partners

Examples: For $k = 0$ with $B_0 = B_0^+ \equiv \mathbb{I}$ it turns out that $L_0^- = a, \ L_0^+ = a^+$ and

$$[H_0, L_0^-] = -L_0^-$$
$$[H_0, L_0^+] = L_0^+$$
$$[L_0^-, L_0^+] = \mathbb{I}$$
$$N(H_0) = H_0 - \frac{1}{2} = N$$

The Heisenberg-Weyl algebra is recovered!

Harmonic oscillator SUSY partners

For $k = 1$ with arbitrary ϵ_1 the quadratic case is found

$$[L_1^-, L_1^+] = (H_1 - \epsilon_1)(3H_1 - \epsilon_1)$$

The analogue of the number operator is cubic in H_1:

$$N(H_1) = \left(H_1 - \tfrac{1}{2}\right)(H_1 - \epsilon_1)(H_1 - \epsilon_1 - 1)$$

For general k a $(2k)$-th degree PHA is obtained, defined by the polynomial $N(H_k)$

Polynomial Heisenberg algebras

Polynomial Heisenberg algebras of $(m-1)$-th degree: deformations of the Heisenberg-Weyl algebra of kind:

$$[H, \mathcal{L}_m^+] = \mathcal{L}_m^+$$

$$[H, \mathcal{L}_m^-] = -\mathcal{L}_m^-$$

$$[\mathcal{L}_m^-, \mathcal{L}_m^+] \equiv N_m(H+1) - N_m(H) \equiv P_{m-1}(H)$$

The analogue of the number operator

$$N_m(H) \equiv \mathcal{L}_m^+ \mathcal{L}_m^-$$

is a m-th degree polynomial in H!

Polynomial Heisenberg algebras

Differential realizations: through one-dimensional Schrödinger-type Hamiltonians

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + V(x)$$

$\mathcal{L}_m^{\pm}$ being differential ladder operators of order m and $N_m(H)$ a m-th degree polynomial in H which can be factorized as

$$N_m(H) = \prod_{i=1}^{m}(H - \mathcal{E}_i)$$

$P_{m-1}(H)$ is a $(m-1)$-th degree polynomial in H

Polynomial Heisenberg algebras

$\mathrm{Sp}(H)$: appears from the study of the Kernel $K_{\mathcal{L}_m^-}$ of $\mathcal{L}_m^-$

$$\mathcal{L}_m^- \psi = 0 \quad \Rightarrow \quad \mathcal{L}_m^+ \mathcal{L}_m^- \psi = \prod_{i=1}^{m} (H - \mathcal{E}_i) \psi = 0$$

Since $K_{\mathcal{L}_m^-}$ in invariant under H,

$$\mathcal{L}_m^- H \psi = (H+1) \mathcal{L}_m^- \psi = 0 \quad \forall \quad \psi \in K_{\mathcal{L}_m^-}$$

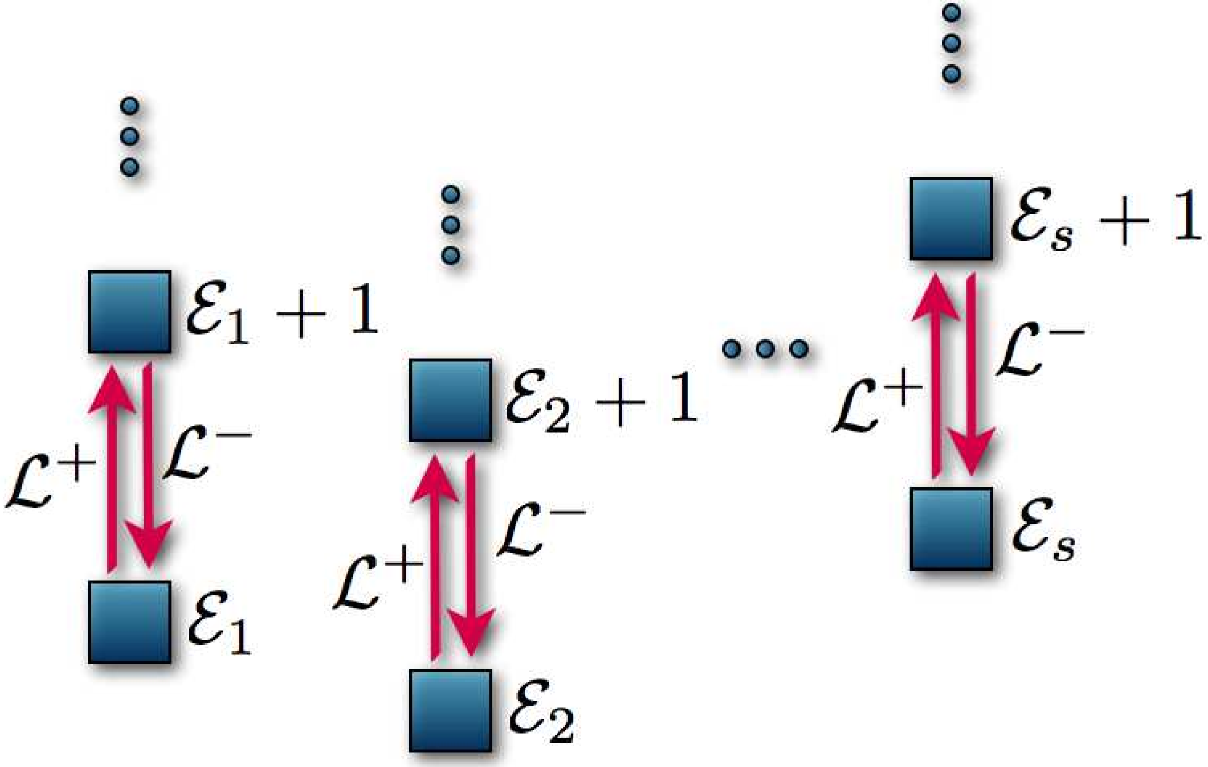
a natural basis choice in $K_{\mathcal{L}_m^-}$ is

$$H \psi_{\mathcal{E}_i} = \mathcal{E}_i \psi_{\mathcal{E}_i}$$

$\psi_{\mathcal{E}_i}$ are the extremal states: by applying $\mathcal{L}_m^+$ onto them m energy ladders with spacing $\Delta E = 1$ will arise

Polynomial Heisenberg algebras

(a) If s extremal states have physical meaning, $\{\psi_{\mathcal{E}_i}, i = 1, \ldots, s\}$, there will be s physical ladders obtained from the iterated action of $\mathcal{L}_m^+$ onto these extremal states



Polynomial Heisenberg algebras

(b) If for the j-th ladder $\exists\ n \in \mathbb{N}$ such that

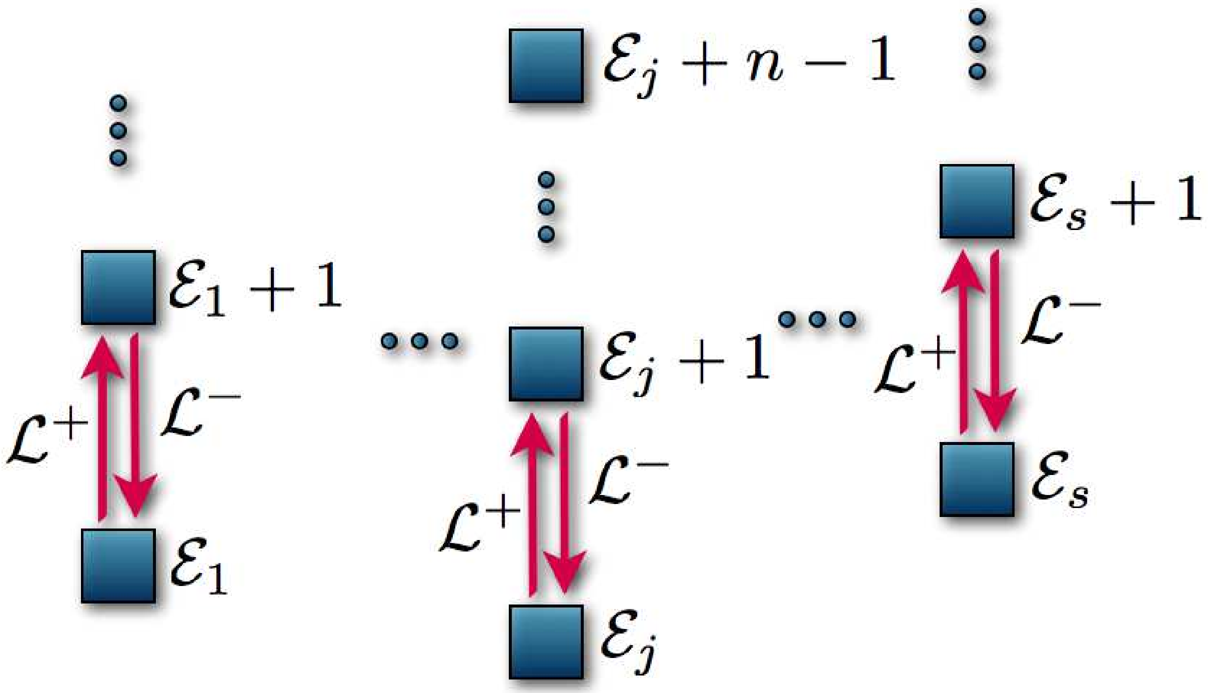
$$\left(\mathcal{L}_m^+\right)^{n-1} \psi_{\mathcal{E}_j} \neq 0, \qquad \left(\mathcal{L}_m^+\right)^{n} \psi_{\mathcal{E}_j} = 0$$

we will have

$$\begin{aligned}
\mathcal{L}_m^-(\mathcal{L}_m^+)^n\psi_{\mathcal{E}_j} &= \mathcal{L}_m^-\mathcal{L}_m^+(\mathcal{L}_m^+)^{n-1}\psi_{\mathcal{E}_j} \\
&= \prod_{i=1}^m (H + 1 - \mathcal{E}_i)\,(\mathcal{L}_m^+)^{n-1}\psi_{\mathcal{E}_j} \\
&= \prod_{i=1}^m (\mathcal{E}_j + n - \mathcal{E}_i)\,(\mathcal{L}_m^+)^{n-1}\psi_{\mathcal{E}_j} = 0
\end{aligned}$$

Polynomial Heisenberg algebras

Therefore $\mathcal{E}_l = \mathcal{E}_j + n$ for some $l \in \{s+1, \ldots, m\}$, $j \in \{1, \ldots, s\} \Rightarrow \mathrm{Sp}(H)$ consists of $s-1$ infinite ladders and one of lenght n which starts from $\mathcal{E}_j$ and ends at $\mathcal{E}_j + n - 1$. Conclusion: $\mathrm{Sp}(H)$ can have up to m infinite ladders



Polynomial Heisenberg algebras

General systems described by PHA of degree $0, 1, 2$

Zeroth-degree PHA ($m = 1$): First-order ladder operators

Let us take

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + V(x)$$

$$\mathcal{L}_1^+ = \frac{1}{\sqrt{2}}\left[-\frac{\mathrm{d}}{\mathrm{d}x} + f(x)\right], \quad \mathcal{L}_1^- = (\mathcal{L}_1^+)^\dagger$$

which satisfy

$$[H, \mathcal{L}_1^+] = \mathcal{L}_1^+$$

A system involving V, f, and their derivatives is obtained

Polynomial Heisenberg algebras

$$f' - 1 = 0$$
$$V' - f = 0$$

Up to coordinate and energy displacements

$$f(x) = x$$
$$V(x) = \frac{x^2}{2}$$

The normalized extremal state is obtained by solving $\mathcal{L}_1^- \psi_{\mathcal{E}_1} = 0$, leading to

$$\psi_{\mathcal{E}_1} = \pi^{-1/4} \exp(-x^2/2)$$

Polynomial Heisenberg algebras

Conclusions.

- The spectrum of H consists of an equidistant infinite energy ladder departing from $\mathcal{E}_1 = 1/2$
- The number operator is linear in H

$$N_1(H) = H - \mathcal{E}_1$$

- We recover the Heisenberg-Weyl algebra through the identification

$$\mathcal{L}_1^- = a, \quad \mathcal{L}_1^+ = a^+$$

Polynomial Heisenberg algebras

First-degree PHA (m=2): Second-order ladder operators

Now let us take

$$\mathcal{L}_2^+ = \frac{1}{2}\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} + g(x)\frac{\mathrm{d}}{\mathrm{d}x} + h(x)\right], \quad \mathcal{L}_2^- = (\mathcal{L}_2^+)^\dagger$$

Then, a system of equations for V, g, h, and their derivatives is obtained

$$\begin{aligned} g' + 1 &= 0 \\ h' + 2V' + g &= 0 \\ h'' + 2V'' + 2gV' + 2h &= 0 \end{aligned}$$

Polynomial Heisenberg algebras

The general solution (up to coordinate and energy displacements)

$$\begin{aligned} g(x) &= -x \\ h(x) &= \frac{x^2}{4} - \frac{\gamma}{x^2} - \frac{1}{2} \\ V(x) &= \frac{x^2}{8} + \frac{\gamma}{2x^2} \end{aligned}$$

γ is an integration constant. Now there are two extremal states, which are annihilated by $\mathcal{L}_2^-$ and are eigenstates of H with eigenvalues $\mathcal{E}_{1,2}$, given by

Polynomial Heisenberg algebras

$$\psi_{\mathcal{E}_1} \propto x^{1/2+\sqrt{\gamma+1/4}} \exp\left(-\frac{x^2}{4}\right), \quad \mathcal{E}_1 = \frac{1}{2} + \frac{1}{2}\sqrt{\gamma + \frac{1}{4}}$$

$$\psi_{\mathcal{E}_2} \propto x^{1/2-\sqrt{\gamma+1/4}} \exp\left(-\frac{x^2}{4}\right), \quad \mathcal{E}_2 = \frac{1}{2} - \frac{1}{2}\sqrt{\gamma + \frac{1}{4}}$$

Now $N_2(H)$ is quadratic in H

$$N_2(H) = (H - \mathcal{E}_1)(H - \mathcal{E}_2)$$

The potentials can be expressed as [$\gamma = \ell(\ell+1)$, $\ell \geq 0$]

$$V(x) = \frac{x^2}{8} + \frac{\ell(\ell+1)}{2x^2}, \quad x > 0, \quad \ell \geq 0$$

Polynomial Heisenberg algebras

Some conclusions:

- The general systems having second-order ladder operators are characterized by the radial oscillator potentials
- They have in general two infinite ladders, departing from $\mathcal{E}_1$ and $\mathcal{E}_2$
- By physical considerations (boundary conditions) in general it is ruled out the one starting from $\mathcal{E}_2$
- The ladder operators of the first-degree PHA (together with the Hamiltonian) generate the $\mathrm{so}(2,1)$ algebra

Polynomial Heisenberg algebras

Second-degree PHA ($m = 3$): Third-order ladder operators

Now both $\mathcal{L}_3^{\pm}$ are third-order differential ladder operators. To build them it is proposed a closed chain of SUSY transformations, i.e.,

$$\begin{aligned}
\mathcal{L}_3^{+} &= A_3^{+}A_2^{+}A_1^{+} = \frac{1}{2^{3/2}}\left(\frac{\mathrm{d}}{\mathrm{d}x} - f_3\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} - f_2\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} - f_1\right) \\
\mathcal{L}_3^{-} &= A_1^{-}A_2^{-}A_3^{-} = \frac{1}{2^{3/2}}\left(-\frac{\mathrm{d}}{\mathrm{d}x} - f_1\right)\left(-\frac{\mathrm{d}}{\mathrm{d}x} - f_2\right)\left(-\frac{\mathrm{d}}{\mathrm{d}x} - f_3\right)
\end{aligned}$$

such that

$$H_{j+1}A_j^{+} = A_j^{+}H_j, \quad H_jA_j^{-} = A_j^{-}H_{j+1}, \quad j = 1,2,3$$

Polynomial Heisenberg algebras

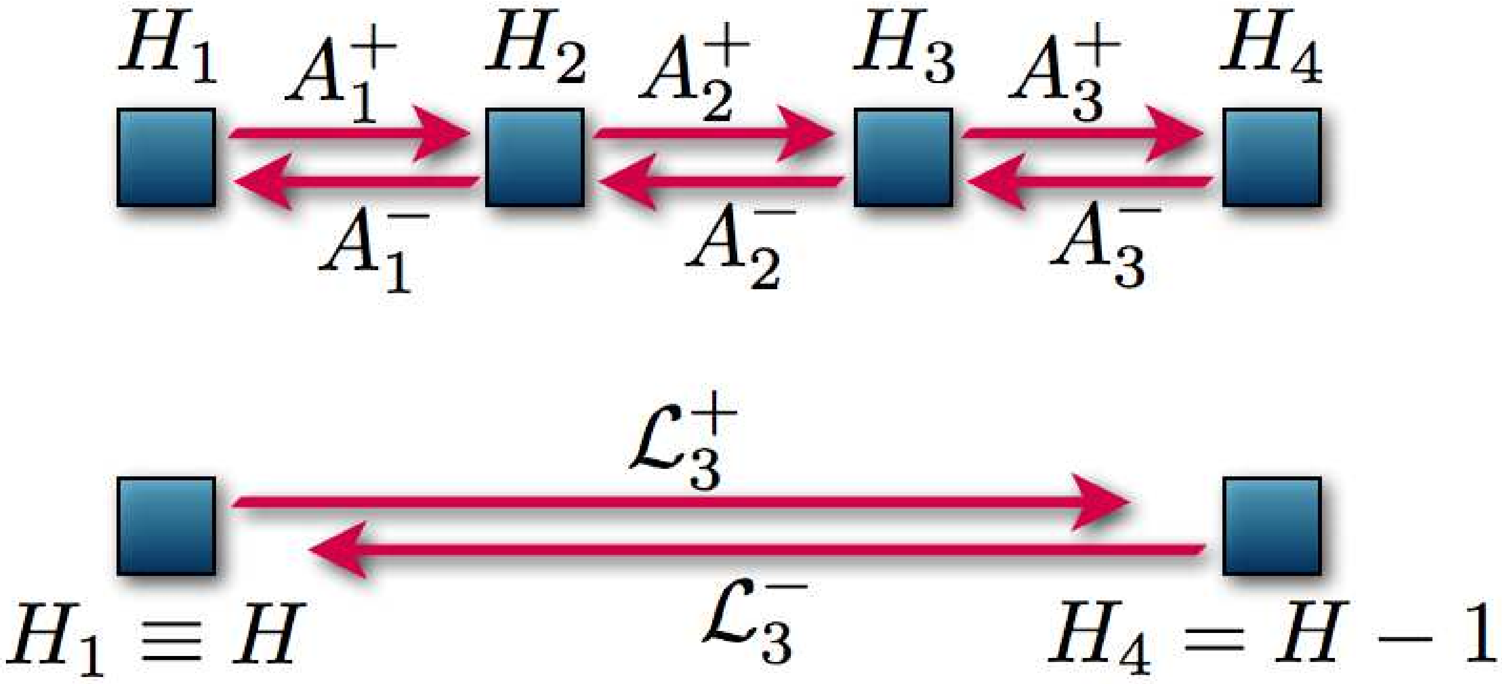
The associated factorizations

$$\begin{aligned} H_1 &= A_1^- A_1^+ + \epsilon_1 \\ H_2 &= A_1^+ A_1^- + \epsilon_1 = A_2^- A_2^+ + \epsilon_2 \\ H_3 &= A_2^+ A_2^- + \epsilon_2 = A_3^- A_3^+ + \epsilon_3 \\ H_4 &= A_3^+ A_3^- + \epsilon_3 \end{aligned}$$

The closure condition

$$H_4 = H_1 - 1 \equiv H - 1$$

Polynomial Heisenberg algebras



Polynomial Heisenberg algebras

An explicit calculation leads to

$$f_1' + f_2' = f_1^2 - f_2^2 + 2(\epsilon_1 - \epsilon_2)$$
$$f_2' + f_3' = f_2^2 - f_3^2 + 2(\epsilon_2 - \epsilon_3)$$
$$f_3' + f_1' = f_3^2 - f_1^2 + 2(\epsilon_3 - \epsilon_1 + 1)$$

If we add the three equations we obtain

$$f_1' + f_2' + f_3' = 1 \quad \Rightarrow$$

$$f_1 + f_2 + f_3 = x$$

Polynomial Heisenberg algebras

Then $f_2 = x - f_1 - f_3$, and substituting this into the first equation and then solve for f_1:

$$f_1 = -\frac{g}{2} + \frac{g'}{2g} + \frac{\epsilon_1 - \epsilon_2}{g}$$

where

$$g \equiv f_3 - x \quad \Rightarrow \quad f_3 = x + g$$

Due to $f_2 = x - f_1 - f_3$ it turns out that

$$f_2 = -\frac{g}{2} - \frac{g'}{2g} - \frac{\epsilon_1 - \epsilon_2}{g}$$

Polynomial Heisenberg algebras

Since f_1, f_2, f_3 are expressed in terms of g, we replace them in the third equation to obtain

$$gg'' = \frac{1}{2}g'^2 + \frac{3}{2}g^4 + 4xg^3 + 2\left(x^2 - a\right)g^2 + b$$

which is the Painlevé IV equation with parameters

$$a = \epsilon_1 + \epsilon_2 - 2\epsilon_3 - 1, \qquad b = -2(\epsilon_1 - \epsilon_2)^2$$

Once we find a solution to this equation the potential can be found through

$$V(x) = \frac{x^2}{2} - \frac{g'}{2} + \frac{g^2}{2} + xg + \epsilon_3 + \frac{1}{2}$$

Polynomial Heisenberg algebras

In addition, since $f_i, \ i = 1, 2, 3$ are expressed in terms of g the ladder operators $\mathcal{L}_3^{\pm}$ are also completely determined. The energies of the extremal states are the roots of the generalized number operator, which is cubic

$$N_3(H) = (H - \mathcal{E}_1)(H - \mathcal{E}_2)(H - \mathcal{E}_3)$$

where $\mathcal{E}_i = \epsilon_i + 1, \ i = 1, 2, 3$. The three extremal states are obtained from

$$\mathcal{L}_3^- \psi_{\mathcal{E}_j} = (H - \mathcal{E}_j)\psi_{\mathcal{E}_j} = 0, \quad j = 1, 2, 3,$$

which leads to the following expressions

Polynomial Heisenberg algebras

$$\psi_{\mathcal{E}_1} \propto \left(\frac{g'}{2g} - \frac{g}{2} - \frac{1}{g}\sqrt{-\frac{b}{2}} - x\right) \exp\left[\int \left(\frac{g'}{2g} + \frac{g}{2} - \frac{1}{g}\sqrt{-\frac{b}{2}}\right) \mathrm{d}x\right]$$

$$\psi_{\mathcal{E}_2} \propto \left(\frac{g'}{2g} - \frac{g}{2} + \frac{1}{g}\sqrt{-\frac{b}{2}} - x\right) \exp\left[\int \left(\frac{g'}{2g} + \frac{g}{2} + \frac{1}{g}\sqrt{-\frac{b}{2}}\right) \mathrm{d}x\right]$$

$$\psi_{\mathcal{E}_3} \propto \exp\left(-\frac{x^2}{2} - \int g \, \mathrm{d}x\right)$$

- The physical ladders are obtained departing from the extremal states with physical meaning
- In this way we determine the spectrum of the Hamiltonian H
- All this discussion concerns what is called as direct approach

Polynomial Heisenberg algebras

Example: let the following solution to the PIV equation

$$g(x) = -x - \alpha(x)$$

where $\mathcal{E}_1 = \mathcal{E}_3$, $\alpha(x) = u'/u$ satisfies the Riccati equation

$$\alpha'(x) + \alpha^2(x) = x^2 - 2\epsilon$$

with $\epsilon = \mathcal{E}_3 - \mathcal{E}_2 + 1/2$, and $u(x)$ is the corresponding Schrödinger solution given by ($|\nu| < 1$)

$$u(x) = \mathrm{e}^{-\frac{x^2}{2}} \left[{}_1F_1\left(\tfrac{1-2\epsilon}{4}, \tfrac{1}{2}; x^2\right) + 2x\nu \frac{\Gamma\left(\frac{3-2\epsilon}{4}\right)}{\Gamma\left(\frac{1-2\epsilon}{4}\right)} \, {}_1F_1\left(\tfrac{3-2\epsilon}{4}, \tfrac{3}{2}; x^2\right) \right]$$

This solution $g(x)$ leads to the harmonic oscillator potential

Polynomial Heisenberg algebras

$$V(x) = \frac{x^2}{2} + \mathcal{E}_2 - \frac{1}{2}$$

The three extremal states become

$$\begin{aligned}
\psi_{\mathcal{E}_1} &= 0 \\
\psi_{\mathcal{E}_2} &\propto \exp\left(-\frac{x^2}{2}\right) \\
\psi_{\mathcal{E}_3} &\propto u(x)
\end{aligned}$$

Thus, the only physical ladder is generated from $\psi_{\mathcal{E}_2}$

Polynomial Heisenberg algebras

There is a straightforward connection between the third extremal state and $g(x)$, the solution to the PIV equation:

$$g(x) = -x - \{\ln[\psi_{\mathcal{E}_3}(x)]\}'$$

Thus, if we would know a system ruled by a second-degree PHA, specifically its extremal states, we could find solutions to the PIV equation. This is the spirit of the inverse problem which we will explore in detail in the last lecture

Solutions of PIV through SUSY QM

First-order SUSY QM

- $L_1^{\pm}$ are third-order differential ladder operators
- Thus, the first-order SUSY partner of the oscillator could provide solutions to the P_{IV} equation
- We just need to idenfity the extremal states of the system as well as their corresponding energies
- The energies are the roots involved in the analogue of the number operator

$$N(H_1) = (H_1 - \tfrac{1}{2})(H_1 - \epsilon_1 - 1)(H_1 - \epsilon_1)$$

Solutions of PIV through SUSY QM

The extremal states associated to $\frac{1}{2}$ and ϵ_1 have been given previously. The one associated to $\epsilon_1 + 1$ has to be built with the help of the seed solution u_1 employed to implement the transformation. All this allows to identify the extremal states of our system in the way

$$
\begin{aligned}
\psi_{\mathcal{E}_1} &\propto A_1^+ e^{-x^2/2}, && \mathcal{E}_1 = \frac{1}{2} \\
\psi_{\mathcal{E}_2} &\propto A_1^+ a^+ u_1, && \mathcal{E}_2 = \epsilon_1 + 1 \\
\psi_{\mathcal{E}_3} &\propto \frac{1}{u_1}, && \mathcal{E}_3 = \epsilon_1
\end{aligned}
$$

Solutions of PIV through SUSY QM

The first-order SUSY partner potential $V_1(x)$ of the harmonic oscillator and the corresponding non-singular solution of PIV are

$$V_1(x) = \frac{x^2}{2} - \{\ln[u_1(x)]\}'$$
$$g_1(x,\epsilon_1) = -x - \{\ln[\psi_{\mathcal{E}_3}(x)]\}' = -x + \{\ln[u_1(x)]\}'$$

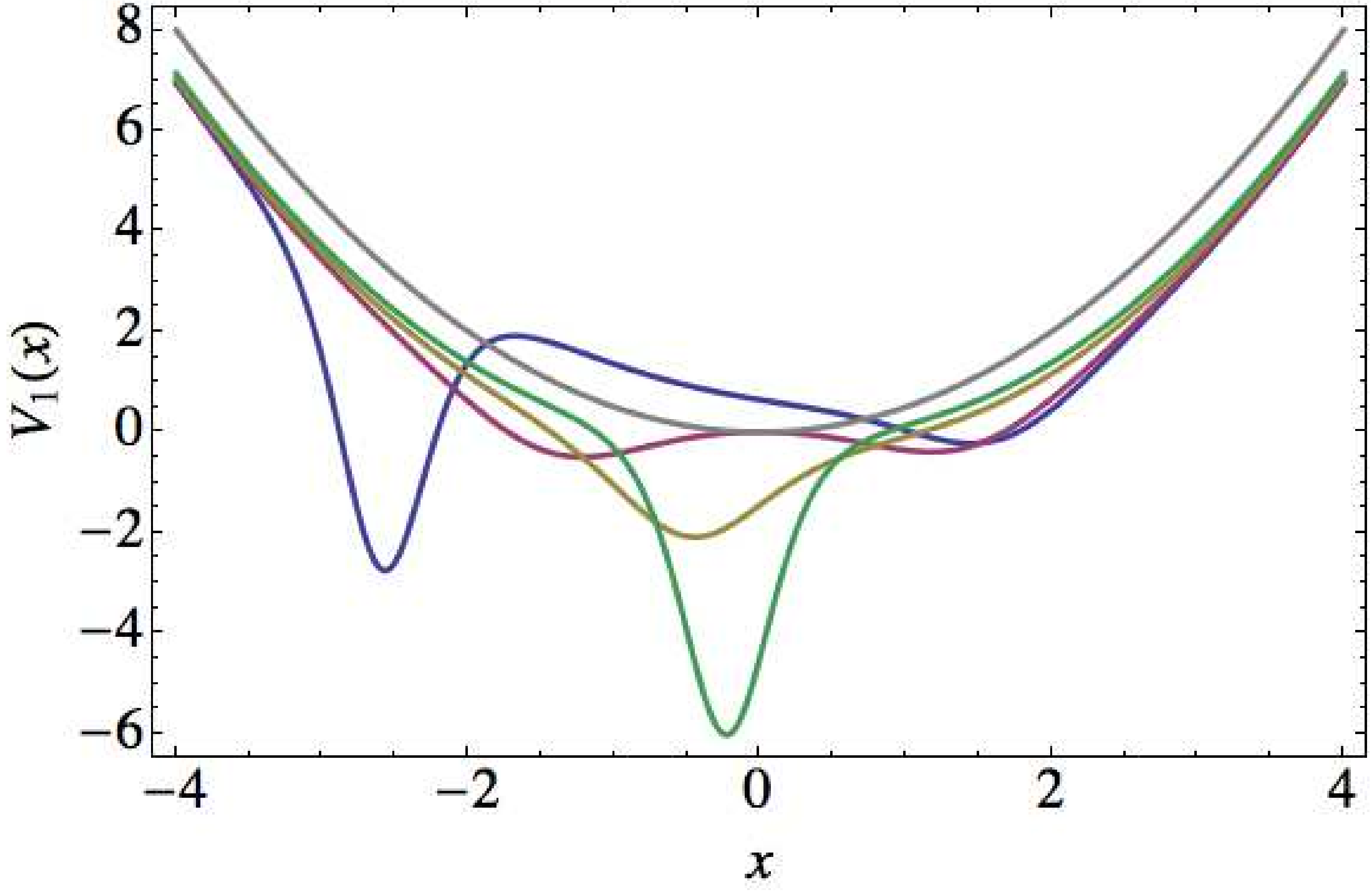
The parameters of the PIV equation are here

$$a = \tfrac{1}{2} - \epsilon_1 \geq 0, \quad b = -2(\epsilon_1 + \tfrac{1}{2})^2 \leq 0 \quad \Rightarrow \quad b = -2(a-1)^2$$

Solutions of PIV through SUSY QM

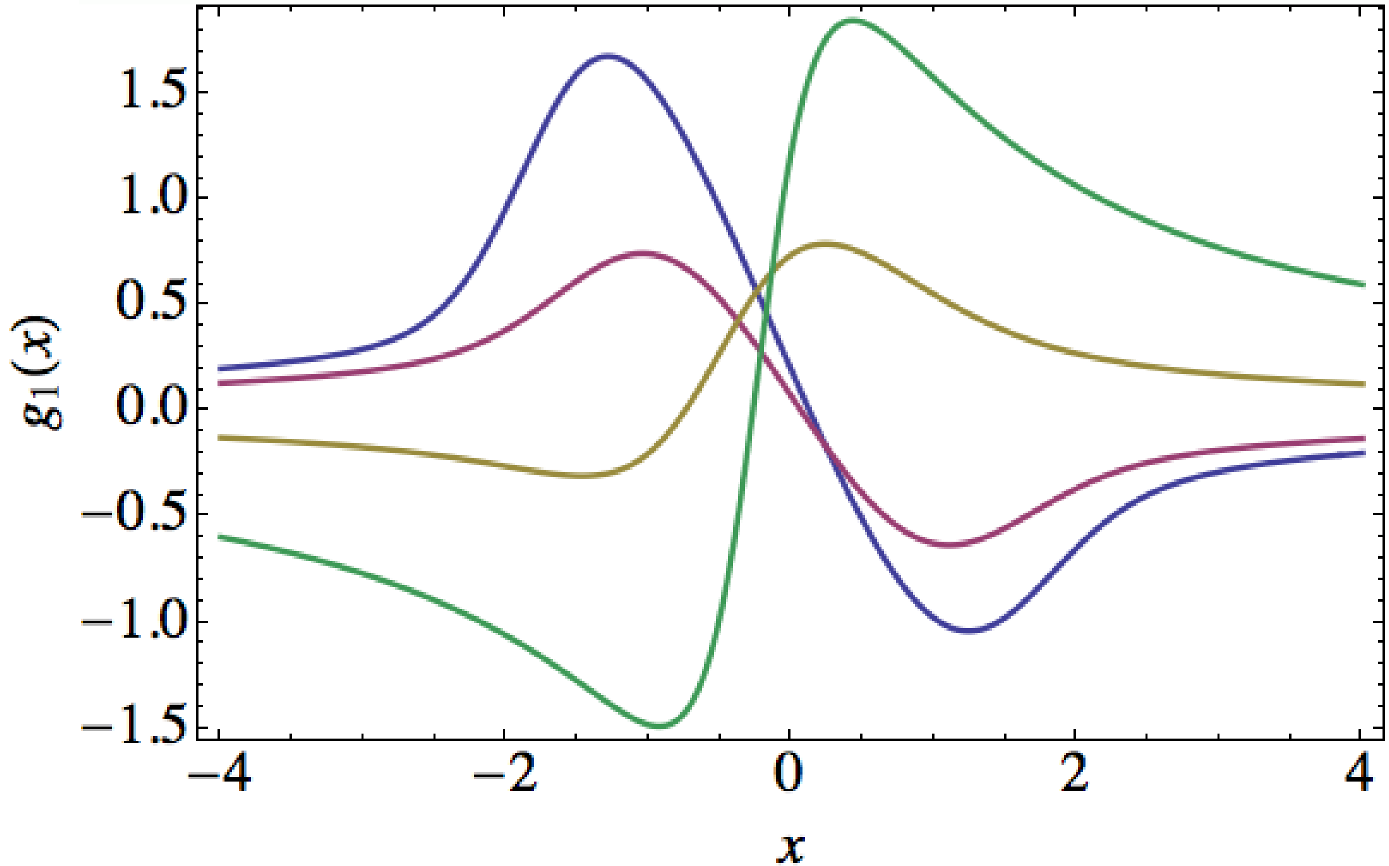
- The index in the PIV solution indicate the order of the transformation
- Two more solutions of the PIV equation are obtained by cyclic permutations of the indices $(1, 2, 3)$. However, they will have singularities

Solutions of PIV through SUSY QM



$\epsilon_1 = 0.25,\ \nu_1 = 0.99$ (blue); $\epsilon_1 = 0,\ \nu_1 = 0.1$ (magenta); $\epsilon_1 = -1,\ \nu_1 = 0.5$ (yellow); and $\epsilon_1 = -4,\ \nu_1 = 0.5$ (green)

Solutions of PIV through SUSY QM



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Solutions of PIV through SUSY QM

Higher-order SUSY QM

- $L_k^{\pm}$ are $(2k+1)$-order differential ladder operators
- Is it possible to reduce somehow the order of this natural ladder operators to three?
- If so we could obtain new systems which perhaps would supply us with new solutions to the PIV equation
- The answer turns out to be positive and it is contained in the following reduction theorem

Solutions of PIV through SUSY QM

Theorem. Suppose that the kth-order SUSY partner H_k of the harmonic oscillator Hamiltonian H_0 is generated by k connected Schrödinger seed solutions

$$u_j = (a^-)^{j-1} u_1, \quad \epsilon_j = \epsilon_1 - (j-1), \quad j = 1, \dots, k$$

$u_1(x)$ is a nodeless solution for $\epsilon_1 < 1/2$ and $|\nu_1| < 1$. Thus, the natural $(2k+1)$th-order ladder operator $L_k^+ = B_k^+ a^+ B_k^-$ of H_k is factorized in the form

$$L_k^+ = P_{k-1}(H_k) l_k^+$$

where $P_{k-1}(H_k) = (H_k - \epsilon_1) \dots (H_k - \epsilon_{k-1})$, and l_k^+ is a third-order differential ladder operator such that

Solutions of PIV through SUSY QM

$$[H_k, l_k^+] = l_k^+$$

$$l_k^+ l_k^- = (H_k - \epsilon_k)\left(H_k - \frac{1}{2}\right)(H_k - \epsilon_1 - 1)$$

Note that

- The operators $l_k^\pm$ connect the eigenstates of the new levels $\epsilon_j, j = 1, \ldots, k$, which form a finite ladder of lenght k starting from $\epsilon_k = \epsilon_1 - (k-1)$ and ending at ϵ_1
- The operator l_k^+ annihilates the eigenstate for ϵ_1

Solutions of PIV through SUSY QM

- From the roots of $l_k^+ l_k^-$ the operator l_k^- annihilates the following three extremal states, two of which are physical:

$$\psi_{\mathcal{E}_1} \propto B_k^+ e^{-x^2/2}, \quad \mathcal{E}_1 = \tfrac{1}{2}$$

$$\psi_{\mathcal{E}_2} \propto B_k^+ a^+ u_1, \quad \mathcal{E}_2 = \epsilon_1 + 1$$

$$\psi_{\mathcal{E}_3} \propto \frac{W(u_1,\ldots,u_{k-1})}{W(u_1,\ldots,u_k)}, \quad \mathcal{E}_3 = \epsilon_k = \epsilon_1 - (k-1)$$

- The operators $l_k^\pm$ and the Hamiltonian H_k fulfill a second-degree PHA, thus we will generate solutions to the PIV equation departing from its extremal states

Solutions of PIV through SUSY QM

The kth-order SUSY partner potential of the harmonic oscillator and the corresponding non-singular solution of the PIV equation become now

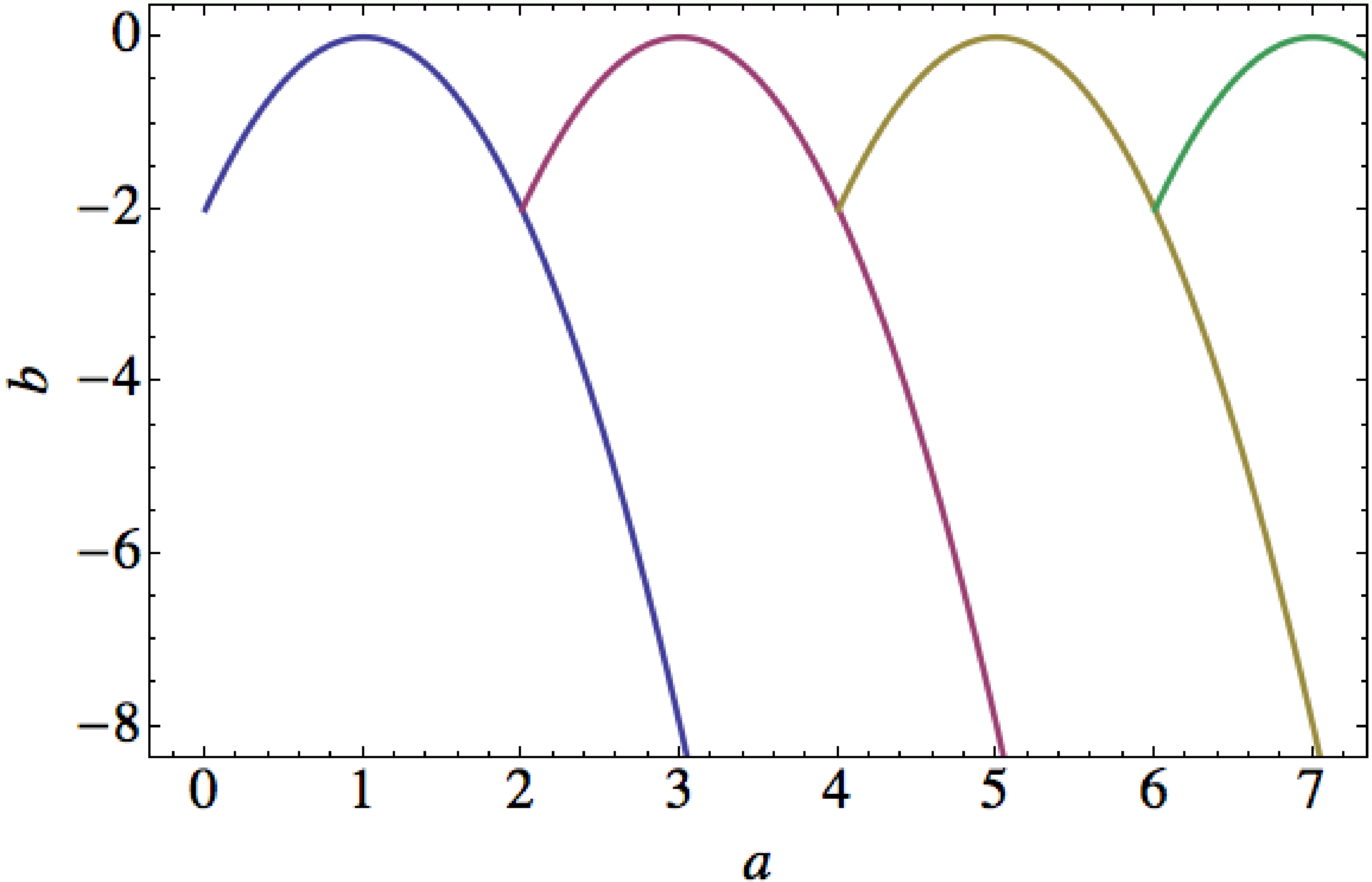
$$V_k(x) = \frac{x^2}{2} - \{\ln[W(u_1,\dots,u_k)]\}'', \quad k \geq 2$$

$$g_k(x,\epsilon_1) = -x - \{\ln[\psi_{\mathcal{E}_3}(x)]\}' = -x - \left\{\ln\left[\frac{W(u_1,\dots,u_{k-1})}{W(u_1,\dots,u_k)}\right]\right\}'$$

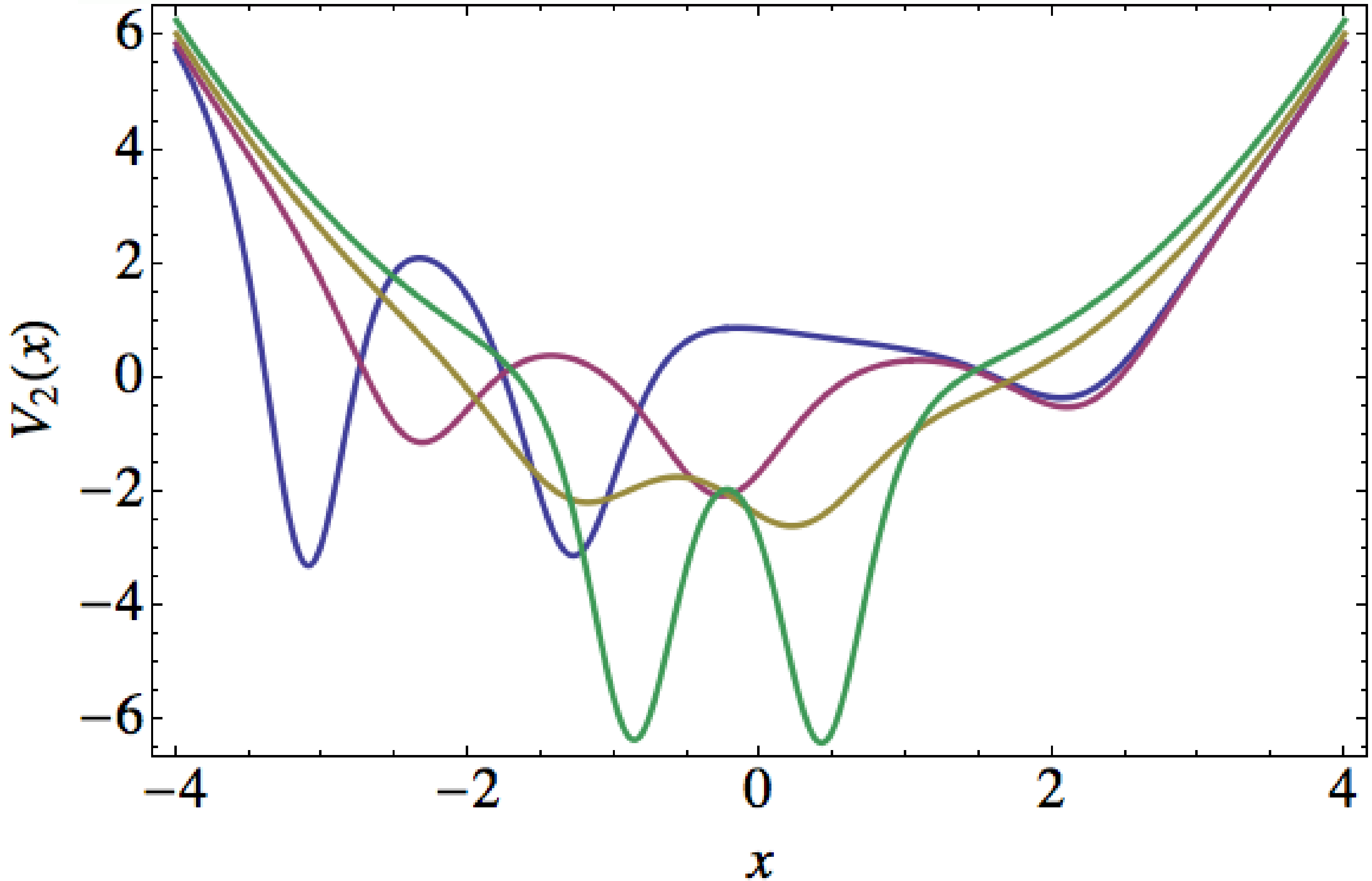
The parameters of the PIV equation are now

$$a = -\epsilon_1 + 2k - \frac{3}{2}, \quad b = -2\left(\epsilon_1 + \frac{1}{2}\right)^2$$

Solutions of PIV through SUSY QM

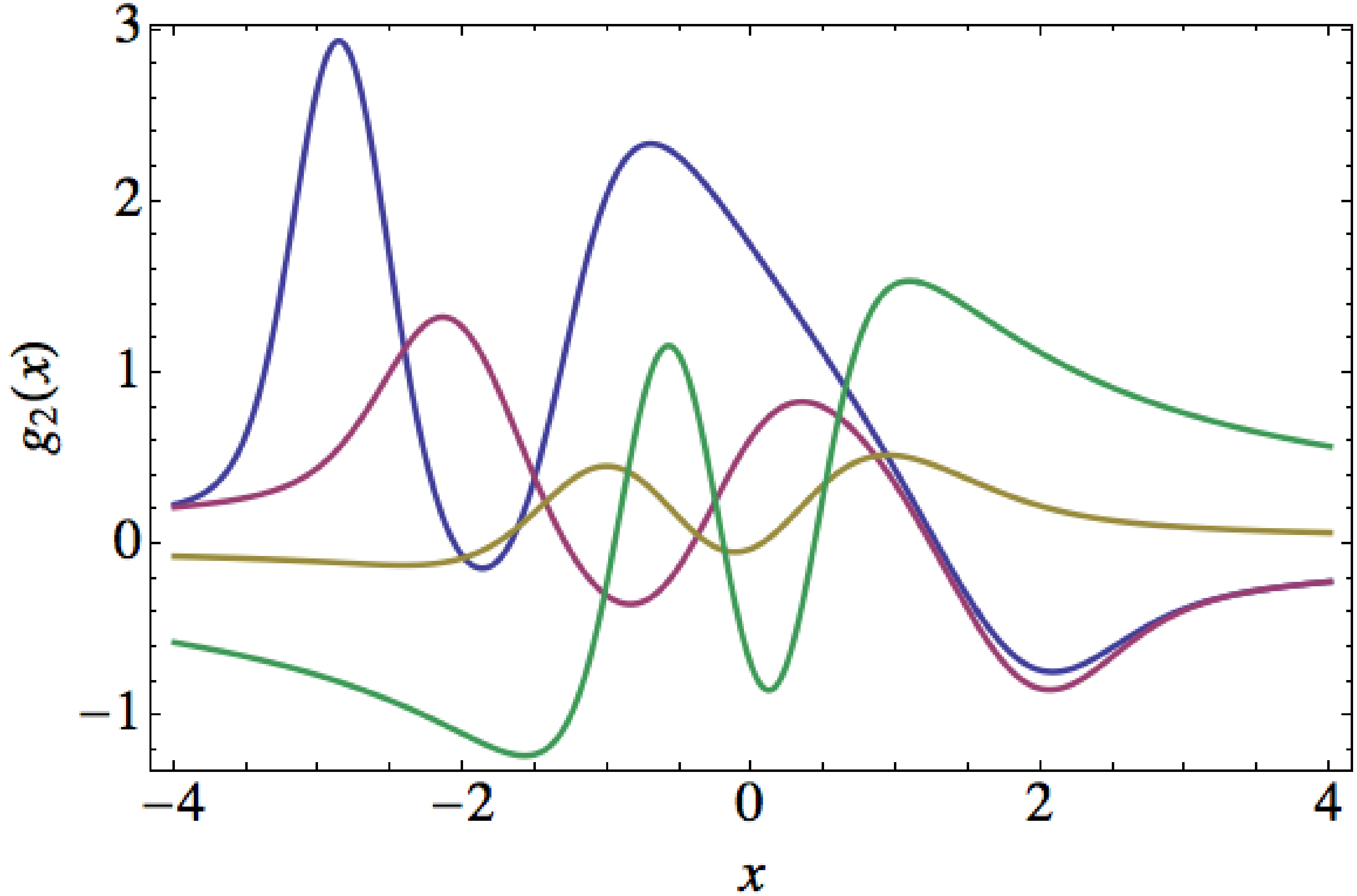


Solutions of PIV through SUSY QM



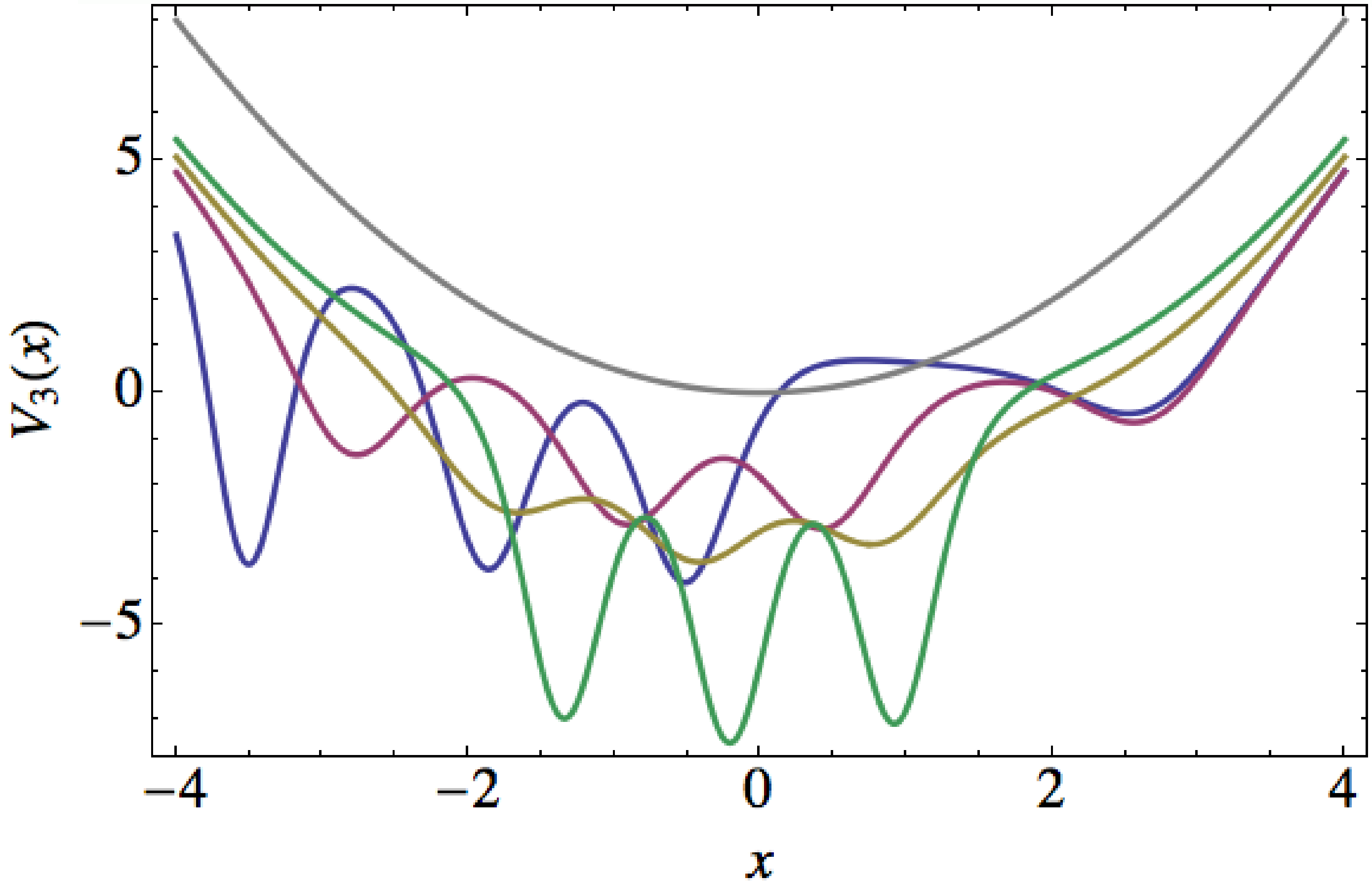
$\epsilon_1 = 0.25, \nu_1 = 0.99$ (blue) $\epsilon_1 = \{0.25$ (magenta), -0.75 (yellow), -2.75 (green)$\}$, $\nu_1 = 0.5$

Solutions of PIV through SUSY QM



$\epsilon_1 = 0.25, \nu_1 = 0.99$ (blue) $\epsilon_1 = \{0.25$ (magenta),-0.75 (yellow),$-2.75(\mathrm{green})\}$, $\nu_1 = 0.5$

Solutions of PIV through SUSY QM



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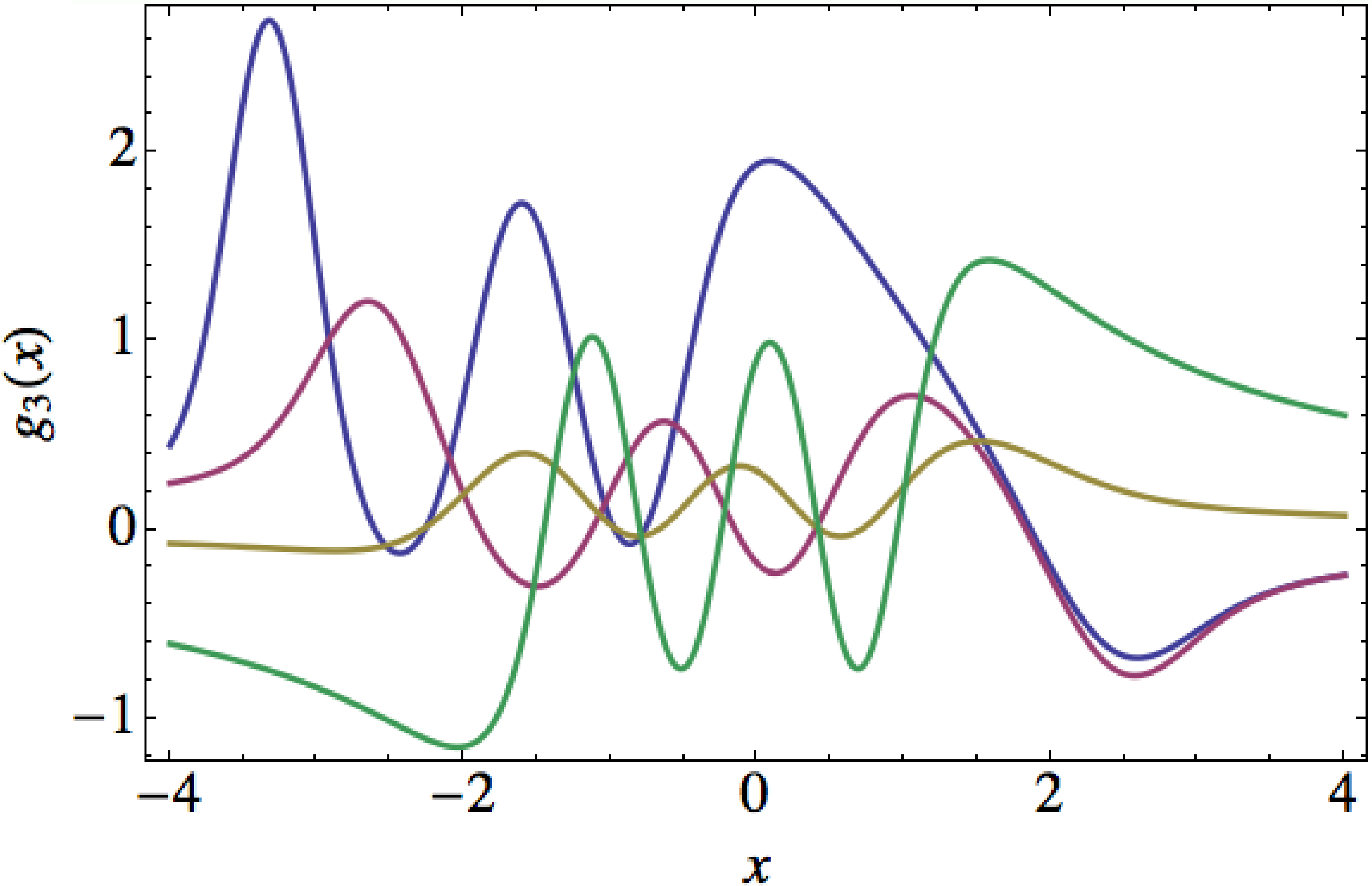
Solutions of PIV through SUSY QM



$\epsilon_1 = 0.25, \nu_1 = 0.99$ (blue) $\epsilon_1 = \{0.25$ (magenta), -0.75 (yellow), $-2.75(\text{green})\}$, $\nu_1 = 0.5$

Solutions of PIV through SUSY QM

Partial conclusions:

- The first-order SUSY partners of the harmonic oscillator provide straightforwardly non-singular real solutions to the PIV equation
- Its higher-order SUSY partners require a reduction process which, once performed, produces new real non-singular solution to the PIV equation
- In the parameters space of solutions $a - b$ we have been able to identify some curves on which one-parametric families of solutions exist

Solutions of PIV through SUSY QM

- We would like to be able to expand the points in the parameter space $a - b$ on which we can find also non-singular solutions to the PIV equation
- Let us tackle this issue

Solutions of PIV through SUSY QM

- In our previous treatment we got the restriction $\epsilon_1 < E_0 = 1/2, \ |\nu_1| < 1$ in order that the new potentials V_k and the corresponding PIV solution would be non-singular
- The previous facts imply that the complete finite ladder of H_k (with k steps) is placed below E_0
- From the spectral design point of view it would be important to surpass this restriction so that it would be possible to place (either partially or totally) the finite ladder of H_k above E_0

Solutions of PIV through SUSY QM

- This can be done, but there is a price to pay: the transformation function u_1 associated to the real factorization energy ϵ_1 will have to be complex
- As a consequence, we will obtain now complex potentials with real energy spectra
- Almost all the previous formulae remain valid, the main change is that in the real case $A_j^- = (A_j^+)^\dagger$, but now this is not true, although $A_j^\pm$ maintain its original form

$$A_j^\pm = \frac{1}{2^{1/2}}\left(\mp\frac{d}{dx} + \alpha_j\right)$$

Solutions of PIV through SUSY QM

- Thus, the normalization factors appearing in the previous formulae are no longer valid although the corresponding eigenfunctions would be square-integrable
- The reduction theorem is still valid, so we will have to supply one complex solution u_1 of the SSE associated to a real ϵ_1 which now can be any real number

Solutions of PIV through SUSY QM

Let us assume that the k seed solutions of the SSE used to implement the SUSY transformation are connected in the way

$$
\begin{aligned}
u_j &= (a^-)^{j-1} u_1, \\
\epsilon_j &= \epsilon_1 - (j-1), \qquad j = 1, \dots, k,
\end{aligned}
$$

where now

$$
u_1(x) = e^{-x^2/2} \left[{}_1F_1 \left(\tfrac{1-2\epsilon_1}{4}, \tfrac{1}{2}; x^2 \right) + \Lambda \, x \, {}_1F_1 \left(\tfrac{3-2\epsilon_1}{4}, \tfrac{3}{2}; x^2 \right) \right]
$$

where $\Lambda = \lambda + i\kappa$ ($\lambda, \kappa \in \mathbb{R}$), $\epsilon_1 \in \mathbb{R}$. Once again, this system has third-order differential ladder operators, the corresponding extremal states being

Solutions of PIV through SUSY QM

$$\psi_{\mathcal{E}_1} \propto B_k^+ e^{-x^2/2}, \quad \mathcal{E}_1 = \tfrac{1}{2}$$
$$\psi_{\mathcal{E}_2} \propto B_k^+ a^+ u_1, \quad \mathcal{E}_2 = \epsilon_1 + 1$$
$$\psi_{\mathcal{E}_3} \propto \tfrac{W(u_1,\ldots,u_{k-1})}{W(u_1,\ldots,u_k)}, \quad \mathcal{E}_3 = \epsilon_k = \epsilon_1 - (k-1)$$

With this labels choice we obtain the following solution to the PIV equation

$$g_k(x) = -x - \{\ln[\psi_{\mathcal{E}_3}(x)]\}' = -x - \left\{\ln\left[\tfrac{W(u_1,\ldots,u_{k-1})}{W(u_1,\ldots,u_k)}\right]\right\}'$$
$$a_i = -\epsilon_1 + 2k - \tfrac{3}{2}, \quad b_i = -2\left(\epsilon_1 + \tfrac{1}{2}\right)^2$$

Solutions of PIV through SUSY QM

Moreover, by making a cyclic permutation of the indices we can obtain now two additional non-singular solutions for different parameters a, b:

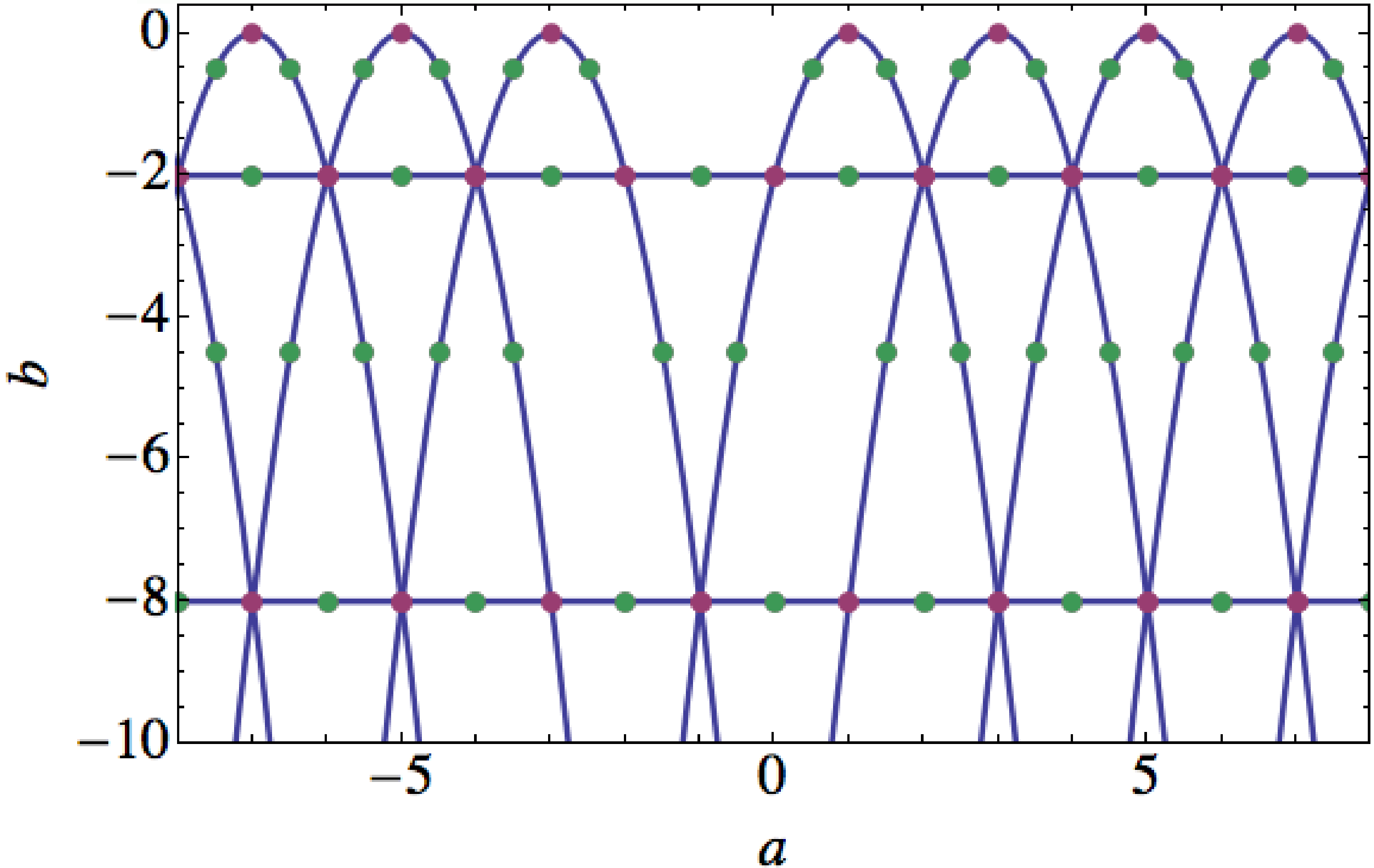
$$g_k(x) = -x - \left\{ \ln \left[B_k^+ e^{-x^2/2} \right] \right\}'$$

$$a_{ii} = 2\epsilon_1 - k, \quad b_{ii} = -2k^2$$

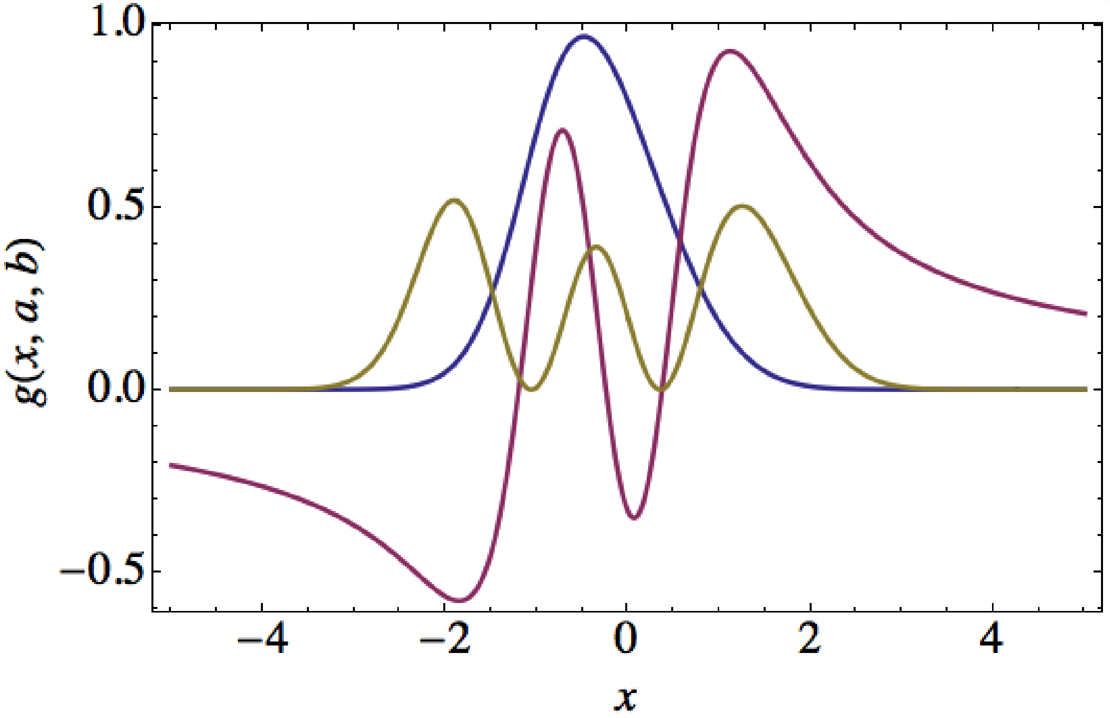
$$g_k(x) = -x - \left\{ \ln \left[B_k^+ a^+ u_1 \right] \right\}'$$

$$a_{iii} = -\epsilon_1 - k - \tfrac{3}{2}, \quad b_{iii} = -2(\epsilon_1 - k + \tfrac{1}{2})^2$$

Solutions of PIV through SUSY QM

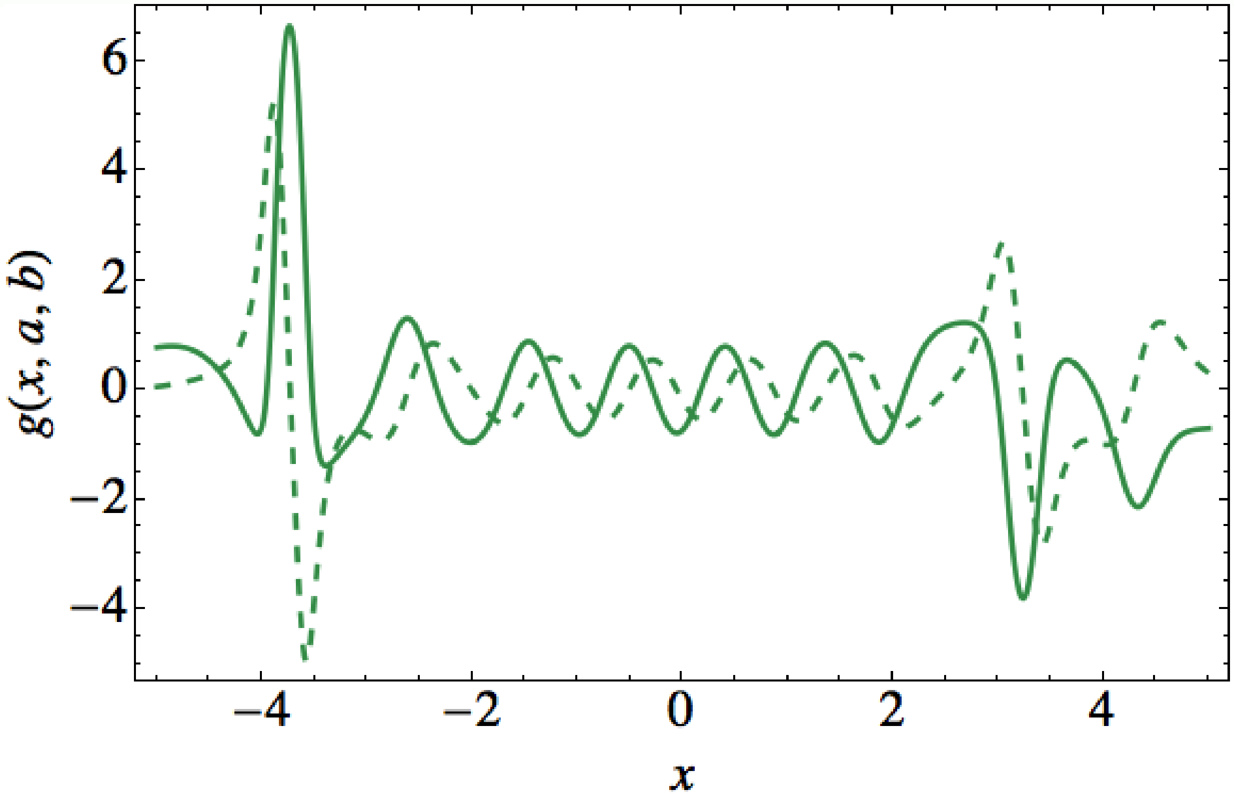


Solutions of PIV through SUSY QM



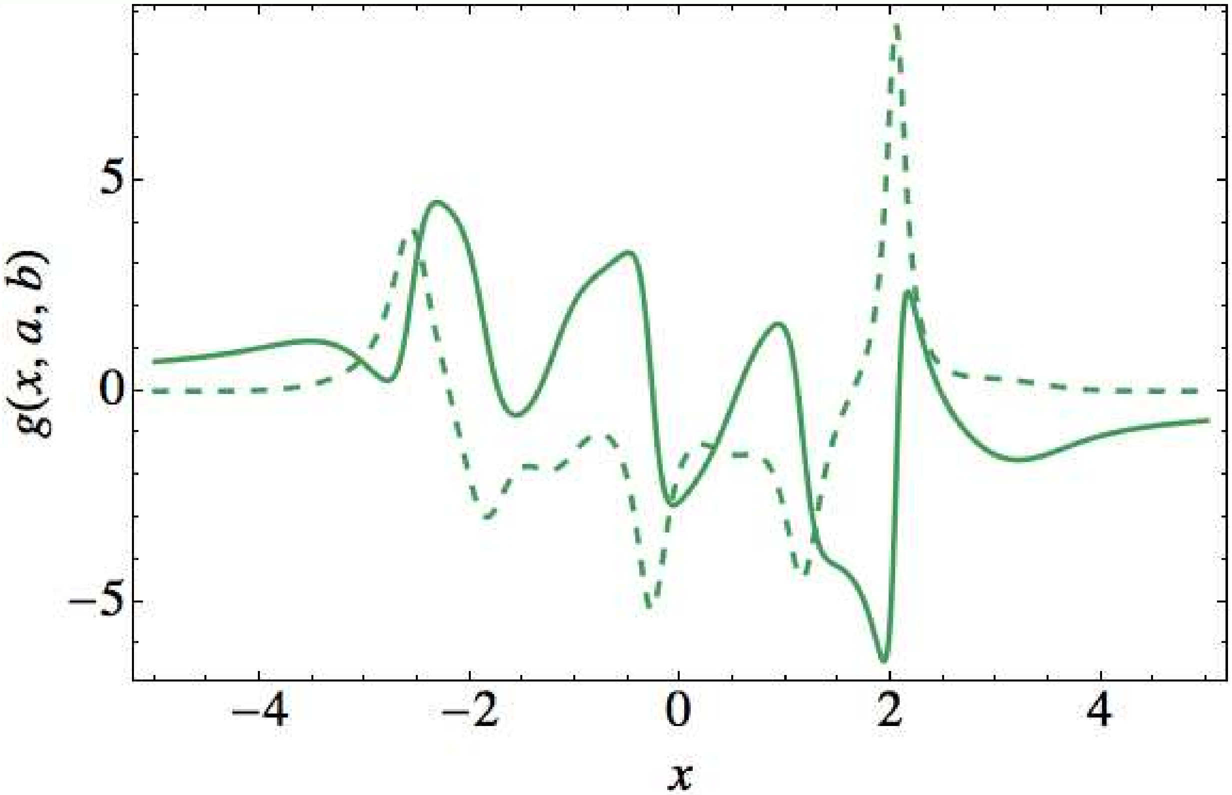
Real PIV solutions corresponding to $a_i = 1$, $b_i = 0$ ($k = 1$, $\epsilon_1 = -1/2$, $\nu = 0.7$) (blue); $a_i = 4$, $b_i = -2$ ($k = 2$, $\epsilon_1 = -3/2$, $\nu = 0.5$) (magenta); $a_i = 7$, $b_i = -8$ ($k = 3$, $\epsilon_1 = -1/2$, $\nu = 0.3$) (yellow)

Solutions of PIV through SUSY QM



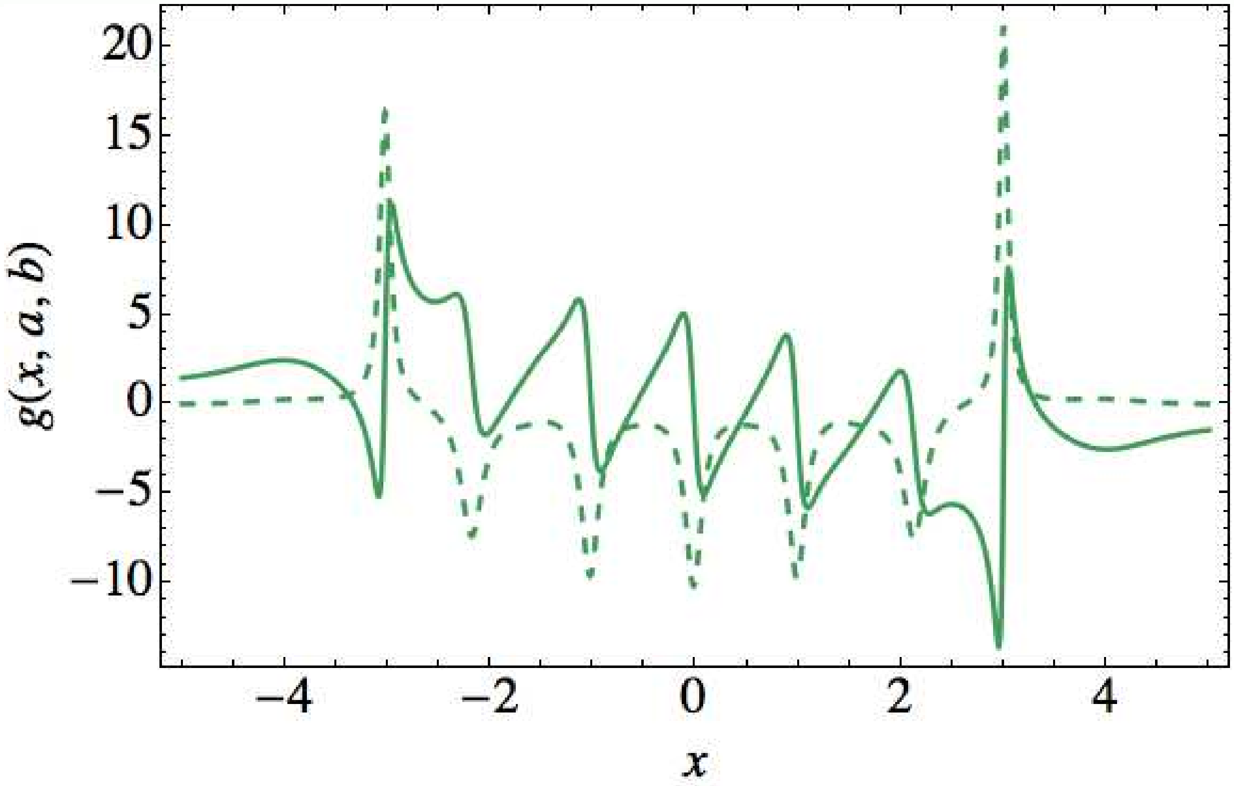
Real (solid line) and imaginary (dashed line) parts of some complex solutions to PIV for $a_{ii} = 12$, $b_{ii} = -8$ ($k = 2$, $\epsilon_1 = 7$, $\lambda = \kappa = 1$)

Solutions of PIV through SUSY QM



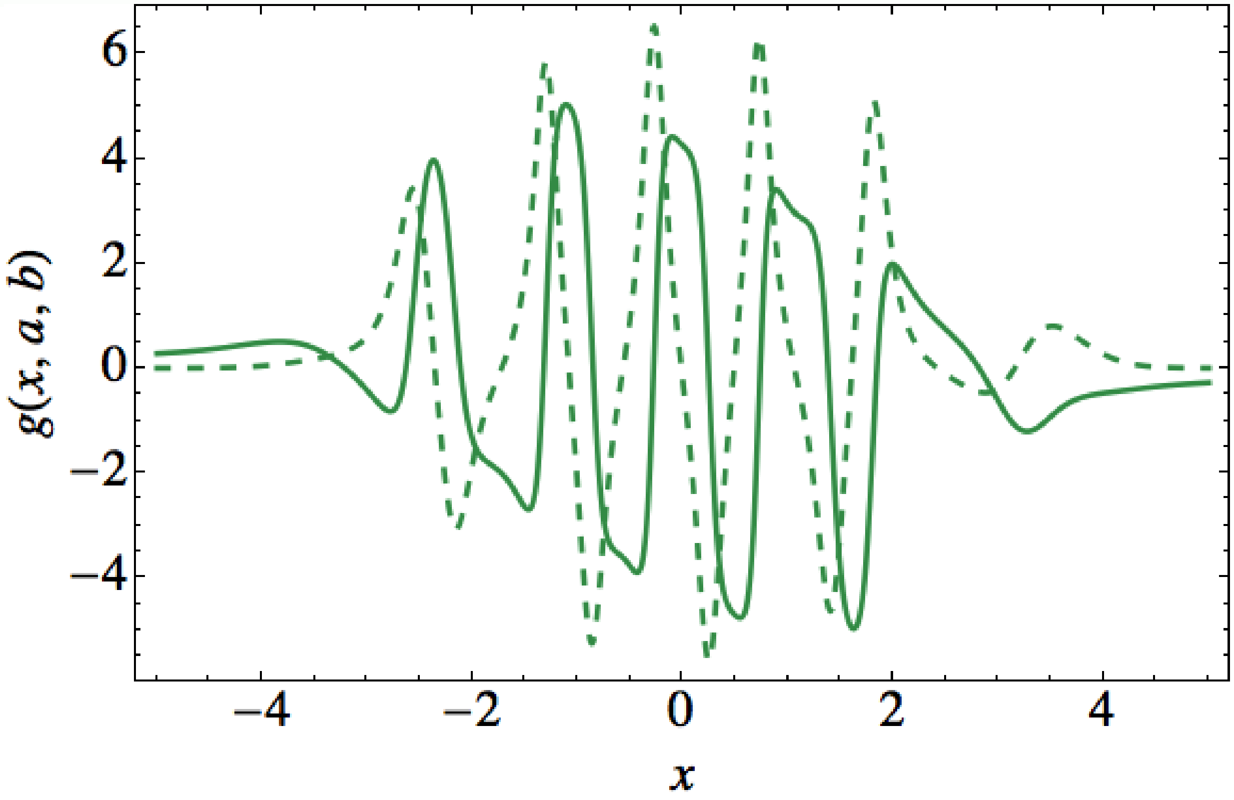
Real (solid line) and imaginary (dashed line) parts of some complex solutions to PIV for $a_{iii} = -5$, $b_{iii} = -8$ ($k = 1$, $\epsilon_1 = 5/2$, $\lambda = \kappa = 1$)

Solutions of PIV through SUSY QM



Real (solid curve) and imaginary (dashed curve) parts of some complex solutions to PIV for $a_{iii} = -5/2$, $b_{iii} = -121/2$ ($k = 2$, $\epsilon_1 = 5$, $\lambda = 1$, $\kappa = 5$)

Solutions of PIV through SUSY QM



Real (solid curve) and imaginary (dashed curve) parts of some complex solutions to PIV for $a_{iii} = 9$, $b_{iii} = -2$ ($k = 1$, $\epsilon_1 = 5$, $\lambda = \kappa = 1$)

Solutions of PIV through SUSY QM

- We have simply derived real non-singular solutions to the PIV equation with real parameters a, b
- This procedure has been as well successfully applied to complex non-singular solutions to the PIV equation with real parameters a, b
- The same technique can be applied to generate non-singular complex solutions to the PIV equation associated to complex parameters a, b
- A similar method can be followed for generating solutions to the PV equation

Conclusions

- SUSY QM is a powerful technique for generating exactly solvable potentials (with modified spectra compared with the initial one)
- System ruled by polynomial Heisenberg algebras are quite interesting
- A confluence between these two subjects led us to implement a simple technique for generating solutions to PIV and PV equations
- Simple methods for addressing complicated matters are available, we hope having contributed to this view of science (life)

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