Quantizing real semisimple Lie groups

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(joint work with J.R. Dzokou Talla)

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Overview

Real semisimple Lie algebras and Lie groups

Quantization of real semisimple Lie groups

Representation theory

The case of quantum $\mathfrak{sl}(2, \mathbb{R})$

Real Lie algebras

Definition

Let $\mathfrak{g}$ be a (finite dimensional) complex Lie algebra.

A real form on $\mathfrak{g}$ is an anti-linear map $\dagger : \mathfrak{g} \to \mathfrak{g}$ such that

$$[X, Y]^\dagger = [Y^\dagger, X^\dagger]$$

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$$\text{Real form } (\mathfrak{g}, \dagger)$$

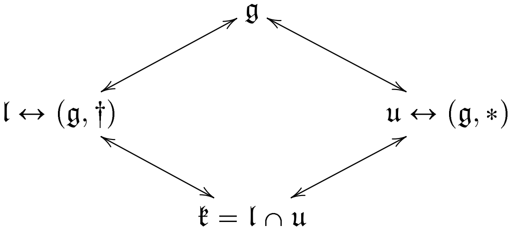
$$\updownarrow$$

$$\text{Real Lie algebra } \mathfrak{l} = \{X \in \mathfrak{g} \mid X^\dagger = -X\}.$$

Real semisimple Lie algebras and Lie groups

Cartan duality for semisimple Lie algebras

Let $\mathfrak{g}$ be complex semisimple and $\mathfrak{l}$ a real form. Then



with $\mathfrak{u}$ a compact form of $\mathfrak{g}$ such that

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$$\begin{array}{ccc} & \mathfrak{g} & \\ \nearrow\!\!\swarrow & & \searrow\!\!\nwarrow \\ \mathfrak{l} \leftrightarrow (\mathfrak{g}, \dagger) & & \mathfrak{u} \leftrightarrow (\mathfrak{g}, *) \\ \nwarrow\!\!\searrow & & \swarrow\!\!\nearrow \\ & \mathfrak{k} = \mathfrak{l} \cap \mathfrak{u} & \end{array}$$

with $\mathfrak{u}$ a compact form of $\mathfrak{g}$ such that

$$*\dagger = \dagger * \, .$$

Then $\sigma = *\dagger : \mathfrak{g} \to \mathfrak{g}$ involutive automorphism:

$$\begin{aligned} \mathfrak{l} \quad &\leftrightarrow \quad \text{compact symmetric pair } \mathfrak{u}^\sigma =: \mathfrak{k} \subseteq \mathfrak{u}. \\ &\leftrightarrow \quad \text{complex symmetric pair } \mathfrak{g}^\sigma =: \mathfrak{k}^{\mathbb{C}} \subseteq \mathfrak{g}. \end{aligned}$$

Cartan duality for semisimple Lie groups

Let $\mathfrak{l} \subseteq \mathfrak{g}$ a semisimple real Lie algebra.

Let G connected, simply connected Lie group with $\mathfrak{g} = T_e G$.
We define the associated Lie group L of $\mathfrak{l}$ as

$$L = \{g \in G \mid g^\dagger = g^{-1}\}, \qquad \mathfrak{l} = T_e L.$$

Warning: L need not be connected or simply connected!

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With $U \subseteq G$ integrating $\mathfrak{u}$, we have the Cartan duality

$$L \quad \leftrightarrow \quad \text{Symmetric space of compact type } U/K, \quad K = U^\sigma.$$

Warning: K is connected with $T_e K = \mathfrak{k}$, but:

- K is not necessarily simply connected!
- $\mathfrak{k}$ in general only reductive, not semisimple!

Real semisimple Lie algebras and Lie groups

Examples

Type A: Fix $G = SL(N,\mathbb{C})$ with $U = SU(N)$.

Type AI:

$$\begin{aligned} L = SL(N,\mathbb{R}) \quad &\leftrightarrow \quad \sigma(g) = (g^T)^{-1} \\ &\leftrightarrow \quad SO(N) \subseteq SU(N) \\ &\leftrightarrow \quad SU(N) \curvearrowright S\Lambda(N). \end{aligned}$$

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Type $AIII$, with $N = m + n$:

$$\begin{aligned} L = SU(m, n) &\leftrightarrow \sigma = \mathrm{Ad}\begin{pmatrix} I_m & 0 \\ 0 & -I_n \end{pmatrix} \\ &\leftrightarrow S(U(m) \times U(n)) \subseteq SU(N) \\ &\leftrightarrow SU(N) \curvearrowright \mathrm{Gr}(m, \mathbb{C}^n). \end{aligned}$$

Quantization of real semisimple Lie groups

Quantizing complex semisimple Lie algebras

Fix complex semisimple $\mathfrak{g}$ with $\mathfrak{h} \subseteq \mathfrak{b} \subseteq \mathfrak{g}$ for

- $\mathfrak{h}$ maximal Cartan, $\mathfrak{h} \ni \{H_i\}$,
- $\mathfrak{b}$ Borel, $\mathfrak{b} \ni \{H_i, E_i\}$.

For $q \in \mathbb{C} \backslash \{0, \text{roots of unity}\}$, we can quantize along $(\mathfrak{h}, \mathfrak{b})$:

$$\underset{E_i, F_i, H_i, \quad \Delta(E_i) = E_i \otimes 1 + 1 \otimes E_i}{\text{Hopf algebra } (U(\mathfrak{g}), \Delta)} \quad \underset{\text{Drinfeld, Jimbo}}{\longrightarrow} \quad \underset{E_i, F_i, q^{H_i}, \quad \Delta(E_i) = E_i \otimes 1 + q^{H_i} \otimes E_i}{\text{Hopf algebra } (U_q(\mathfrak{g}), \Delta)}.$$

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We can choose compact form with $\mathfrak{h}^* = \mathfrak{h}$ and $\mathfrak{b}^* = \mathfrak{b}^-$:

$$\underset{E_i^* = F_i}{\text{Hopf } *\text{-algebra } (U(\mathfrak{g}), *, \Delta)} \longrightarrow \underset{E_i^* = F_i q^{H_i}}{\text{Hopf } *\text{-algebra } (U_q(\mathfrak{g}), *, \Delta)}.$$

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We write

$$U_q(\mathfrak{u}) = (U_q(\mathfrak{g}), *).$$

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First choice: $\mathfrak{h}^\dagger = \mathfrak{h}, \mathfrak{b}^\dagger = \mathfrak{b}^-$ (always possible). Then

$$\nu = *\dagger \quad \Rightarrow \quad \text{Vogan form}.$$

Example: standard inclusion $\mathfrak{su}(m,n) \subseteq \mathfrak{sl}(m+n,\mathbb{C})$.

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Theorem (Twietmeyer, '92)

$\exists$ *Hopf $*$-algebra involution $\nu : U_q(\mathfrak{u}) \to U_q(\mathfrak{u})$ quantizing ν.*

In particular, we have the quantization

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Concrete form: $\nu(E_i) = \varepsilon_i E_{\tau(i)}$, so $E_i^\dagger = \varepsilon_i F_{\tau(i)} q^{H_{\tau(i)}}$.

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Problems

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In rank 1: Extended quantum $SU(1,1)$ (Koelink-Kustermans '03).

Second quantization approach

Second choice: Position $\mathfrak{l} \subseteq \mathfrak{g}$ with $\mathfrak{b}^{\dagger} \cap \mathfrak{b}^{-}$ minimal dimension!
Then

$$\theta = *\dagger \quad \Rightarrow \quad \text{Satake form}.$$

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Put $\mathfrak{k} = \mathfrak{u}^\theta = \mathfrak{u} \cap \mathfrak{l}$.

Theorem (Letzter '99, Kolb '14, DC-Neshveyev-Tuset-Yamashita '19)
There exists a left coideal $*$-subalgebra $U_q(\mathfrak{k}) \subseteq U_q(\mathfrak{u})$:

$$\Delta(U_q(\mathfrak{k})) \subseteq U_q(\mathfrak{u}) \otimes U_q(\mathfrak{k}).$$

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What about the full Cartan duality?

Drinfeld duality

For finite-dimensional $U_q(\mathfrak{u})$-representations, we ask $q^{H_i} \geqslant 0$.

$\Rightarrow$ Hopf $*$-algebra $\mathcal{O}_q(U)$ of matrix coefficients.

In particular, we have a pairing

$$\tau : \mathcal{O}_q(U) \times U_q(\mathfrak{u}) \to \mathbb{C}, \qquad (f, x) \mapsto \tau(f, x).$$

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Theorem (Drinfeld '87)

Let $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be the Iwasawa decomposition. Then

$$\mathcal{O}_q(U) = U_q(\mathfrak{a} \oplus \mathfrak{n}).$$

Drinfeld doubles

Put $(\mathcal{O}_q(U), \Delta)$ and $(U_q(\mathfrak{u}), \Delta^{\mathrm{op}})$ together via Drinfeld double:

$$xf = \tau(f_{(3)}, x_{(1)}) f_{(2)} x_{(2)} \tau(f_{(1)}, S^{-1}(x_{(3)})).$$

This gives Hopf $*$-algebra $\mathscr{D}(\mathcal{O}_q(U), U_q(\mathfrak{u}), \tau)$.

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Theorem (Drinfeld '87)

We have

$$\mathscr{D}(\mathcal{O}_q(U), U_q(\mathfrak{u}), \tau) = U_q(\mathfrak{g}_{\mathbb{R}}),$$

with $\mathfrak{g}_{\mathbb{R}}$ the Lie algebra $\mathfrak{g}$ considered as a real Lie algebra.

Quantizing real semisimple Lie groups

Orthogonal to $U_q(\mathfrak{k}) \subseteq U_q(\mathfrak{u})$ we have right coideal $*$-subalgebra

$$\mathcal{O}_q(U) \supseteq \mathcal{O}_q(K\backslash U) = \{f \mid \forall x \in U_q(\mathfrak{k}) : \tau(x, f_{(1)})f_{(2)} = \varepsilon(x)f\}.$$

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Definition (De Commer-Dzokou Talla '21)

We define $U_q(\mathfrak{l})$ as the right coideal $*$-subalgebra

$$U_q(\mathfrak{l}) = \mathscr{D}(\mathcal{O}_q(K\backslash U), U_q(\mathfrak{k}), \tau) \subseteq U_q(\mathfrak{g}_\mathbb{R})$$

generated by $U_q(\mathfrak{k})$ and $\mathcal{O}_q(K\backslash U)$.

Representation theory

Unitary representations

Only consider finite-dimensional $U_q(\mathfrak{k})$-representations π with

$$\pi \subseteq \pi'_{|U_q(\mathfrak{k})}, \qquad U_q(\mathfrak{u})\text{-representation } \pi'.$$

Set of (fixed representatives) irreducible $U_q(\mathfrak{k})$-representations:

$$\widehat{K} = \{V_\tau \mid \tau \in \widehat{K}\}$$

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Definition

A unitary L_q-representation consists of pre-Hilbert space $\mathcal{H}_0$ with $*$-homomorphism $\pi : U_q(\mathfrak{l}) \to \mathrm{End}_*(\mathcal{H}_0)$ such that

$$\mathcal{H}_0 = \bigoplus_{\tau \in \widehat{K}}^{\mathrm{alg}} V_\tau \otimes \mathcal{M}_\tau$$

as $U_q(\mathfrak{k})$-representations, with ‘multiplicity Hilbert spaces’ $\mathcal{M}_\tau$.

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Tensor products

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The unitary L_q-representations allow an associative tensor product.

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Proof (sketch).

With $A = \mathcal{O}_q(U)$ and $B = \mathcal{O}_q(K \backslash U)$, we have A-module coalgebra

$$\mathcal{O}_q(K) = C := A/AB_+ \qquad B_+ = \mathrm{Ker}(\varepsilon_{|B}),$$

and

$$U_q(\mathfrak{k})\text{-representations} \quad \leftrightarrow \quad \text{unitary } C\text{-comodules.}$$

Now add pre-Hilbert space structure to Takeuchi equivalence

$$_B\mathrm{Mod}^C \cong {}^C_A\mathrm{Mod}^C. \qquad \square$$

The case of quantum $\mathfrak{sl}(2,\mathbb{R})$

Example

Let $0 < q < 1$.

- $\mathscr{U} = U_q(\mathfrak{su}(2))$: generated by $K^{\pm 1}E, F$ with

$$KE = q^2 EK, \qquad KF = q^{-2} FK, \qquad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

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- $\mathscr{K} = U_q(\mathfrak{k}) = \mathbb{C}[iB] \subseteq U_q(\mathfrak{su}(2))$ with $iB = q^{-1/2} i(E - FK)$.

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$$\mathcal{D}(\mathscr{A}, \mathscr{U}) = U_q(\mathfrak{sl}(2,\mathbb{C})_\mathbb{R}) \qquad \mathcal{D}(\mathscr{B}, \mathscr{K}) = U_q(\mathfrak{sl}(2,\mathbb{R})).$$

The case of quantum $\mathfrak{sl}(2,\mathbb{R})$

Casimir element

The Podleś sphere $*$-algebra $\mathcal{O}_q(S^2)$ admits generators X, Y, Z.

Then inside $U_q(\mathfrak{sl}(2,\mathbb{R}))$ lives the central Casimir element

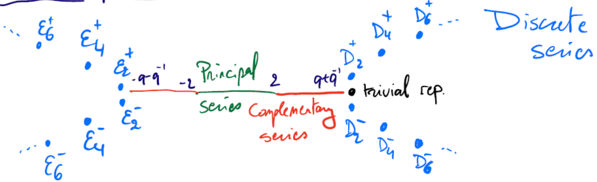
$$\Omega = iq^{-1}X + (q - q^{-1})iZB - iqY.$$

Using Ω, the irreducible $SL_q(2,\mathbb{R})$-representations can be classified.

The case of quantum $\mathfrak{sl}(2, \mathbb{R})$

Irreducible representations of quantum $SL(2, \mathbb{R})$

Even representations: $\mathrm{Spec}(iB) \subseteq [2\mathbb{Z}]_q$



Odd representations: $\mathrm{Spec}(iB) \subseteq [2\mathbb{Z}+1]_q$