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Hidden Symmetries of the Type D Solutions of the EE

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Bogdan Mielnik and Pirate

Acapulco, 2001

Bogdan Mielnik and Geodesics

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A STUDY OF GEODESIC MOTION IN THE FIELD OF SCHWARZSCHILD'S SOLUTION*

By Bogdan Mielnik and Jerzy Plebański

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Geodesic equations are integrated in terms of elliptic functions. A classification of possible types of motion is given. Exact formulae for the perihelion motion, the third Kepler law, *etc.* are investigated. The problem of the physical validity of the results derived *i.e.* their independence of the coordinates used, is investigated. A method of studying the physical meaning of the formulae obtained is proposed; this consists in "translating" the data pertaning the motion under consideration into directly observable quantities, when using a telescope at a finite distance. A gravitational equivalent of Rutherford scattering is also investigated.

He studied geodesics of the Schwarzschild metric in 1961, as part of his PhD adviced by Jerzy Plebañski. This pioneer work is cited in Gravitation by MTW.

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R = 8\pi T_{ab},$$

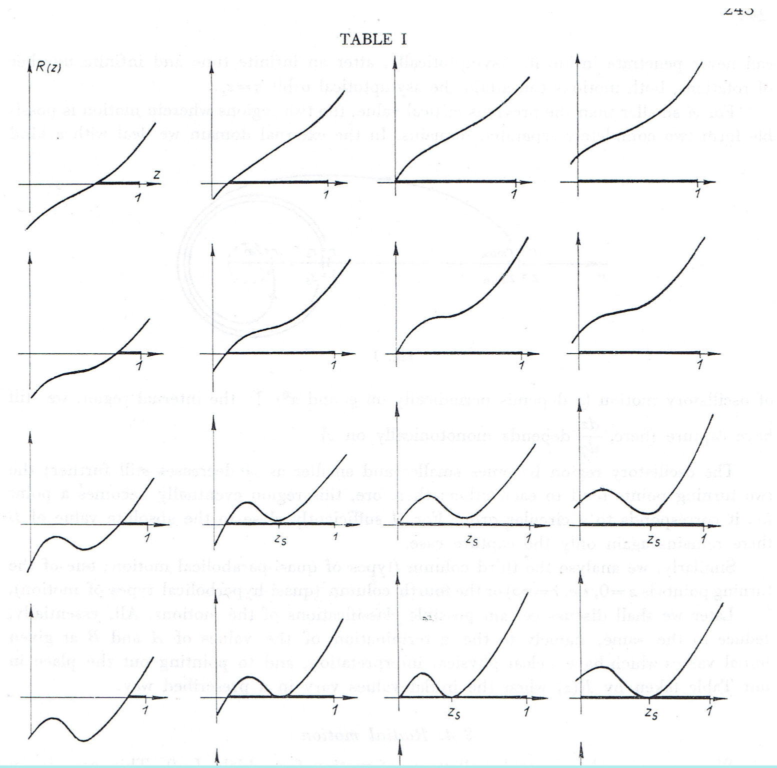
Spacetime with two Killing vectors, ∂_τ, ∂_σ,
Stationary and axisymmetric (including spherical symmetry)

$$g_{\mu\nu} = \begin{pmatrix} g_{tt}(r) & 0 & 0 & 0 \\ 0 & g_{rr}(r) & 0 & 0 \\ 0 & 0 & g_{\theta\theta}(r) = r^2 & 0 \\ 0 & 0 & 0 & g_{\varphi\varphi}(r,\theta) = r^2 \sin^2\theta \end{pmatrix},$$

$$ds^2 = g_{ab} dx^a dx^b = -f(r) dt^2 + f(r)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2 \right),$$

$$f(r) = 1 - \frac{2M}{r},$$

Bogdan Mielnik drawings



These are the potential curves of the radial motion of a test particle near a Schwarzschild BH. He was an artist illustrating his ideas (not the best example).

Geodesics in curved spacetimes

can never penetrate inside it. Asymptotically, after an infinite time and infinite number of rotations, both motions can attain the asymptotical orbit $z=z_s$.

For A smaller than the previous critical value, the two regions wherein motion is possible form two completely separated domains. In the external domain we deal with a kind

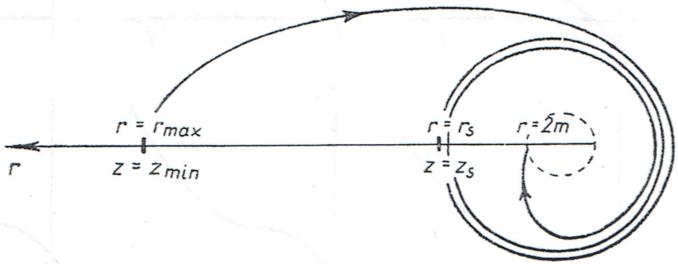


Fig. 1

Jerzy's solutions of Einstein Equations

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A Class of Solutions of Einstein–Maxwell Equations

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A metric defined by $ds^2 = [(p^2 + q^2)/\mathscr{P}]\, dp^2 + [\mathscr{P}/(p^2 + q^2)](d\tau + q^2\, d\sigma)^2 + [(p^2+q^2)/\mathscr{Q}]\, dq^2 - [\mathscr{Q}/(p^2 + q^2)](d\tau - p^2\, d\sigma)^2$, with $\mathscr{P} = \mathscr{P}(p)$, $\mathscr{Q} = \mathscr{Q}(q)$, is studied; the first sections investigate its connections and curvature; the metric is of type D, with Einstein tensor of the electromagnetic algebraic type. Metrics with $R =$ const are characterized by $\mathscr{P}$ and $\mathscr{Q}$ being polynomials of 4th order. In Section 5, by applying Rainich–Wheeler procedure, the electromagnetic field associated with the studied metric is constructed.

During the 70's Jerzy worked (among other things) on finding exact solutions of Einstein-Maxwell Equations. He derived the most general solution of type D metrics with 7 parameters, particular examples are Schwarzschild and Kerr metrics.

Test particle trajectories

The parametric curve: $t(\tau), \quad x(\tau), \quad y(\tau), \quad z(\tau)$, that is solution of

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma}\left(\frac{dx^\beta}{d\tau}\right)\left(\frac{dx^\gamma}{d\tau}\right) = 0, \quad \ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma = 0,$$

These are 4 EQUATIONS; the eq. for $t(\tau)$ is given by

$$\ddot{t} + \Gamma^t_{tt}\dot{t}^2 + \Gamma^t_{xx}\dot{x}^2 + \Gamma^t_{yy}\dot{y}^2 + \Gamma^t_{zz}\dot{z}^2 +$$
$$2\left\{\Gamma^t_{xt}\dot{x}\dot{t} + \Gamma^t_{yt}\dot{y}\dot{t} + \Gamma^t_{zt}\dot{z}\dot{t} + \Gamma^t_{xy}\dot{x}\dot{y} + \Gamma^t_{xz}\dot{x}\dot{z} + \Gamma^t_{yz}\dot{y}\dot{z}\right\} = 0,$$

and so on for $x(\tau)$, $y(\tau)$ and $z(\tau)$. And remember that

$$\Gamma^a_{bc} = \frac{1}{2}g^{ad}(\partial_b g_{dc} + \partial_c g_{db} - \partial_d g_{bc}),$$

We need symmetries to solve the set of geodesic equations
Two cyclic coordinates t, φ, i.e. $g_{\mu\nu}(r, \theta)$.
$\Rightarrow$ spacetime with two Killing vectors, $\partial_t, \partial_\varphi$, stationary and axisymmetric,
Conserved quantities:
Stationarity $\Longrightarrow p_t = E$
axialsymm. $\Longrightarrow p_\varphi = L$
Besides the conservation of the mass particle μ ,

$$g_{\alpha\beta} p^\alpha p^\beta + \mu^2 = 0, \quad p^\alpha = \mu u^\alpha = \mu \frac{dx^\alpha}{d\tau},$$
$$g_{tt}(p^t)^2 + g_{\varphi\varphi}(p^\varphi)^2 + g_{rr}(p^r)^2 + g_{\theta\theta}(p^\theta)^2 = -\mu^2, \tag{1}$$

Schwarzschild case:

$$\begin{aligned} \dot{r}^2 &= E^2 - \left(1 - \frac{2M}{r}\right)\left(\mu^2 + \frac{L^2}{r^2}\right), \quad \theta = \frac{\pi}{2}, \\ \tau &= \int d\tau = \int \frac{dr}{\sqrt{E^2 - \left(1 - \frac{2M}{r}\right)\left(\mu^2 + \frac{L^2}{r^2}\right)}}, \end{aligned}$$

Petrov Type D metrics

Algebraic symmetries of the Weyl tensor,

$$\frac{1}{2}C^{ab}_{mn}\chi^{mn} = \lambda\chi^{ab},$$

The multiplicity of the eigenbivectors indicates an algebraic symmetry of the Weyl tensor.
The eigenbivectors are associated with the null vectors called Principal Null Directions (PND);
The Petrov classification are the six possible types of degeneracy of the PND.
I: 4 different PND; II: 1 double and two simple PND; D: two double PND
III; IIII; O, Weyl$= 0$, conformally flat space.

The stationary and axisymmetric type D metric

$$g = \Omega^2 \left[\frac{\Sigma}{\mathcal{P}(x)} dx^2 + \frac{\mathcal{P}(x)}{\Sigma} (d\tau + y^2 d\varphi)^2 + \frac{\Sigma}{\mathcal{Q}(y)} dy^2 - \frac{\mathcal{Q}(y)}{\Sigma} (d\tau - x^2 d\varphi)^2 \right],$$

$\Omega = 1 - xy, \quad \Sigma = x^2 + y^2$
in a null tetrad $\{e^1, e^2, e^3, e^4\}$ we align (e^3, e^4) with the PND
To couple with an electromagnetic field

$$T^{\mathrm{em}}_{ab} = \frac{1}{4\pi}(g^{cd} F_{ac} F_{bd} + \frac{1}{4} g_{ab} F_{cd} F^{cd}),$$

you should align the eigenvectors of F_{ab} with e^3, e^4;
then the only nonvanishing componentes of F_{ab} are
$F_{12} = E$ and $F_{34} = B$

The geodesic eqs. in a stationary and axisymmetric type D metric

$$g = \Omega^2 \left[\frac{\Sigma}{\mathcal{P}(x)} dx^2 + \frac{\mathcal{P}(x)}{\Sigma} (d\tau + y^2 d\varphi)^2 + \frac{\Sigma}{\mathcal{Q}(y)} dy^2 - \frac{\mathcal{Q}(y)}{\Sigma} (d\tau - x^2 d\varphi)^2 \right],$$

$\Omega = 1 - xy, \quad \Sigma = x^2 + y^2$

The geodesic eqs. for the coordinates x and y,

$$\begin{aligned} \mathcal{P}(x) \left(\frac{dU}{dx} \right)^2 + \frac{1}{\mathcal{P}(x)} \left[x^2 c_\tau - xg + c_\sigma \right] + x^2 \mu^2 &= C_0, \\ \mathcal{Q}(y) \left(\frac{dV}{dy} \right)^2 - \frac{1}{\mathcal{Q}(y)} \left[y^2 c_\tau - ye - c_\sigma \right] + y^2 \mu^2 &= -C_0, \end{aligned} \tag{2}$$

being C_0 the separation constant, related to the conservation of the total angular momentum (BH + test particle)

Hidden symmetries

Not in the spacetime but in the phase space.
EXAMPLE: A rotating BH, Kerr metric characterized by two parameters: BH mass, m, and BH angular momentum, $J = am$,

$$
\begin{aligned}
ds^2 &= -\frac{\Delta}{\Sigma}(dt - a\sin^2\theta d\varphi)^2 + \frac{\sin^2}{\Sigma}[(r^2+a^2)d\varphi - adt]^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2, \\
\Delta &= r^2 + a^2 - 2mr, \quad \Sigma = r^2 + a^2\cos^2\theta,
\end{aligned}
$$

B. Carter (1968) found that Kerr metric admits a fourth constant of motion, that allows to integrate the FOUR coordinates, $t(\tau), \quad x(\tau), \quad y(\tau), \quad z(\tau)$
This fourth constant of motion is related to the existence of the KT, Q_{bc}, that satisfies: $\nabla_{(a}Q_{bc)} = 0$,
it is a kind of generalization of the Killing vector, $\nabla_{(a}K_{b)} = 0$, and the CONSERVED QUANTITY is given by the contraction of the KT with the test particle 4-momentum

$$
\begin{aligned}
Q^{ab}p_a p_b &= \mathcal{C}, \quad g^{ab}p_a p_b = -\mu^2, \\
\mathcal{C} &= p_\theta^2 + \cos^2\theta\left[a^2(\mu^2 - E^2) + \left(\frac{L_z}{\sin\theta}\right)^2\right],
\end{aligned}
$$

KT is a symmetric tensor

The PD metric (7-parameter) admits the conformal generalization of the Killing-Yano tensor (KYT) [Kubiznak, PRD 2007],

$$\begin{aligned}\nabla_\mu K_{\alpha\beta} &= \nabla_{[\mu} K_{\alpha\beta]} + 2 g_{\mu[\alpha}\xi_{\beta]},\\ \xi_\beta &= \frac{1}{3}\nabla_\mu K^\mu{}_\beta,\end{aligned}$$

$\xi_\beta = 0$ then the CKYT is either a KY or a KT; If K is the CKYT of g then $\Omega^3 K$ is the CKYT of $\Omega^2 g$

The (C)KYT implies the (C)KT,

$$\begin{aligned}\nabla_{(\mu} Q_{\alpha\beta)} &= g_{(\mu\alpha} Q_{\beta)},\\ Q_{\alpha\beta} &= K_{\alpha s} K_\beta{}^s,\end{aligned}$$

Associated to the KT is the conserved quantity, $Q_{\alpha\beta}p^\alpha p^\beta = \mathcal{C}$. The metric itself is a KT, and the conserved quantity is the test particle mass $g_{\alpha\beta}p^\alpha p^\beta = -\mu^2$.

The KT of the stationary and axisymmetric type D metric

$$g = \Omega^2 \left[\frac{\Sigma dx^2}{\mathcal{P}(x)} + \frac{\mathcal{P}(x)}{\Sigma}(d\tau + y^2 d\varphi)^2 + \frac{\Sigma dy^2}{\mathcal{Q}(y)} - \frac{\mathcal{Q}(y)}{\Sigma}(d\tau - x^2 d\varphi)^2 \right],$$

$\Omega = 1 - xy, \quad \Sigma = x^2 + y^2$

can also be obtained multiplying the antisymmetric Killing-Yano tensor,

$$k = \Omega^3 \left[\omega x dy \wedge (d\tau - \omega x^2 d\varphi) + y dx \wedge (\omega d\tau + y^2 d\varphi) \right] \tag{3}$$

$h = *k$ is also a KY tensor; k and h are CKYT for the Plebañski metric.
The KT is given by $Q_{ab} = Y_{ac} Y_b^{\ c}$ with $Y = k$ or $Y = h$
KT are as well responsible for the separability in typeD spaces of several eqs. like H-J, KG, Dirac,...

Conserved Quantities

For massless particle ($\mu = 0$), with conformal factor $\Omega = 1 - \nu xy$
The geodesic equations integrate

$$
\begin{aligned}
\dot{p}^2 &= -\frac{C\mathcal{P}}{\Delta^2} - \frac{(Ep^2\omega)^2}{\Delta^2}, \\
\dot{q}^2 &= \frac{C\mathcal{Q}}{\Delta^2} - \frac{(Eq^2 - L_z\omega)^2}{\Delta^2}, \\
\Delta &= \Omega^2\Sigma, \quad \Omega = 1 - \alpha pq,
\end{aligned}
$$

$$
\begin{aligned}
(Q_k)_{\mu\nu}\dot{x}^\mu\dot{x}^\nu &= -C, \\
(Q_h)_{\mu\nu}\dot{x}^\mu\dot{x}^\nu &= -C, \\
(Q_{hk})_{\mu\nu}\dot{x}^\mu\dot{x}^\nu &= 0,
\end{aligned}
$$

Conserved Quantities

For massive particle, without conformal factor $\Omega = 1$
The geodesic equations integrate

$$\begin{aligned}\dot{p}^2 &= \frac{\mathcal{P}}{\Sigma^2}(C + \mu^2 p^2 \omega^2) - \frac{(Ep^2\omega)^2}{\Sigma^2}, \\ \dot{q}^2 &= \frac{\mathcal{Q}}{\Sigma^2}(C - \mu^2 q^2) - \frac{(Eq^2 - L_z\omega)^2}{\Sigma^2}, \\ \Sigma &= p^2 + q^2,\end{aligned}$$

$$\begin{aligned}(Q_k)_{\mu\nu}\dot{x}^\mu \dot{x}^\nu &= -C, \\ (Q_h)_{\mu\nu}\dot{x}^\mu \dot{x}^\nu &= -C - \mu^2(x^2\omega^2 - y^2),\end{aligned} \quad (4)$$

CONCLUSION:

We confirm that the conserved quantity in the type D metric (PD) can be derived from the contraction of the KT with the 4-momentum of the test particle.

THANK YOU FOR ATTENDING!