Factorization method: Bogdan Mielnik contributions

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Factorization method

- The method consists in factorizing the Hamiltonian

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + V(x)$$

as a product of two first-order differential operators

- The aim: to solve the eigenvalue problem for H, i.e.,

$$H\psi(x) = E\psi(x)$$

with the appropriate boundary conditions

Origins of the method

- In 1935 Dirac factorizes the harmonic oscillator
- In 1940 Schrödinger factorizes the hydrogen atom
- Between 1941 and 1951 Infeld contributes strongly to the subject. In 1951 his classical review paper including a wide classification of potentials solvable through factorization is published. After this work the subject was believed to be exhausted

Factorization method rebirth

- In 1984 Mielnik discovers the generalized factorization

Factorization method rebirth

- In 1984 I applied Mielnik's factorization to the hidrogen atom, Andrianov et al related it with Darboux transformaton and MM Nieto with SUSY QM
- In 1985 Sukumar generalized the methods for arbitrary potentials and factorization energies
- In 1993 Andrianov et al introduced the higher-order generalizations of the method
- In 1995 Bogdan and me tried to implement the second-order generalization. In 1997 I found numerically that such generalization was possible
- In 1998 our widen collaboration (including LM Nieto and LW Glasser) succeeded in finding the way to implement analytically the second-order method. Mielnik's role was crucial for achieving this goal

Harmonic oscillator factorization

The Hamiltonian

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{x^2}{2}$$

fulfills the two factorizations

$$H = aa^+ - \frac{1}{2} = a^+a + \frac{1}{2}$$

$$a = \frac{1}{\sqrt{2}}\left(\frac{d}{dx} + x\right), \quad a^+ = \frac{1}{\sqrt{2}}\left(-\frac{d}{dx} + x\right)$$

thus the two intertwining relations

$$(H+1)a = aH$$

$$(H-1)a^+ = a^+H$$

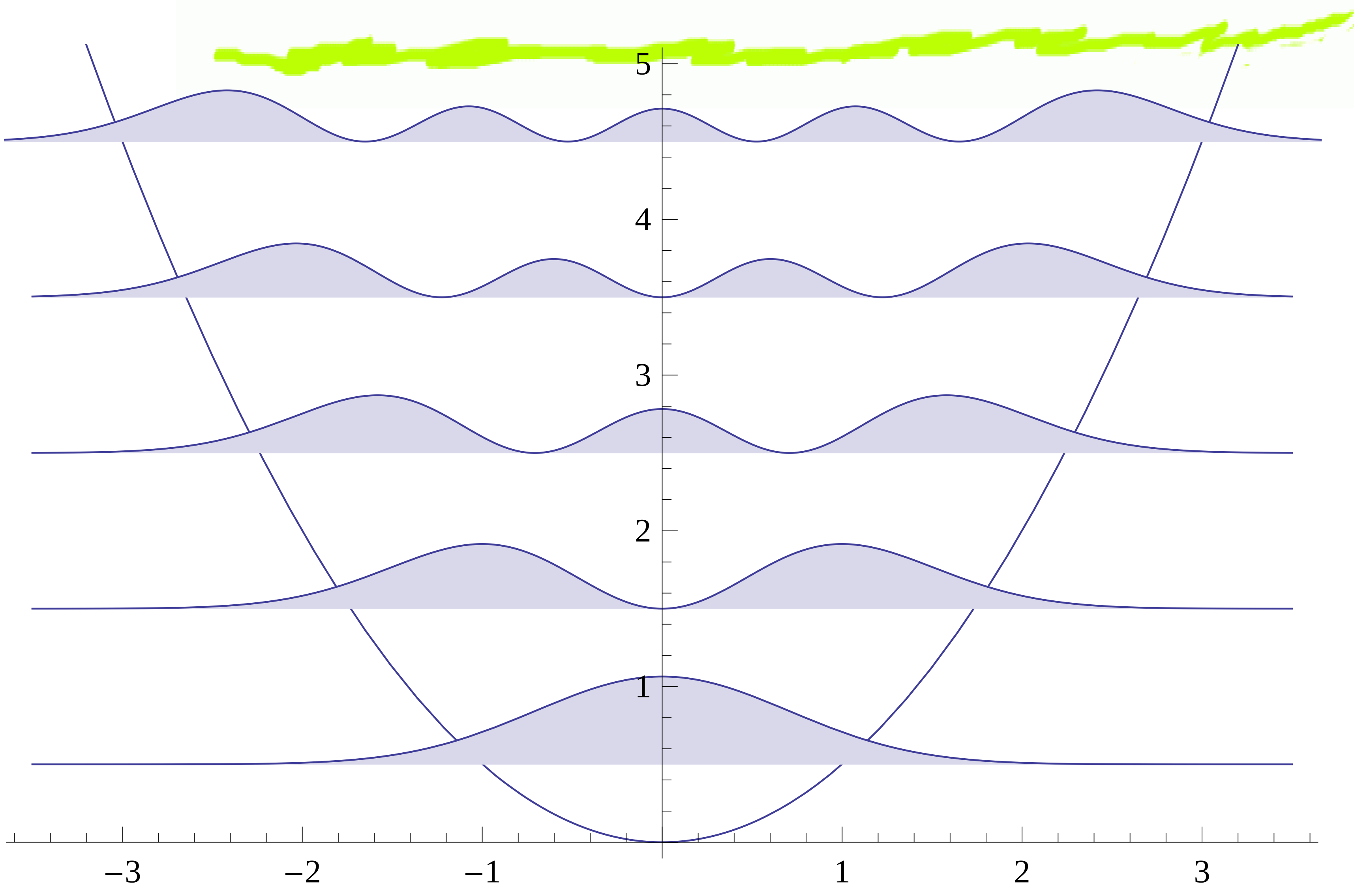
and commutation relationships (Heisenberg-Weyl algebra)

$$[H, a] = -a, \quad [H, a^+] = a^+, \quad [a, a^+] = 1$$

Harmonic oscillator factorization

- Due to aa^+ and a^+a are positive definite operators, then the eigenvalues of H fulfill $E_n \geq 1/2$
- The action of a (a^+) on an eigenfunction ψ_n of H associated to E_n produces a new eigenfunction of H with eigenvalue $E_n - 1$ ($E_n + 1$)
- The iterated action of a could lead to a contradiction $\Rightarrow$ there must exists an eigenfunction $\psi_0(x)$ which is annihilated by a. Thus $\psi_0(x) = (\pi)^{-1/4} e^{-x^2/2}$ and $E_0 = 1/2$
- The iterated action of a^+ on $\psi_0(x)$ produces the eigenfunctions $\psi_n(x) = (a^+)^n \psi_0(x)/\sqrt{n!}$ of H associated to $E_n = n + 1/2$

Harmonic oscillator factorization



Mielnik factorization

In 1984 Mielnik proposed that

$$H = bb^+ - \tfrac{1}{2}$$

$$b = \tfrac{1}{\sqrt{2}}\left[\tfrac{d}{dx} + \beta(x)\right], \quad b^+ = \tfrac{1}{\sqrt{2}}\left[-\tfrac{d}{dx} + \beta(x)\right]$$

thus $\beta(x)$ must fulfill $\beta' + \beta^2 = x^2 + 1$ whose general solution is

$$\beta = x + \frac{e^{-x^2}}{\lambda + \int_0^x e^{-y^2}dy} = x + \left[\log\left(\lambda + \int_0^x e^{-y^2}dy\right)\right]'$$

The key point is that

$$H_\lambda = b^+b + \tfrac{1}{2} = -\tfrac{1}{2}\tfrac{d^2}{dx^2} + V_\lambda(x)$$

$$V_\lambda(x) = \tfrac{x^2}{2} - \left[\frac{e^{-x^2}}{\lambda + \int_0^x e^{-y^2}dy}\right]'$$

Mielnik factorization

The following intertwining relations are relevant,

$$(H_\lambda - 1)b^+ = b^+ H$$
$$(H + 1)b = bH_\lambda$$

The eigenfunctions of H_λ are obtained from those of H,

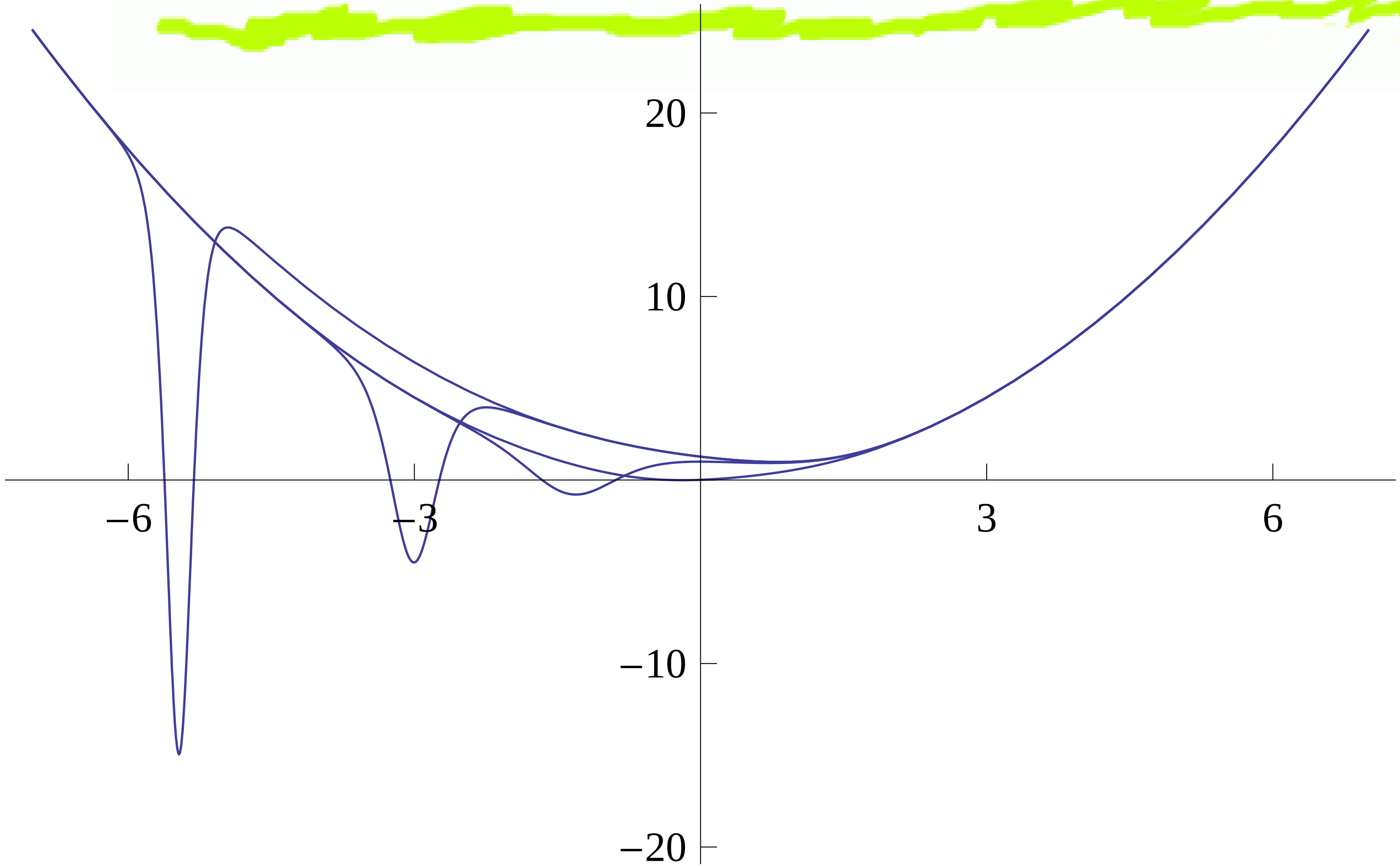
$$\theta_{n+1} = \frac{b^+\psi_n}{\sqrt{n+1}}, \quad n = 0, 1, \ldots$$

plus an eigenfunction associated to $E_0 = 1/2$,

$$\theta_0 \propto \exp\left[-\int_0^x \beta(y)dy\right] \propto \frac{e^{-\frac{x^2}{2}}}{\lambda + \int_0^x e^{-y^2} dy}$$

For $|\lambda| > \sqrt{\pi}/2$ the potential $V_\lambda(x)$ and $\theta_n(x),\ n = 0, 1, \ldots$ are non-singular, i.e., in this domain H_λ is a new Hamiltonian isospectral to the harmonic oscillator one

Mielnik factorization



$$\lambda \in \left\{ \frac{\sqrt{\pi}}{2} + 10^{-14}, 0.88625, 1, 10 \right\}$$

Second-order technique

In 1995 Bogdan and me started to explore the second-order intertwining technique

$$H_2 B_2^+ = B_2^+ H_0$$

$$H_i = -\frac{1}{2}\frac{d^2}{dx^2} + V_i(x), \quad i = 0, 2,$$

$$B_2^+ = \frac{1}{2}\left(\frac{d^2}{dx^2} - \eta(x)\frac{d}{dx} + \gamma(x)\right)$$

After some work we found that

$$V_2 = V_0 - \eta'$$

$$\gamma = \frac{\eta'}{2} + \frac{\eta^2}{2} - 2V_0 + d$$

$$\frac{\eta\eta''}{2} - \frac{\eta'^2}{4} + \frac{\eta^4}{4} + \eta^2\eta' - 2V_0\eta^2 + d\eta^2 + c = 0, \quad c, d \in \mathbb{R}$$

Given V_0, the potential V_2 and γ will be obtained if we can determine η

Second-order technique

In 1997, I was able to find non-singular solutions η for $V_0 = \frac{x^2}{2}$, $d = -2,\ c = 1$. A bit later, with Glasser and Luismi Nieto in 1998, we were able to find the corresponding analytic solutions. Almost immediately Luismi and me were able to find the analytic solution in the general case by means of the following ansatz:

$$\eta' = -\eta^2 + 2\beta\eta + 2\xi$$

where β y ξ must fulfill

$$\xi_{1,2} = \pm\sqrt{c}, \quad \epsilon_j = \frac{d+\xi_j}{2}, \quad \beta_j' + \beta_j^2 = 2[V_0 - \epsilon_j], \quad j = 1, 2$$

which implies that

$$\eta' = -\eta^2 + 2\beta_j\eta + 2\xi_j, \quad j = 1, 2$$

Second-order technique

Later, together with Bogdan and other members of the widened group (Fernández, Hussin, Mielnik 1998, Mielnik, Nieto, Rosas-Ortiz 2000) we derived rigorously the explicit expressions for the second and higher-order cases. However, in the second-order case, since $\xi_{1,2} = \pm\sqrt{c}$ a classification scheme based on the sign of c arises

(i) Real case with $c > 0$

In this case $\epsilon_1, \epsilon_2 \in \mathbb{R}$, $\epsilon_1 \neq \epsilon_2$. The ansatz leads to

$$\eta' = -\eta^2 + 2\beta_1\eta + 2(\epsilon_1 - \epsilon_2)$$
$$\eta' = -\eta^2 + 2\beta_2\eta + 2(\epsilon_2 - \epsilon_1)$$

By subtracting them it is obtained η in terms of $\beta_{1,2}$:

$$\eta = -\frac{2(\epsilon_1 - \epsilon_2)}{\beta_1 - \beta_2}$$

Second-order technique

(ii) Confluent case with $c = 0$

For $\xi = 0$ it turns out that $\epsilon \equiv \epsilon_1 = \epsilon_2 \in \mathbb{R}$. Given the Riccati solution β associated to ϵ, it must be solved the resulting Bernoulli equation for η

$$\eta' = -\eta^2 + 2\beta\eta$$

Thus

$$\eta = \frac{e^{2\int \beta(x)dx}}{w_0 + \int e^{2\int \beta(x)dx}dx}, \quad w_0 \in \mathbb{R}$$

An alternative expression in terms of the derivative with respect to the factorization energy appears:

$$\eta = -2\left(\frac{\partial \beta}{\partial \epsilon}\right)^{-1}$$

Second-order technique

(iii) Complex case with $c < 0$

Now $\epsilon_{1,2} \in \mathbb{C}$, $\epsilon_2 = \bar{\epsilon}_1$. As we want that V_2 would be real $\Rightarrow$ $\beta_2 = \bar{\beta}_1$. Hence the previous expression reduces to:

$$\eta = -\frac{2\mathrm{Im}(\epsilon_1)}{\mathrm{Im}[\beta_1]}$$

Along the years we were applying the technique to several potentials with a discrete part of the spectrum (harmonic oscillator, Coulomb potential, among others). Specially interesting were systems with purely continuous spectrum, as the free particle. Are there other potentials to which we could apply the technique?

Periodic potentials

In 2001 B Samsonov visited Cinvestav. We wanted to apply the factorization techniques to periodic systems, as the Lamé potentials $V(x) = n(n+1)\, m\, \mathrm{sn}^2(x,m),\ n \in \mathbb{N}$. The main results in the first-order case were:

- If we take as seed solution a Bloch function belonging to the forbidden energy band which is below the lowest band edge it turns out that the new potential are periodic, with the same period as the initial potentials
- However, if we take a nodeless linear combinations of the two Bloch functions we will create an isolated bound state at the factorization energy chosen, and the new potential will be just asymptotically periodic, with zones where the periodicity is broken. This can be interpreted as if lattice periodicity defects would be created in the material

Periodic potentials

In the second-order case the main results were:

- If we take as seed solutions two Bloch functions whose factorization energies belong to the same forbidden band it turns out that the new potentials are as well periodic, with the same period as the initial potentials and the same band structure
- However, if we take two appropriate linear combinations of the two Bloch functions with factorization energies belonging to the same forbidden band we can create two isolated bound states at the two factorization energies chosen. The new potential in this case will be just asymptotically periodic, with zones in which the periodicity is broken. Once again, this can be interpreted as if lattice periodicity defects would be created in the material

State of the art up to 2004

- In 2003 we invited Bogdan to write a review paper on the factorization method. It appeared as the opening article in the special issue of the “International Conference on Progress in Supersymmetric Quantum Mechanics” published in J. Phys. A: Math. Gen. 37, Number 43 (2004)
- What a wise decision: such an article is considered nowadays a classic on the factorization techniques due to the highest and widespread impact it has have in the scientific community, as well as the very special issue

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