Deformation quantization on the cotangent bundle of a Lie group

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- The description of a quantum system should use a similar mathematical language as used for classical systems.
- We want to receive a complete theory of quantum mechanics in the language of deformation quantization.
- The theory should be non-formal.
- Operator representation on a Hilbert space is received as an appropriate representation of the quantum system.

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Starting point: classical Hamiltonian system (T^*G, ω, H)

Configuration space G — Lie group

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We can add magnetic field by modifying the symplectic structure:

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where $F = \mathrm{d}A$ is an exact 2-form on G representing magnetic field (Faraday tensor) and A is a 1-form on G representing magnetic potential.

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Example

Rigid body in a magnetic field:

$$G = SO(3), \quad H(q,p) = \frac{1}{2} I^{ij} p_i p_j = \frac{p_1^2}{2I_1} + \frac{p_2^2}{2I_2} + \frac{p_3^2}{2I_3},$$

where I_1, I_2, I_3 are the principal moments of inertia and p_1, p_2, p_3 are fiber variables (angular momenta) corresponding to a basis $X_1, X_2, X_3 \in \mathfrak{g} = \mathfrak{so}(3)$: $p_j(q,p) = \langle p, X_j \rangle$.

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The Fourier transform of $f\colon T^*G \to \mathbb{C}$ in the momentum variable:

$$\tilde{f}(q, X) = \frac{1}{(2\pi)^n} \int_{\mathfrak{g}^*} f(q,p) e^{i\langle p, X\rangle}\, \mathrm{d}p \quad \text{for } q \in G,\ X \in \mathfrak{g}$$

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$\mathcal{F}_0(T^*G)$ is a pre-C^*-algebra with respect to the norm $\|f\|_0 = \sup_{x \in T^*G} |f(x)|$. Its completion is equal $C_0(T^*G)$.

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- all information about the Poisson manifold T^*G is encoded in $\mathcal{F}_0(T^*G)$,
- states can be defined as continuous positive-definite linear functionals on $\mathcal{F}_0(T^*G)$ normalized to unity:

$$\|\Lambda\| = 1,$$
$$\Lambda(\bar{f} \cdot f) \geq 0 \text{ for every } f \in \mathcal{F}_0(T^*G).$$

Then from Riesz representation theorem $\Lambda(f) = \int_{T^*G} f \, \mathrm{d}\mu$.

Quantization

For $\hbar \in \mathbb{R}$, $\hbar \neq 0$ let $\mathcal{F}_\hbar(T^*G) \subset L^2(T^*G, \mathrm{d}l)$ be an associative noncommutative algebra with a product denoted $\star_\hbar$, with a Lie bracket

$$[\![f, g]\!]_\hbar = \frac{1}{i\hbar}(f \star_\hbar g - g \star_\hbar f),$$

and with an involution being a complex-conjugation.

Scalar product and L^2-norm on $\mathcal{F}_\hbar(T^*G)$

$$(f, g) = \int_{T^*G} \overline{f(x)} g(x) \, \mathrm{d}l(x), \quad \|f\|_2 = \int_{T^*G} |f(x)|^2 \, \mathrm{d}l(x),$$

where $\mathrm{d}l = \dfrac{\mathrm{d}x}{|2\pi\hbar|^n}$.

Let us assume that the following property holds

$$\|f \star_\hbar g\|_2 \leq \|f\|_2 \|g\|_2, \quad f, g \in \mathcal{F}_\hbar(T^*G).$$

We will denote by $\mathcal{L}_\hbar(T^*G)$ the closure of $\mathcal{F}_\hbar(T^*G)$ in $L^2(T^*G, \mathrm{d}l)$. The space $\mathcal{L}_\hbar(T^*G)$ is a Hilbert subspace of $L^2(T^*G, \mathrm{d}l)$.

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C^*-norm on $\mathcal{F}_\hbar(T^*G)$

For $f \in \mathcal{F}_\hbar(T^*G)$ we define $\|f\|_\hbar$ as the operator norm of the bounded linear operator $h \mapsto f \star_\hbar h$ defined on a dense subspace of the Hilbert space $\mathcal{L}_\hbar(T^*G)$:

$$\|f\|_\hbar = \sup\{\|f \star_\hbar g\|_2 \mid g \in \mathcal{F}_\hbar(T^*G), \|g\|_2 = 1\}.$$

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Completion of $\mathcal{F}_\hbar(T^*G)$ with respect to the C^*-norm $\|\cdot\|_\hbar$ will be denoted $\mathcal{A}_\hbar(T^*G)$. The space $\mathcal{A}_\hbar(T^*G)$ is a C^*-algebra of observables.

By quantization of the algebra $\mathcal{F}_0(T^*G)$ we mean a family of pre-C^*-algebras $\mathcal{F}_\hbar(T^*G)$, where $\hbar \in \mathbb{R}$, $\hbar \neq 0$, satisfying the following condition:

For any $f, g \in \mathcal{F}_0(T^*G)$ there exists $\hbar_0 > 0$ such that for every $\hbar \in (-\hbar_0, \hbar_0)$ the functions $f, g \in \mathcal{F}_\hbar(T^*G)$ and

- the map $\hbar \mapsto \|f\|_\hbar$ is continuous on $(-\hbar_0, \hbar_0)$,
- $\|f \star_\hbar g - f \cdot g\|_\hbar \to 0$ as $\hbar \to 0$,
- $\|[\![f, g]\!]_\hbar - \{f, g\}\|_\hbar \to 0$ as $\hbar \to 0$.

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Examples: $(\mathbb{R}^n, +)$, $SO(3, \mathbb{R})$, $SU(2)$, $SL(2, \mathbb{C})$. However, $SL(2, \mathbb{R})$ is not weakly exponential.

The rotation group $SO(3)$:

Norm on $\mathfrak{so}(3)$

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$\mathcal{O} = \{X \in \mathfrak{so}(3) \mid \|X\| < \pi\}$
$\mathcal{U} = \exp(\mathcal{O}) = \{R \in SO(3) \mid \operatorname{Tr} R \neq -1\}$ consists of all rotations except ones with angle π.

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$$\mathcal{U} = \exp(\mathcal{O}) = SU(2) \setminus \{-I\}$$

Realization of the algebras $\mathcal{F}_\hbar(T^*G)$:

A space of square integrable functions f on T^*G whose momentum Fourier transforms $\tilde{f}$ have support in $G \times \hbar^{-1}\overline{\mathcal{O}}$ and for which the functions

$$\mathsf{f}(a,b) = |\hbar|^{-n} \tilde{f}\big(a \exp(\tfrac{1}{2} V_a(b)), \hbar^{-1} V_a(b)\big) e^{\frac{i}{\hbar}\Phi(a,b)} F(\hbar^{-1} V_a(b))^{-1},$$

defined on a dense subset $\{(a,b) \in G \times G \mid a^{-1}b \in \mathcal{U}\}$ of $G \times G$, extend to smooth compactly supported functions on $G \times G$.

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For $X \in \mathfrak{g}$

$$F(X) = \sqrt{|\det \phi(\hbar \operatorname{ad}_X)|},$$

where $\phi(x) = \dfrac{1 - e^{-x}}{x}$.

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For $q \in G$

$$V_q = (L_q \circ \exp|_{\mathcal{O}})^{-1}, \text{ so that } V_q(a) = \exp^{-1}(q^{-1}a) \text{ for } a \in L_q(\mathcal{U}).$$

For $a, b, c \in G$

$$\Phi(a,b,c) = \int_{\Delta(a,b,c)} F, \quad \Phi(a,b) = \Phi(e,a,b),$$

where $\Delta(a,b,c)$ is a surface enclosed by one-parameter subgroups connecting a with b, b with c, and c with a.

$$(f \star_\hbar g)(q,p) = \int_{\mathfrak{g}\times\mathfrak{g}} \tilde{f}\big(q\exp(-\tfrac{\hbar}{2}(X \diamond Y))\exp(\tfrac{\hbar}{2}X), X\big)$$
$$\times\, \tilde{g}\big(q\exp(\tfrac{\hbar}{2}(X \diamond Y))\exp(-\tfrac{\hbar}{2}Y), Y\big) e^{\frac{i}{\hbar}\Phi_q(X,Y)} e^{-i\langle p, X\diamond Y\rangle} L(X,Y)\,\mathrm{d}X\,\mathrm{d}Y$$

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$$\begin{aligned} X \diamond Y &= \hbar^{-1}\exp^{-1}\big(\exp(\hbar X)\exp(\hbar Y)\big) \\ &= X + Y + \frac{\hbar}{2}[X,Y] + \frac{\hbar^2}{12}([X,[X,Y]] + [Y,[Y,X]]) + \cdots \end{aligned}$$

for $X, Y \in \hbar^{-1}\mathcal{O}$ such that $\exp(\hbar X)\exp(\hbar Y) \in \mathcal{U}$.

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Extension of $\mathcal{F}_\hbar(T^*G)$ to an algebra of distributions:

Algebra of distributions $\mathcal{F}_\star(T^*G)$

Treating $\mathcal{F}_\hbar(T^*G)$ as the space of test functions we define the space of distributions $\mathcal{F}'_\hbar(T^*G)$ as the space of linear functionals on $\mathcal{F}_\hbar(T^*G)$. The space

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All smooth functions polynomial in fiber variables p_j

$$f(q,p) = \sum_{l=0}^{k} f^{i_1 i_2 \ldots i_l}(q) p_{i_1} p_{i_2} \cdots p_{i_l}$$

for $k \geq 0$ and $f^{i_1 i_2 \ldots i_l} \in C^\infty(G)$, are in $\mathcal{F}_\star(T^*G)$.

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2. $\int_{T^*G} \rho \, \mathrm{d}l = 1$,
3. $\int_{T^*G} \bar{f} \star_\hbar f \star_\hbar \rho \, \mathrm{d}l \geq 0$ for every $f \in \mathcal{F}_\hbar(T^*G)$.

Operator representation

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Let $\mathcal{H} = L^2(G, \mathrm{d}m)$. Then the representation $f \mapsto \hat{f}$ on $\mathcal{F}_\hbar(T^*G)$ is expressed by the formula

$$\hat{f}\psi(q) = \int_{\mathfrak{g}} \tilde{f}\left(q\exp(\tfrac{\hbar}{2}X), X\right)\psi\big(q\exp(\hbar X)\big)e^{\frac{i}{\hbar}\Phi(q,q\exp(\hbar X))}F(X)\,\mathrm{d}X.$$

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The integral kernel of $\hat{f}$:

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$$\begin{array}{lcll} \mathcal{L}_\hbar(T^*G) & \leftrightarrow & \mathcal{B}_2(\mathcal{H}) & \text{(space of Hilbert-Schmidt operators)} \\ \mathcal{A}_\hbar(T^*G) & \leftrightarrow & \mathcal{K}(\mathcal{H}) & \text{(space of compact operators)} \\ \mathcal{F}_\star(T^*G) & \leftrightarrow & \widehat{\mathcal{F}}_\star(T^*G) & \end{array}$$

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Take as $\mathcal{F}_\hbar(T^*SO(3))$ the subalgebra of $\mathcal{F}_\hbar(T^*SU(2))$ consisting of symmetric functions: $f(-a,p) = f(a,p)$. Such functions can be identified with functions on $T^*SO(3)$.

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The algebra $\mathcal{F}_\hbar(T^*SO(3))$ from the previous quantization is a subalgebra of the new algebra $\mathcal{F}_\hbar(T^*SO(3))$.

In the operator representation

$$\mathcal{H} = L^2(SU(2), \mathrm{d}m) = \mathcal{H}_+ \oplus \mathcal{H}_-,$$

where

$$\mathcal{H}_\pm = \{\psi \in \mathcal{H} \mid \psi(-x) = \pm\psi(x)\}.$$

The Hilbert space $\mathcal{H}_+$ is naturally isomorphic to $L^2(SO(3), \mathrm{d}m)$.

Hamilton operator of a rigid body in a magnetic field:

$$H(q,p) = \frac{1}{2} I^{ij} p_i p_j = \frac{p_1^2}{2I_1} + \frac{p_2^2}{2I_2} + \frac{p_3^2}{2I_3}$$

$$\downarrow$$

$$\begin{aligned}\hat{H} &= \frac{1}{4} I^{ij} \hat{p}_i \hat{p}_j + \frac{1}{2} \hat{p}_i I^{ij} \hat{p}_j + \frac{1}{4} \hat{p}_i \hat{p}_j I^{ij} - \frac{\hbar^2}{24} C_{il}^k C_{jk}^l I^{ij} \\ &= \frac{\hat{p}_1^2}{2I_1} + \frac{\hat{p}_2^2}{2I_2} + \frac{\hat{p}_3^2}{2I_3} + \frac{\hbar^2}{24} \left(\frac{1}{I_1} + \frac{1}{I_2} + \frac{1}{I_3} \right),\end{aligned}$$

where

$$\begin{aligned}\hat{p}_j &= i\hbar L_{X_j} - A_j, \\ [\hat{p}_i, \hat{p}_j] &= i\hbar (C_{ij}^k \hat{p}_k + F_{ij}).\end{aligned}$$

Magnetic field of a magnetic monopole of magnetic charge k:

$$F = \mathrm{d}A$$
$$A = -k(\langle \mathbf{q}, u_1\rangle \alpha^1 + \langle \mathbf{q}, u_2\rangle \alpha^2 + \langle \mathbf{q}, u_3\rangle \alpha^3),$$

where $\mathbf{q} \in \mathbb{R}^3$ is the center of charge describing the direction of the magnetic field, $u_1, u_2, u_3 \in \mathbb{R}^3$ are vectors corresponding to X_1, X_2, X_3 via the natural isomorphism of Lie algebras $\mathfrak{so}(3) \cong \mathbb{R}^3$ and $\alpha^1, \alpha^2, \alpha^3$ are left invariant 1-forms on $G = SO(3)$ such that $\alpha^i(L_{X_j}) = \delta^i_j$.

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Spectrum of $\hat{H}$:

$$\mathrm{Spec}(\hat{H}) = \{E_{j,\ell} \mid j \in \tfrac{1}{2}\mathbb{Z},\ j \geq 0,\ \ell = -j, -j+1, \dots, j-1, j\},$$

where for $I_1 = I_2 = I_3 = I$ (spherical top)

$$E_{j,\ell} = \frac{\hbar^2}{2I} j(j+1) - \hbar \frac{k\|\mathbf{q}\|}{I}\ell + \frac{k^2\|\mathbf{q}\|^2}{2I} + \frac{\hbar^2}{8I}$$

and for $I_1 \neq I_2 = I_3 = I$ (symmetric top)

$$E_{j,\ell} = \frac{\hbar^2}{2I} j(j+1) + \frac{\hbar^2}{2}\left(\frac{1}{I_1} - \frac{1}{I}\right)\ell^2 - \hbar \frac{k\|\mathbf{q}\|}{I_1}\ell + \frac{k^2\|\mathbf{q}\|^2}{2I_1} + \frac{\hbar^2}{24}\left(\frac{1}{I_1} + \frac{2}{I}\right).$$

Thank you :)