Generalization of the concept of classical r-matrix to Lie algebroids

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XXXIX Workshop on Geometric Methods in Physics
June 19-25, 2022, Białystok

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Lie Algebroid $(A, [\cdot,\cdot]_A, a_A)$

Definition

A Lie algebroid $(A, [\cdot,\cdot]_A, a_A)$ is a vector bundle $A \longrightarrow M$ over a manifold M, together with a vector bundle map $a_A : A \longrightarrow TM$, called the anchor map, and a Lie bracket $[\cdot,\cdot]_A : \Gamma(A) \times \Gamma(A) \longrightarrow \Gamma(A)$, such that the following Leibniz rule is satisfied

$$[\alpha, f\beta]_A = f[\alpha,\beta]_A + a_A(\alpha)(f)\beta,$$

for all $\alpha, \beta \in \Gamma(A)$, $f \in C^\infty(M)$.

The anchor map is a Lie algebra homomorphism

$$a\left([\alpha,\beta]_A\right) = [a(\alpha), a(\beta)].$$

Examples

$$[\alpha, f\beta]_A = f[\alpha, \beta]_A + a_A(\alpha)(f)\beta,$$

Example

Any tangent bundle $A = TM$ of a manifold M, with $a_A = id$ and the usual Lie bracket of vector fields, is a Lie algebroid.

Example

Any Lie algebra $A = \mathfrak{g}$, with trivial anchor $a_A = 0$, is a Lie algebroid.

Linear fiber-wise Poisson structure

If $(A, [\cdot,\cdot]_A, a)$ is a Lie algebroid then on the total space A^* of dual bundle $A^* \stackrel{q}{\to} M$ there exists a Poisson structure given by

$$\{f \circ q, g \circ q\} = 0,$$

$$\{l_X, g \circ q\} = a(X)(g) \circ q,$$

$$\{l_X, l_Y\} = l_{[X,Y]_A},$$

where $X, Y \in \Gamma^\infty(A),\;\; l_X(v) = \langle v, X(q(v))\rangle,\; v \in A^*$ and $f, g \in C^\infty(M)$.

Let $(M, \{.,.\})$ be a Poisson manifold, then its cotangent bundle $q^* : T^*M \to M$ possesses a Lie algebroid structure

$$\begin{array}{ccc} A = T^*M & \xrightarrow{a_{T^*M}} & TM \\ \downarrow{\scriptstyle q^*} & & \downarrow{\scriptstyle q} \\ M & \xrightarrow{id} & M \end{array}$$

given by

$$a_{T^*M}(df)(\cdot) = \{f, \cdot\},$$

$$[df, dg]_{T^*M} = d\{f, g\},$$

where $f, g \in C^\infty(M)$.

Lifting of a Poisson structure from M to TM

If $(M, \{,\}) = (M, \pi)$ is a Poisson manifold, then the manifold TM possesses a Poisson structure given by

$$\{f \circ q, g \circ q\}_{TM} = 0,$$

$$\{l_{df}, g \circ q\}_{TM} = \{f, g\} \circ q,$$

$$\{l_{df}, l_{dg}\}_{TM} = l_{d\{f,g\}},$$

where $l_{df}(v) = \langle v, df(q_M(v))\rangle,\ v \in TM$ and $f, g \in C^\infty(M)$. Let $\mathbf{x} = (x_1, \ldots, x_N)$ be a system of local coordinates on M. Then the Poisson tensor π^C on the manifold TM associated with π has the form

$$\pi^C(\mathbf{x}, \mathbf{y}) = \left(\begin{array}{c|c} 0 & \pi(\mathbf{x}) \\ \hline \pi(\mathbf{x}) & \sum_{s=1}^N \frac{\partial \pi}{\partial x_s}(\mathbf{x}) y_s \end{array}\right),$$

in the system of local coordinates $(\mathbf{x}, \mathbf{y}) = (x_1, \ldots, x_N, y_1 = l_{dx_1}, \ldots, y_N = l_{dx_N})$ on TM.

The complete lift, the vertical lift

$$\pi(\mathbf{x}) = \sum_{1\le i<j}^{N} \pi^{ij}(\mathbf{x}) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j},$$

$$\Downarrow$$

$$\pi^C(\mathbf{x},\mathbf{y}) = \sum_{1\le i<j}^{N} \Bigg(\pi^{ij}(\mathbf{x}) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial y^j} + \pi^{ij}(\mathbf{x}) \frac{\partial}{\partial y^i} \wedge \frac{\partial}{\partial x^j}$$

$$+ \sum_{s=1}^{N} \frac{\partial \pi^{ij}}{\partial x^s}(\mathbf{x}) y^s \frac{\partial}{\partial y^i} \wedge \frac{\partial}{\partial y^j} \Bigg) \Longrightarrow \left(\begin{array}{c|c} 0 & \pi(\mathbf{x}) \\ \hline \pi(\mathbf{x}) & \sum_{s=1}^{N} \dfrac{\partial \pi}{\partial x_s}(\mathbf{x}) y_s \end{array} \right)$$

$$\pi^V(\mathbf{x},\mathbf{y}) = \sum_{1\le i<j}^{N} \pi^{ij}(\mathbf{x}) \frac{\partial}{\partial y^i} \wedge \frac{\partial}{\partial y^j} \Longrightarrow \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \pi(\mathbf{x}) \end{array} \right)$$

Lifting of Casimir functions from M to TM

Theorem

Let $c_1, \ldots, c_r$, where $r = \dim M - \operatorname{rank} \pi$, be Casimir functions for the the Poisson structure π, then the functions

$$c_i \quad \textit{and} \quad l_{dc_i} = \sum_{s=1}^{N} \frac{\partial c_i}{\partial x_s} y_s, \quad i = 1, \ldots r,$$

are the Casimir functions for the Poisson tensor π^C.

Lifting of functions in involution from M to TM

Theorem

Let functions $\{H_i\}_{i=1}^{k}$ be in involution with respect to the Poisson bracket generated by π, then the functions

$$\{H_i, \quad l_{dH_i} = \sum_{s=1}^{N} \frac{\partial H_i}{\partial x_s}(\mathbf{x}) y_s \ \}_{i=1}^{k}$$

are in involution with respect to the Poisson tensor π^C.

Bi-Hamiltonian structures

Let M be a manifold with two non-proportional Poisson brackets $\{\cdot,\cdot\}_1$, $\{\cdot,\cdot\}_2$. If their linear combination $\alpha\{\cdot,\cdot\}_1 + \beta\{\cdot,\cdot\}_2$, $\alpha, \beta \in \mathbb{R}$, is also a Poisson bracket, we say that the brackets are compatible and we call M the bi-Hamiltonian manifold.
By analogy we will say that two Poisson tensors π_1 and π_2 are compatible if their Schouten–Nijenhuis bracket vanishes

$$[\pi_1, \pi_2]_{SN} = 0,$$

$$\frac{\partial \pi_1^{ij}}{\partial x^s}\pi_2^{sk} + \frac{\partial \pi_2^{ij}}{\partial x^s}\pi_1^{sk} + \frac{\partial \pi_1^{ki}}{\partial x^s}\pi_2^{sj} + \frac{\partial \pi_2^{ki}}{\partial x^s}\pi_1^{sj} + \frac{\partial \pi_1^{jk}}{\partial x^s}\pi_2^{si} + \frac{\partial \pi_2^{jk}}{\partial x^s}\pi_1^{si} = 0.$$

Theorem

If (M, π_1, π_2) is a bi-Hamilton manifold then (TM, π_1^C, π_2^C) is a bi-Hamilton manifold.

In the case of a linear Poisson structure, when $M = \mathfrak{g}^*$, we have additionally a Lie-Poisson structure on TM.

Theorem

Let π be the Lie-Poisson structure on $\mathfrak{g}^*$. Then the tensor

$$\tilde{\pi}_{T\mathfrak{g}^*}(\mathbf{x}, \mathbf{y}) = \left(\begin{array}{c|c} \lambda\pi(\mathbf{y}) & \pi(\mathbf{x}) \\ \hline \pi(\mathbf{x}) & \pi(\mathbf{y}) \end{array} \right)$$

gives the Poisson structure on $T\mathfrak{g}^*$ for any $\lambda \in \mathbb{R}$.

Theorem

Let $c_1, \ldots, c_r$, where $r = \dim M - \operatorname{rank} \pi$, be Casimir functions for the Poisson structure π with $\lambda \neq 0$, then the functions

$$c_i(\mathbf{t}) + c_i(\mathbf{w}) \quad c_i(\mathbf{t}) - c_i(\mathbf{w}), \quad i = 1, \ldots r,$$

where $\mathbf{t} = \left(x_1 - \sqrt{\lambda} y_1, \ldots, x_N - \sqrt{\lambda} y_N\right)$,
$\mathbf{w} = \left(x_1 + \sqrt{\lambda} y_1, \ldots, x_N + \sqrt{\lambda} y_N\right)$, *are the Casimir functions.*

Example: Bi-Hamiltonian structure related to $\mathfrak{so}(3)$

Let us consider the Lie algebra $\mathfrak{so}(3)$ of skew-symmetric matrices. We will now construct two Lie brackets on $\mathfrak{so}(3)$ given by two choices of the matrix S

$$[A, B] = AB - BA, \quad [A, B]_S = ASB - BSA,$$

where $S = \mathrm{diag}(s_1, s_2, s_3)$.
The Poisson tensors can be written in the form

$$\pi_1(X) = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}, \pi_2(X) = \begin{pmatrix} 0 & -s_3x_3 & s_2x_2 \\ s_3x_3 & 0 & -s_1x_1 \\ -s_2x_2 & s_1x_1 & 0 \end{pmatrix}.$$

In this case, the Casimirs for these structures assume the following form

$$c_1(X) = x_1^2 + x_2^2 + x_3^2, \quad c_2(X) = s_1 x_1^2 + s_2 x_2^2 + s_3 x_3^2.$$

Choosing as the Hamiltonian the Casimir c_2 we obtain Euler's equation, which describes the rotation of a rigid body

$$\frac{d\vec{x}}{dt} = \{c_2, \vec{x}\}_1 = \{c_1, \vec{x}\}_2 = 2\,(S\vec{x}) \times \vec{x},$$

where $\vec{x} = (x_1, x_2, x_2)$ and $S = \mathrm{diag}\,(s_1, s_2, s_3)$.

Example: Lifting of a Poisson structure from $\mathfrak{so}(3)$

The Poisson structures on $T\mathfrak{so}(3)$ are given by tensors

$$\pi^C(X,Y) = \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & -x_3 & x_2 \\ 0 & 0 & 0 & x_3 & 0 & -x_1 \\ 0 & 0 & 0 & -x_2 & x_1 & 0 \\ \hline 0 & -x_3 & x_2 & 0 & -y_3 & y_2 \\ x_3 & 0 & -x_1 & y_3 & 0 & -y_1 \\ -x_2 & x_1 & 0 & -y_2 & y_1 & 0 \end{array}\right).$$

Moreover the Casimirs are given by

$$c_1(X) = x_1^2 + x_2^2 + x_3^2, \qquad \frac{1}{2} l_{dc_1} = x_1 y_1 + x_2 y_2 + x_3 y_3.$$

In this case we recognize the Lie-Poisson structure of
$\mathfrak{e}(3) \cong T\mathfrak{so}(3)$.

We have another Poisson structure on $T\mathfrak{so}(3)$

$$\tilde{\pi}_{TM}(X,Y) = \left(\begin{array}{ccc|ccc} 0 & -\lambda y_3 & \lambda y_2 & 0 & -x_3 & x_2 \\ \lambda y_3 & 0 & -\lambda y_1 & x_3 & 0 & -x_1 \\ -\lambda y_2 & \lambda y_1 & 0 & -x_2 & x_1 & 0 \\ \hline 0 & -x_3 & x_2 & 0 & -y_3 & y_2 \\ x_3 & 0 & -x_1 & y_3 & 0 & -y_1 \\ -x_2 & x_1 & 0 & -y_2 & y_1 & 0 \end{array}\right).$$

In this case, we recognize the Lie-Poisson structure of $\mathfrak{so}(4) \cong T\mathfrak{so}(3)$. The Casimir functions now are given by the formulas

$$c_1(X+Y) + c_1(X-Y) = 2\left(x_1^2 + x_2^2 + x_3^2 + y_1^2 + y_2^2 + y_3^2\right),$$
$$c_1(X+Y) - c_1(X-Y) = 4\left(x_1y_1 + x_2y_2 + x_3y_3\right).$$

There is an additional Poisson tensor

$$\tilde{\pi}_{S,TM}(X,Y) = \left(\begin{array}{c|c} \lambda\pi_1(Y) & S\pi_1(X) \\ \hline \pi_1(X)S & \pi_1(SY) \end{array}\right) =$$

$$= \left(\begin{array}{ccc|ccc} 0 & -\lambda y_3 & \lambda y_2 & 0 & -s_1 x_3 & s_1 x_2 \\ \lambda y_3 & 0 & -\lambda y_1 & s_2 x_3 & 0 & -s_2 x_1 \\ -\lambda y_2 & \lambda y_1 & 0 & -s_3 x_2 & s_3 x_1 & 0 \\ \hline 0 & -s_2 x_3 & s_3 x_2 & 0 & -s_3 y_3 & s_2 y_2 \\ s_1 x_3 & 0 & -s_3 x_1 & s_3 y_3 & 0 & -s_1 y_1 \\ -s_1 x_2 & s_2 x_1 & 0 & -s_2 y_2 & s_1 y_1 & 0 \end{array}\right).$$

Furthermore, the Poisson structures π^C, $\tilde{\pi}_{TM}$ and π^C, $\tilde{\pi}_{S,TM}$ are pairwise compatible. If we take as the Hamiltonian c_1 for $\{\cdot,\cdot\}^C$ then we obtain the equations of the Clebsch system

$$\frac{d\vec{x}}{dt} = \{c_1, \vec{x}\}_{\tilde{\pi}_{S,TM}} = 2\lambda\vec{x}\times\vec{y},$$
$$\frac{d\vec{y}}{dt} = \{c_1, \vec{y}\}_{\tilde{\pi}_{S,TM}} = 2\,(S\vec{x})\times\vec{x}.$$

Classical R-matrix

One of the important tools of the integrable systems theory is the so-called classical R-matrix. Given a Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$, a linear operator $R : \mathfrak{g} \longrightarrow \mathfrak{g}$ is called a classical R-matrix if the R–bracket

$$[X, Y]_R = \frac{1}{2}\left([R(X), Y] + [X, R(Y)]\right)$$

is a Lie bracket. The Lie algebra $\mathfrak{g}$ equipped with two Lie brackets: $[\cdot, \cdot]$ and R-bracket $[\cdot, \cdot]_R$ is called a double Lie algebra. A certain class of R-matrices can be obtained from the modified Yang-Baxter equation

$$R([R(X), Y] + [X, R(Y)]) - [R(X), R(Y)] = c[X, Y].$$

Lie algebroids with a Poisson structure

Let $(A, [\cdot,\cdot]_A, a_A)$ be a Lie algebroid and assume that $\pi \in \Gamma\left(\bigwedge^2 A\right)$ satisfies $[\pi,\pi]_A = 0$. Then (A,π) is called a Lie algebroid with a Poisson structure.
Let us define

$$[\alpha,\beta]_\pi = \pounds_{\pi^\sharp\alpha}\beta - \pounds_{\pi^\sharp\beta}\alpha - d\left(\pi(\alpha,\beta)\right),$$

for $\alpha,\beta \in \Gamma\left(A^*\right)$, where $\pounds$ denotes the Lie derivation defined by

$$\pounds_X\alpha(Y) = a_A(X)\alpha(Y) - \alpha\left([X,Y]_A\right),$$

for $X, Y \in \Gamma(A)$ and $\pi^\sharp : A^* \longrightarrow A$ is defined by $\pi^\sharp\alpha(\cdot) = \pi(\alpha,\cdot)$, and set $a_{A^*} = a_A \circ \pi^\sharp$.
Then $(A^*, [\cdot,\cdot]_\pi, a_{A^*})$ is a Lie algebroid.

Substitution $\pi = X \wedge Y$

We rewrite

$$[\alpha,\beta]_\pi = \pounds_{\pi^\sharp\alpha}\beta - \pounds_{\pi^\sharp\beta}\alpha - d\left(\pi(\alpha,\beta)\right),$$

for π of the form $\pi = X \wedge Y$

$$\begin{aligned}[\alpha,\beta]_\pi =& \beta(Y)\pounds_X\alpha - \alpha(Y)\pounds_X\beta - (\beta(X)\pounds_Y\alpha - \alpha(X)\pounds_Y\beta)\\ =&[\alpha,\beta]_{X,Y} - [\alpha,\beta]_{Y,X}.\end{aligned}$$

General situation

$$[\alpha,\beta]_{X,Y} + \lambda\,[\alpha,\beta]_{Y,X}.$$

On some constructions of Lie algebroids on the cotangent bundle of a manifold

It is well known that if M is a manifold then TM is the tangent algebroid of M, with the identity map as the anchor map and the standard commutator of vector fields. However, we will use these fields and give the construction of another algebroid structures.

Theorem

Suppose that M is a manifold and $X, Y \in \Gamma(TM)$ are vector fields such that $[X, Y] = cY$, $c \in \mathbb{R}$. Then $(T^*M, [\cdot, \cdot]_{X,Y}, a_{X,Y})$ is a Lie algebroid, where the Lie bracket and the anchor map are given by

$$[\alpha, \beta]_{X,Y} = \beta(Y)\pounds_X\alpha - \alpha(Y)\pounds_X\beta,$$
$$a_{X,Y}(\alpha) = -\alpha(Y)X,$$

where $\alpha, \beta \in \Gamma(T^*M)$.

The Poisson structure on the tangent bundle TM

In local coordinates $(\mathbf{x},\mathbf{y})$ when $X=\sum_{i=1}^N v^i(\mathbf{x})\frac{\partial}{\partial x^i}$ and $Y=\sum_{i=1}^N w^i(\mathbf{x})\frac{\partial}{\partial x^i}$ the Poisson tensor is given by formula

$$\pi_{X,Y}=\left(\begin{array}{c|c} 0 & v(\mathbf{x})w^\top(\mathbf{x}) \\ \hline -w(\mathbf{x})v^\top(\mathbf{x}) & \sum_{s=1}^N \left(\dfrac{\partial v}{\partial x^s}(\mathbf{x})w^\top(\mathbf{x}) - w(\mathbf{x})\left(\dfrac{\partial v}{\partial x^s}(\mathbf{x})\right)^\top\right) y^s \end{array}\right),$$

where $\mathbf{v}^\top=(v^1,\ldots,v^N)$ and $\mathbf{w}^\top=(w^1,\ldots,w^N)$.

On some constructions of Lie algebroids on the cotangent bundle of a manifold

In addition, we will get a similar structure by swapping vector fields X, Y. Moreover, if we take a linear combination of these structures, we will again obtain a Poisson structure. The same thing also happens on the level of the Lie algebroid.

Theorem

Let $X, Y \in \Gamma(TM)$ be such that $[X, Y] = 0$, then a structure $\left(T^*M, [\cdot,\cdot]^{\lambda}_{X,Y}, a^{\lambda}_{X,Y}\right)$ is a Lie algebroid, where the Lie bracket and the anchor map are given by

$$\begin{aligned}
[\alpha,\beta]^{\lambda}_{X,Y} &= [\alpha,\beta]_{X,Y} + \lambda[\alpha,\beta]_{Y,X} \\
&= \beta(Y)\pounds_X\alpha - \alpha(Y)\pounds_X\beta + \lambda\left(\beta(X)\pounds_Y\alpha - \alpha(X)\pounds_Y\beta\right), \\
a^{\lambda}_{X,Y}(\alpha) &= a_{X,Y}(\alpha) + \lambda a_{Y,X}(\alpha) = -\alpha(Y)X - \lambda\alpha(X)Y
\end{aligned}$$

and λ is a real parameter.

Remark

In the case when $\lambda = -1$, the assumption of $[X, Y] = 0$ can be weakened. It is sufficient to assume that $[X, Y] = bX + cY$, where $b, c \in \mathbb{R}$.

The Poisson structure on the tangent bundle TM

This structure also leads to the Poisson bracket. In the local coordinates expression of the Poisson structure is the following tensor

$$\pi^{\lambda}_{X,Y}(\mathbf{x},\mathbf{y}) =$$

$$\left(\begin{array}{c|c} 0 & v(\mathbf{x})w^\top(\mathbf{x}) + \lambda w(\mathbf{x})v^\top(\mathbf{x}) \\ \hline -w(\mathbf{x})v^\top(\mathbf{x}) & \sum_{s=1}^{N}\left(\frac{\partial v}{\partial x^s}(\mathbf{x})w^\top(\mathbf{x}) - w(\mathbf{x})\left(\frac{\partial v}{\partial x^s}(\mathbf{x})\right)^\top\right. \\ -\lambda v(\mathbf{x})w^\top(\mathbf{x}) & \left. +\ \lambda\ \left(\frac{\partial w}{\partial x^s}(\mathbf{x})v^\top(\mathbf{x}) - v(\mathbf{x})\left(\frac{\partial w}{\partial x^s}(\mathbf{x})\right)^\top\right)\right) y^s \end{array}\right).$$

In this construction, the block $vw^\top + \lambda wv^\top$ is symmetric for $\lambda = 1$ in contrast to the construction of the Poisson bracket from the algebroid bracket of differential forms. Moreover, this block is antisymmetric for $\lambda = -1$ and it is also a Poisson tensor on manifolds M. In this case it is a complete lift of $\pi = X \wedge Y$.

Example

Let us consider again the Lie algebra $\mathfrak{so}(3)$ of skew-symmetric matrices. Thus on $\mathfrak{so}(3)$ we have the linear Poisson structure

$$\pi(X) = -x^3 \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} + x^2 \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^3} - x^1 \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^3}.$$

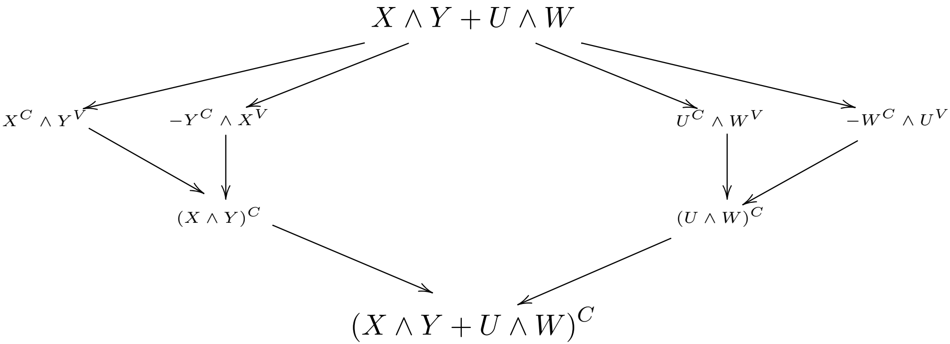
Observe that defining the vector fields

$$X = x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2}, \quad Y = \frac{\partial}{\partial x^3}, \quad U = -x^3 \frac{\partial}{\partial x^1}, \quad W = \frac{\partial}{\partial x^2}.$$

we can split the above Poisson tensor into two terms $\pi(X) = X \wedge Y + U \wedge W$.

Example

We obtain the following splitting



The particular case of above construction $A = \mathfrak{g}$

Because a Lie algebra $\mathfrak{g}$ can be thought of as a Lie algebroid over a point, so we have the opportunity to construct a Lie bracket on the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$.

Corollary

If $(\mathfrak{g}, [\cdot,\cdot])$ *is a Lie algebra and* $X, Y \in \mathfrak{g}$ *such that* $[X, Y] = cY$ *(or* $[X, Y] = 0$*) are fixed, then* $(\mathfrak{g}^*, [\cdot,\cdot]_{X,Y})$ *is a Lie algebra, where*

$$[\alpha, \beta]_{X,Y} = \alpha(Y) ad_X^* \beta - \beta(Y) ad_X^* \alpha,$$

(or $\left(\mathfrak{g}_*, [\cdot,\cdot]_{X,Y}^{\lambda}\right)$ *is a Lie algebra, where the commutator is constructed as follows*

$$[\alpha, \beta]_{X,Y}^{\lambda} = \alpha(Y) ad_X^* \beta - \beta(Y) ad_X^* \alpha + \lambda \left(\alpha(X) ad_Y^* \beta - \beta(X) ad_Y^* \alpha\right),)$$

for $\alpha, \beta \in \mathfrak{g}^*$

Note that when $\lambda = -1$ the bracket can be rewritten as

$$[\alpha, \beta]^{-1}_{X,Y} = -\alpha(X) ad^*_Y \beta + \alpha(Y)\rangle ad^*_X \beta$$

$$+\beta(X) ad^*_Y \alpha - \beta(Y) ad^*_X \alpha = [\alpha, \beta]_r.$$

It is a formula for the r-bracket or classical r-matrix. If $r = Y \wedge X$ the assumptions of corollary can be weakened. In this case we obtain a Lie bracket if r satisfies the Yang-Baxter equation or some of its modifications (modified Yang-Baxter equation). It means that $r^\sharp : \mathfrak{g}^* \longrightarrow \mathfrak{g}$ given by $r^\sharp(\alpha)(\beta) = r(\alpha, \beta)$ satisfies the condition

$$\langle \alpha | [r^\sharp(\beta), r^\sharp(ad^*_Z \gamma)] \rangle + \langle \beta | [r^\sharp(ad^*_Z \gamma), r^\sharp(\alpha)] \rangle + \langle ad^*_Z \gamma | [r^\sharp(\alpha), r^\sharp(\beta)] \rangle = 0$$

for all $\alpha, \beta, \gamma \in \mathfrak{g}^*$ and $Z \in \mathfrak{g}$. Then we can think about the formula as a generalization of the notion of classical r-matrices by introducing a parameter $\lambda \in \mathbb{R}$.

Generalization of the concept of classical r-matrix

Ultimately this concept can be extended to the level of arbitrary $r \in \mathfrak{g} \otimes \mathfrak{g}$. If we define mappings $\underline{r}, \overline{r} : \mathfrak{g}^* \longrightarrow \mathfrak{g}$ such that $\underline{r}(\alpha) = r(\alpha, \cdot)$, $\overline{r}(\alpha) = r(\cdot, \alpha)$ then we obtain the following generalization:

Theorem

Assume that the map r satisfies the condition

$$\langle \alpha | [\overline{r}(\gamma), \underline{r}(\beta)] \rangle + \langle \beta | [\underline{r}(\alpha), \overline{r}(\gamma)] \rangle$$

$$+ \langle \gamma | [\underline{r}(\alpha), \underline{r}(\beta)] \rangle = 0,$$

for all $\alpha, \beta, \gamma \in \mathfrak{g}^*$. Then

$$[\alpha, \beta]_{\underline{r}} = ad^*_{\underline{r}(\alpha)} \beta - ad^*_{\underline{r}(\beta)} \alpha$$

is a Lie bracket on $\mathfrak{g}^*$.

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Thank you for your attention

Poisson manifold $(M, \{\cdot,\cdot\})$

Definition

A Poisson manifold $(M, \{\cdot,\cdot\})$ is a smooth manifold M (equipped with a Poisson structure) with a fixed bilinear and antisymmetric mapping $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$, which satisfies Jacobi identity and Leibniz rule

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0,$$

$$\{f,gh\} = \{f,g\}h + g\{f,h\},$$

where $f,g,h \in C^\infty(M)$.

Poisson bracket can be written in terms of **Poisson tensor** ($\pi \in \Gamma^\infty\left(\bigwedge^2 TM\right)$ such that $[\pi,\pi]_{SN} = 0$) as follows

$$\{f,g\} = \pi(df,dg).$$

Poisson tensor, Hamilton's equations

In the local coordinates $x_1, x_2, \ldots, x_N$ on M

$$\{f, g\} = \sum_{i,j=1}^{N} \pi_{ij}(x) \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}.$$

Components of Poisson tensor are given by the formula

$$\pi_{ij}(x) = \{x_i, x_j\}$$

and satisfy

- $\pi_{ij} = -\pi_{ji}$,
- $\frac{\partial \pi_{ij}}{\partial x_s}\pi_{sk} + \frac{\partial \pi_{ki}}{\partial x_s}\pi_{sj} + \frac{\partial \pi_{jk}}{\partial x_s}\pi_{si} = 0.$

Choosing the function H as a Hamiltonian we can define a dynamics on M using Hamilton's equations

$$\frac{dx_i}{dt} = \{x_i, H\}, \qquad i = 1, 2, \ldots, N,$$

$$\frac{dx}{dt} = \pi \nabla H,$$

Lifting of a bi-Hamiltonian structure from M to TM

Corollary

Let (M, π_1, π_2) be a bi-Hamiltonian manifold and let $\mathbf{x} = (x_1, \ldots, x_N)$ be a system of local coordinates on M. Then the Poisson tensor $\pi_{TM,\lambda}$ related to (M, π_1, π_2) takes form

$$\pi_{TM,\lambda}(\mathbf{x}, \mathbf{y}) = \left(\begin{array}{c|c} 0 & \pi_1(\mathbf{x}) \\ \hline \pi_1(\mathbf{x}) & \sum_{s=1}^{N} \dfrac{\partial \pi_1}{\partial x_s}(\mathbf{x}) y_s + \lambda \pi_2(\mathbf{x}) \end{array} \right),$$

in the system of local coordinates $(\mathbf{x}, \mathbf{y}) = (x_1, \ldots, x_N, y_1, \ldots, y_N)$ on TM.

Lifting of Casimir functions from M to TM

Theorem

Let $c_1, \ldots, c_r$, where $r = \dim M - \operatorname{rank} \pi$, be Casimir functions for the Poisson structure π_1 and functions f_i, $i = 1, \ldots, r$, satisfy the conditions $\{f_i, x_j\}_1 = \{x_j, c_i\}_2, \quad for \quad j = 1, \cdots, n$, then the functions

$$c_i \circ q \quad \textit{and} \quad \tilde{c}_i = \sum_{s=1}^{N} \frac{\partial c_i}{\partial x_s}(\mathbf{x}) y_s + \lambda f_i(\mathbf{x}), \quad i = 1, \ldots r,$$

are the Casimir functions for the Poisson tensor $\pi_{TM,\lambda}$.

Lifting of functions in involution from M to TM

Theorem

Let functions $\{H_i\}_{i=1}^{k}$ *be in involution with respect to the Poisson brackets given by* π_1 *and* π_2 *and let functions* g_i*,* $i = 1, \dots, k$*, satisfy the conditions* $\{H_i(x), g_j(x)\}_1 = \{H_j(x), g_i(x)\}_1$*, for* $i, j = 1, \dots, k$*. Then the functions*

$$H_i \circ q_M^* \quad and \quad \tilde{H}_i = \sum_{s=1}^{N} \frac{\partial H_i}{\partial x_s}(\mathbf{x}) y_s + \lambda g_i(\mathbf{x}), \quad i = 1, \dots r,$$

are in involution with respect to the Poisson tensor $\pi_{TM,\lambda}$*.*

Corollary

If the functions $\{H_i\}$ *are in involution with respect to the Poisson tensor* π *then the functions* $\{H_i \circ q, \ \tilde{H}_i = \sum_{s=1}^{N} \frac{\partial H_i}{\partial x_s}(\mathbf{x}) y_s\}$ *are in involution with respect to the Poisson tensor* $\pi_{TM,\lambda}$*.*

Toda lattice — bi-Hamiltonian system

The Hamiltonian

$$H = \sum_{i \in \mathbb{Z}} \left(\frac{1}{2} p_i^2 + e^{q_{i-1} - q_i} \right).$$

Hamilton's equations

$$\begin{cases} \dot{q}_i = \{q_i, H\} = p_i \\ \dot{p}_i = \{p_i, H\} = e^{q_{i-1} - q_i} - e^{q_i - q_{i+1}} \end{cases}.$$

Under Flaschka's transformation

$$a_i = \frac{1}{2} e^{\frac{(q_{i-1} - q_i)}{2}}, \quad b_i = -\frac{1}{2} p_{i-1}$$

the system transforms to

$$\frac{da_i}{dt} = a_i \left(b_{i+1} - b_i \right),$$
$$\frac{db_i}{dt} = 2 \left(a_i^2 - a_{i-1}^2 \right).$$

The Toda lattice is equivalent to the Lax equation

$$\frac{dL}{dt} = [A, L],$$

where

$$Lf_i = a_i f_{i+1} + b_i f_i + a_{i-1} f_{i-1},$$

$$Af_i = a_i f_{i+1} - a_{i-1} f_{i-1}$$

are linear operators in the Hilbert space of square summable sequences $l^2(\mathbb{Z})$.

The Toda lattice is a bi-Hamiltonian system. There exist another Poisson bracket, which we denote by π_2, and another function H_1, which will play the role of the Hamiltonian for the π_2 bracket, such that $\pi_1 + \pi_2$ is Poisson tensor and $\pi_1 \nabla H = \pi_2 \nabla H_1$ $(H = \sum_i \left(2b_i^2 + 4a_i^2\right))$. The Poisson tensor π_1 is given by the relations

$$8\{a_i, b_i\}_1 = -a_i, \quad 8\{a_i, b_{i+1}\}_1 = a_i.$$

For the Toda lattice the π_2 bracket (which appeared in a paper of M. Adler) is quadratic in the variables b_i,a_i and it is given by the relations

$$\{a_i, a_{i+1}\}_2 = \frac{1}{2} a_i a_{i+1}, \quad \{a_i, b_i\}_2 = -a_i b_i,$$

$$\{a_i, b_{i+1}\}_2 = a_i b_{i+1}, \quad \{b_i, b_{i+1}\}_2 = 2a_i^2$$

and all other brackets are zero.

Example: Extended Toda Lattice

Functions $H_k = TrL^k$ are the functions in involutions with respect to the both brackets. The above functions for $k = 1, 2, 3$ have the expressions

$$H_1 = trL = \sum_{i\in\mathbb{Z}} b_i,\ H_2 = 2H = trL^2 = \sum_{i\in\mathbb{Z}} \left(b_i^2 + 2a_i^2\right),$$

$$H_3 = trL^3 = \sum_{i\in\mathbb{Z}} \left(b_i^3 + 3a_i^2 b_i + 3a_i^2 b_{i+1}\right), \ldots$$

Now deformed tangent Poisson structure $\pi_{TM,\lambda}$ in local coordinates a_i, b_i, n_i, m_i, $i \in \mathbb{Z}$, is given by the relation

$$\{a_i, m_i\}_{TM,\lambda} = -\frac{1}{4}a_i, \qquad \{a_i, m_{i+1}\}_{TM,\lambda} = \frac{1}{4}a_i,$$

$$\{b_i, n_i\}_{TM,\lambda} = \frac{1}{4}a_i, \qquad \{b_{i+1}, n_i\}_{TM,\lambda} = -\frac{1}{4}a_i,$$

$$\{n_i, n_{i+1}\}_{TM,\lambda} = \frac{\lambda}{2}a_i a_{i+1}, \qquad \{n_i, m_i\}_{TM,\lambda} = -\frac{1}{4}n_i - \lambda a_i b_i,$$

$$\{n_i, m_{i+1}\}_{TM,\lambda} = \frac{1}{4}n_i + \lambda a_i b_{i+1}, \qquad \{m_i, m_{i+1}\}_{TM,\lambda} = 2\lambda a_i^2.$$

From the last theorem we transform the functions $H_k = TrL^k$ into the functions $H_k \circ q_M^* = TrL^k \circ q_M^*$ and $\tilde{H}_k = \sum_{s\in\mathbb{Z}} \left(\frac{\partial H_k}{\partial a_s} n_s + \frac{\partial H_k}{\partial b_s} m_s \right)$, i.e.

$$H_1 = \sum_{i\in\mathbb{Z}} b_i, \qquad \tilde{H}_1 = \sum_{i\in\mathbb{Z}} m_i,$$

$$H_2 = \sum_{i\in\mathbb{Z}} \left(b_i^2 + 2a_i^2\right), \qquad \tilde{H}_2 = \sum_{i\in\mathbb{Z}} \left(2b_i m_i + 4a_i n_i\right),$$

$$H_3 = \sum_{i\in\mathbb{Z}} \left(b_i^3 + 3a_i^2 b_i + 3a_i^2 b_{i+1}\right), \qquad \tilde{H}_3 = \sum_{i\in\mathbb{Z}} \Big(3b_i^2 m_i + 3a_i^2 m_i + 3a_i^2 m_{i+}$$

$$+6a_i b_i n_i + 6a_i b_{i+1} n_i\Big),$$

$\dots \qquad \dots$

Now if we take as the Hamiltonian

$$H = \alpha H_2 + \beta \tilde{H}_2 = \sum_{i \in \mathbb{Z}} \left(\alpha b_i^2 + 2\alpha a_i^2 + 2\beta b_i m_i + 4\beta a_i n_i \right)$$

then Hamilton's equations are in the form

$$\begin{aligned}
\frac{da_i}{dt} &= \frac{1}{2}\beta a_i \left(b_{i+1} - b_i\right), \\
\frac{db_i}{dt} &= \beta \left(a_i^2 - a_{i-1}^2\right), \\
\frac{dn_i}{dt} &= \frac{1}{2}\alpha a_i \left(b_{i+1} - b_i\right) + \frac{1}{2}\beta a_i \left(m_{i+1} - m_i\right) + \frac{1}{2}\beta n_i \left(b_{i+1} - b_i\right) + \\
&\quad + 2\beta\lambda a_i \left(a_{i+1}^2 - a_{i-1}^2 - b_i^2 + b_{i+1}^2\right), \\
\frac{dm_i}{dt} &= \alpha \left(a_i^2 - a_{i-1}^2\right) + 2\beta \left(a_i n_i - a_{i-1} n_{i-1}\right) + \\
&\quad + 4\beta\lambda \left(a_i^2 b_{i+1} + a_i^2 b_i - a_{i-1}^2 b_i - a_{i-1}^2 b_{i-1}\right).
\end{aligned}$$

We can interpret this integrable system as an extension of the Toda lattice. If we put $\alpha = \lambda = 0, \beta = 2$ and we take $n_i = m_i = 0$ then we observe that we reduce it to Toda lattice.

Example

On $\mathbb{R}^3$ with coordinates (x_1, x_2, x_3) we consider the linear Poisson structure given by following Poisson tensor

$$\pi_1(x_1, x_2, x_3) = x_1 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3}$$

associated to Lie algebra $\mathcal{A}_{3,1}$. The second Poisson tensor is related to Euclidean Lie algebra $\mathcal{A}_{3,6} = \mathfrak{e}(2)$
$([e_1, e_3] = -e_2, \quad [e_2, e_3] = e_1)$ and defined by

$$\pi_2(x_1, x_2, x_3) = -x_2 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_3} + x_1 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3}.$$

In this case, the Casimir c_1 for π_1 and Casimir d_1 for π_2 assume the following form

$$c_1(x_1, x_2, x_3) = x_1, \quad d_1(x_1, x_2, x_3) = x_1^2 + x_2^2.$$

We get linear Poisson structure on $T\mathbb{R}^3 \cong \mathbb{R}^6$ given by

$$\pi_{TM,\lambda}(\mathbf{x},\mathbf{y}) = \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_1 \\ 0 & 0 & 0 & 0 & -x_1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -\lambda x_2 \\ 0 & 0 & x_1 & 0 & 0 & y_1 + \lambda x_1 \\ 0 & -x_1 & 0 & \lambda x_2 & -(y_1 + \lambda x_1) & 0 \end{array}\right),$$

where $(x_1, x_2, x_3, y_1, y_2, y_3)$ are coordinates on $\mathbb{R}^6$.
For $\lambda \neq 0$, we recognize the Lie-Poisson structure related to the Lie algebra $\mathcal{A}_{6,17}$. The Casimir functions are given by the formulas

$$c_1(x_1, x_2, x_3, y_1, y_2, y_3) = x_1, \quad \tilde{c}_1(x_1, x_2, x_3, y_1, y_2, y_3) = y_1 + \lambda \frac{x_2^2}{2x_1}.$$

For $\lambda = 0$, we obtain the Lie-Poisson structure of $\mathcal{A}_{6,4}$. Moreover the Casimirs are given by

$$c_1(x_1, x_2, x_3, y_1, y_2, y_3) = x_1, \quad \tilde{c}_1(x_1, x_2, x_3, y_1, y_2, y_3) = y_1.$$

The case of a linear Poisson structure

In the case of a linear Poisson structure, when $M = \mathfrak{g}^*$ is the dual to Lie algebra $\mathfrak{g}$, we have additionally a Lie-Poisson structure on TM.

Theorem

Let $(\mathfrak{g}^*, \pi_1, \pi_2)$ be a bi-Hamiltonian manifold. If at least one of the following conditions is satisfied

1. $\lambda = 0$;
2. $\mu = \epsilon = 0$;
3. $\mu = 0$ *and* $\kappa = 1$

then we can construct the following Poisson structure on $T\mathfrak{g}^*$:

$$\tilde{\pi}_{T\mathfrak{g}^*,\lambda}(\mathbf{x}, \mathbf{y}) = \left(\begin{array}{c|c} \epsilon\pi_1(\mathbf{x}) + \mu\pi_1(\mathbf{y}) & \kappa\pi_1(\mathbf{x}) \\ \hline \kappa\pi_1(\mathbf{x}) & \kappa\pi_1(\mathbf{y}) + \lambda\pi_2(\mathbf{x}) - \lambda\epsilon\pi_2(\mathbf{y}) \end{array} \right).$$

We write the elements of the Lie algebra $T\mathfrak{g} = \mathfrak{g} \times \mathfrak{g}$ that generate the Lie-Poisson structures on $T\mathfrak{g}^*$, as pairs (X, Y), where $X, Y \in \mathfrak{g}$. The commutators related to the above Lie-Poisson structures have the form

$$\begin{aligned}
[(X_1, Y_1), (X_2, Y_2)]_{TM} =& ([X_1, Y_2]_1 + [Y_1, X_2]_1 + \lambda[Y_1, Y_2]_2, [Y_1, Y_2]_1), \\
[(X_1, Y_1), (X_2, Y_2)]_{T\mathfrak{g}^*} =& ([X_1, Y_2]_1 + [Y_1, X_2]_1 + \lambda[Y_1, Y_2]_2 + \epsilon[X_1, X_2]_1, \\
& [Y_1, Y_2]_1 - \lambda\epsilon[Y_1, Y_2]_2).
\end{aligned}$$

It is easy to see that these commutators are compatible, i.e. their linear combination $\alpha[\cdot, \cdot]_1 + \beta[\cdot, \cdot]_2$ is again a Lie bracket. They generate a Lie bundle.

Some special case of this Lie bundle was considered by Bolsinov and Fedorov. They restricted their considerations to the case $\mathfrak{g} = \mathfrak{so}(n)$, where the first commutator $[\cdot, \cdot]_1$ is a standard commutator and the second commutator $[\cdot, \cdot]_2$ has the form

$$[X_1, X_2]_2 = X_1 S X_2 - X_2 S X_1,$$

where S is a symmetric matrix. This Lie bundle is related to the Steklov–Lyapunov cases.

Example

If we take Euclidean Lie algebra $\mathcal{A}_{3,6} = \mathfrak{e}(2)$, then in above construction we obtain the following Poisson structure on $T\mathcal{A}_{3,6}$ $(\epsilon = \lambda = 0)$

$$\pi_{T\mathfrak{g}^*,\lambda}(\mathbf{x},\mathbf{y}) = \left(\begin{array}{ccc|ccc} 0 & 0 & -\mu y_2 & 0 & 0 & 0 \\ 0 & 0 & \mu y_1 & 0 & 0 & 0 \\ \mu y_2 & -\mu y_1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

We recognize the Lie-Poisson structure related to direct sum $\mathcal{A}_{5,1} \oplus \langle y_3 \rangle$.

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Thank you for your attention