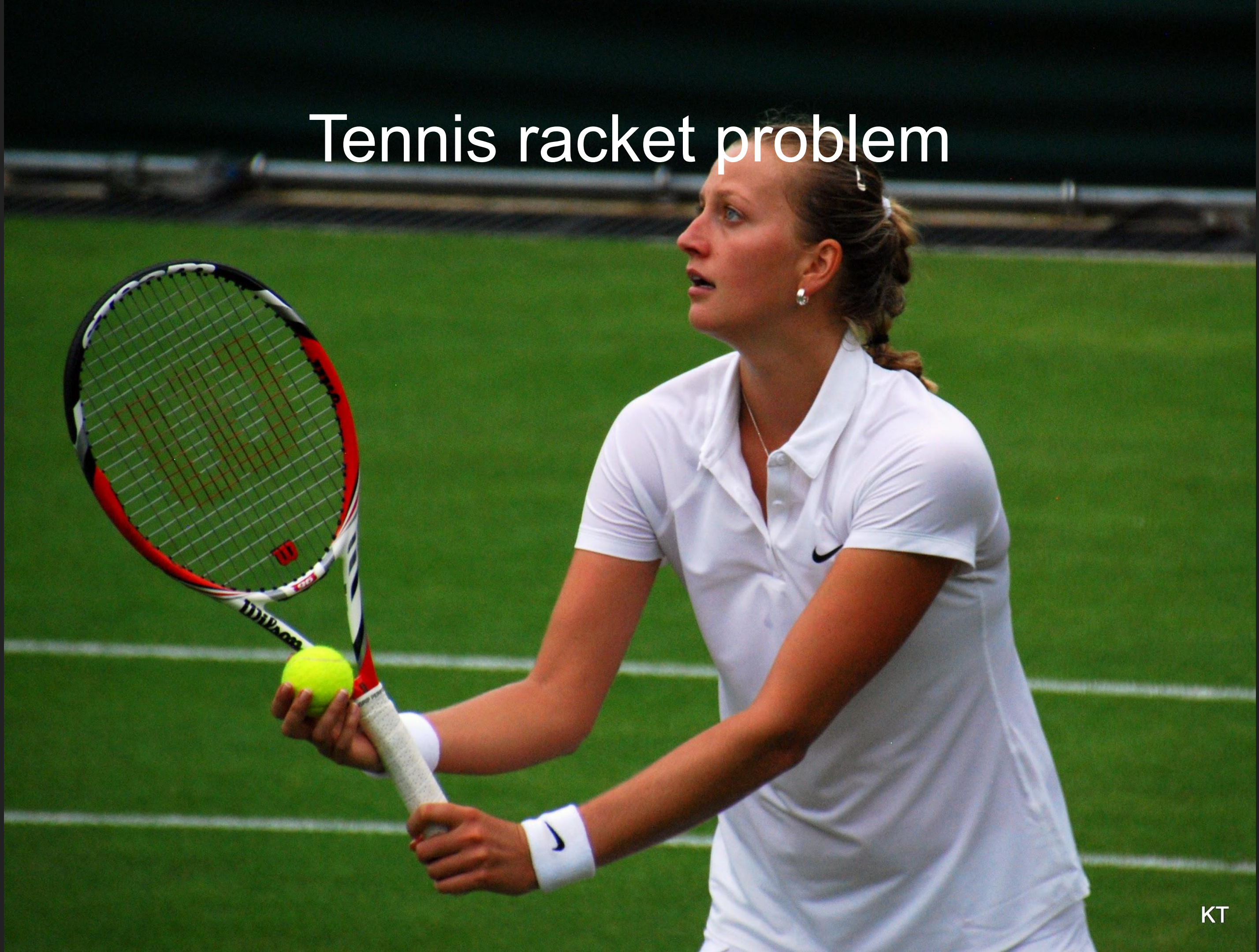
Pedal coordinates, Dipole drive orbits and other force problems

Białystok 24. 6. 2022

Petr Blaschke

Tennis racket problem



KT

Tennis racket problem

Find a curve of a given length fixed on both ends that sweeps maximal volume when rotated around the x-axis.

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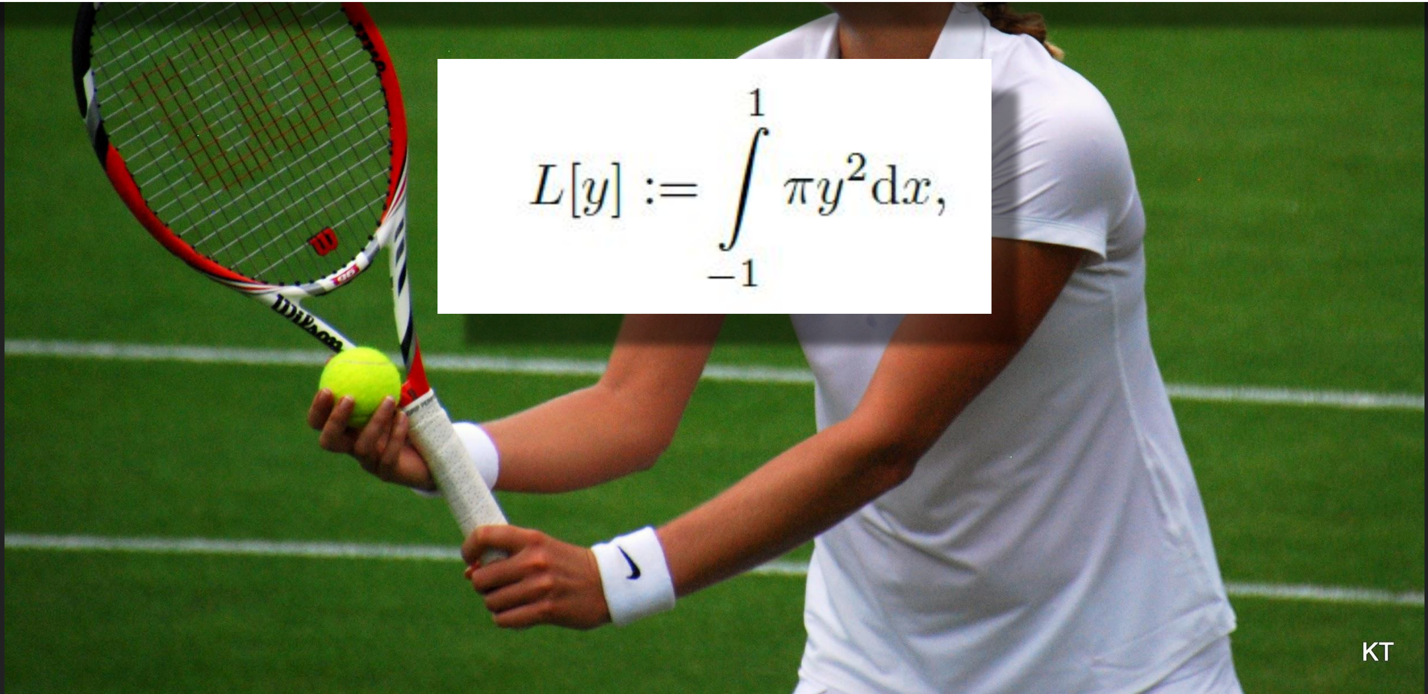
$$L[y] := \int_{-1}^{1} \pi y^2 \mathrm{d}x,$$



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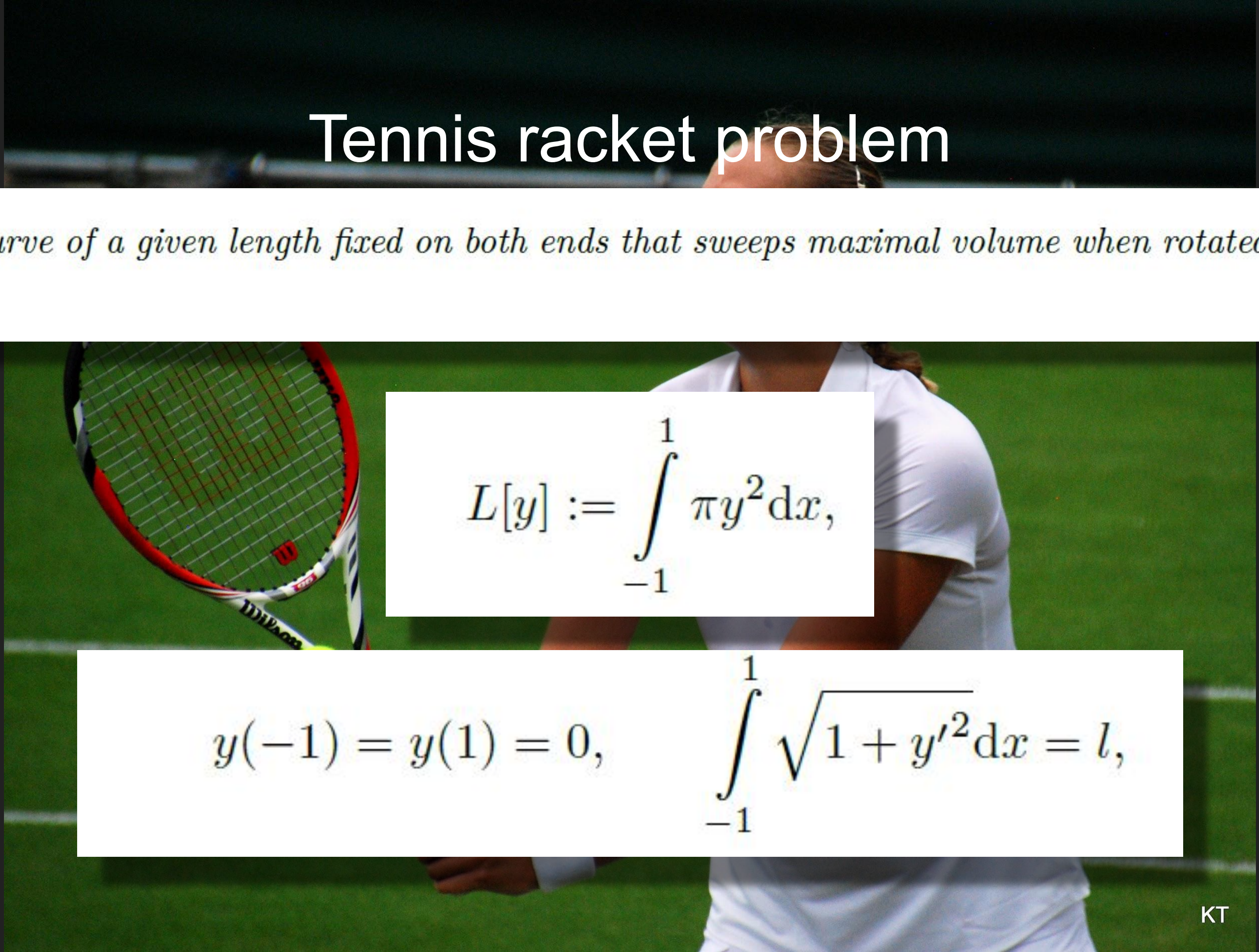
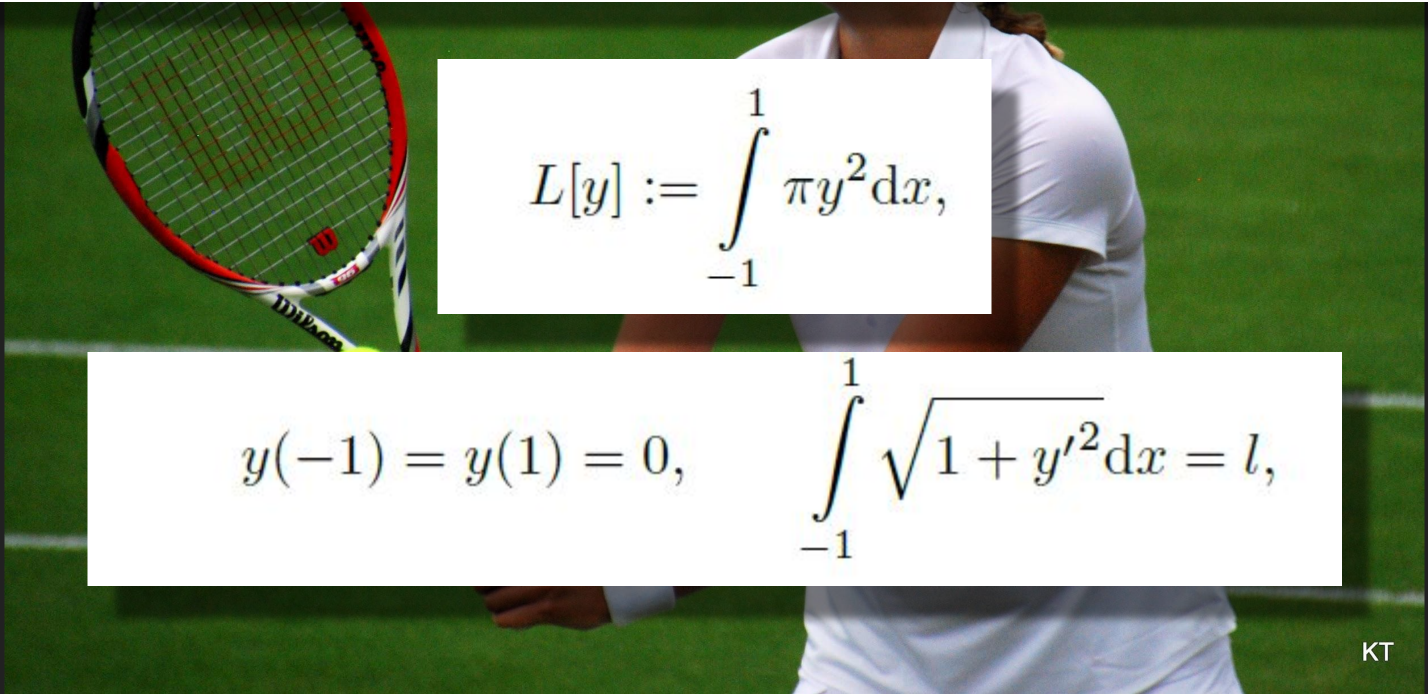


$$L[y] := \int_{-1}^{1} \pi y^2 \mathrm{d}x,$$

$$y(-1) = y(1) = 0, \qquad \int_{-1}^{1} \sqrt{1 + y'^2} \mathrm{d}x = l,$$

Beltrami identity:

$$\pi y^2 + \lambda\sqrt{1+y'^2} - \lambda\frac{y'^2}{\sqrt{1+y'^2}} = C$$



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$$\sqrt{\frac{\lambda - C}{\pi}}F\left(y\sqrt{\frac{\pi}{C+\lambda}}, k\right) - \sqrt{\frac{\lambda - C}{\pi}}E\left(y\sqrt{\frac{\pi}{C+\lambda}}, k\right) + Cy = x - d.$$

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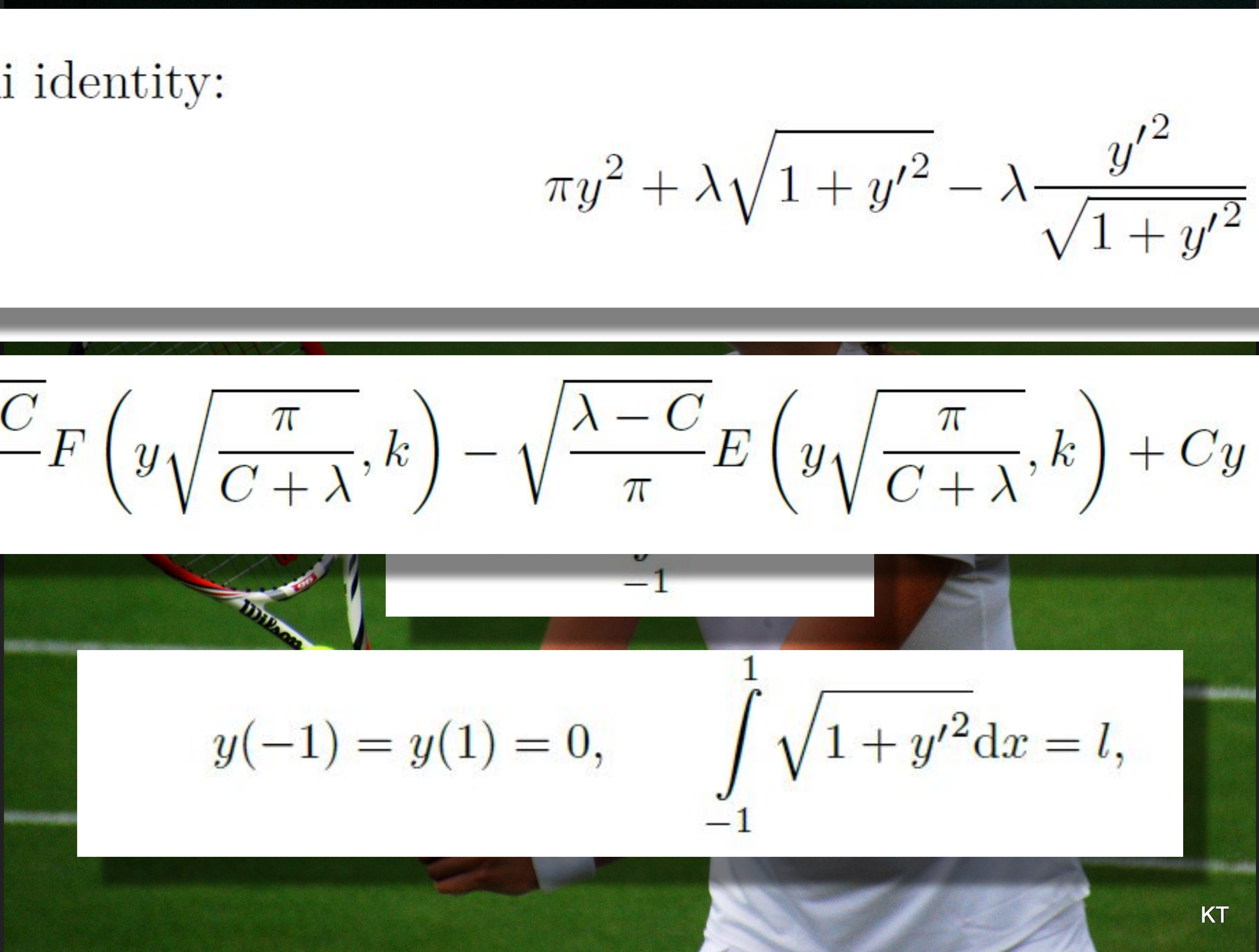

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Euler-Lagrange equation:

$$\frac{y''}{\left(1+y'^2\right)^{\frac{3}{2}}} = \frac{2\pi}{\lambda} y.$$

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$$\kappa \propto y.$$

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Beltrami identity:

$$\sqrt{\frac{\lambda - C}{\pi}} F\left(y\sqrt{\frac{\pi}{C+\lambda}}, k\right) - \sqrt{}$$

Euler-Lagrange equation:

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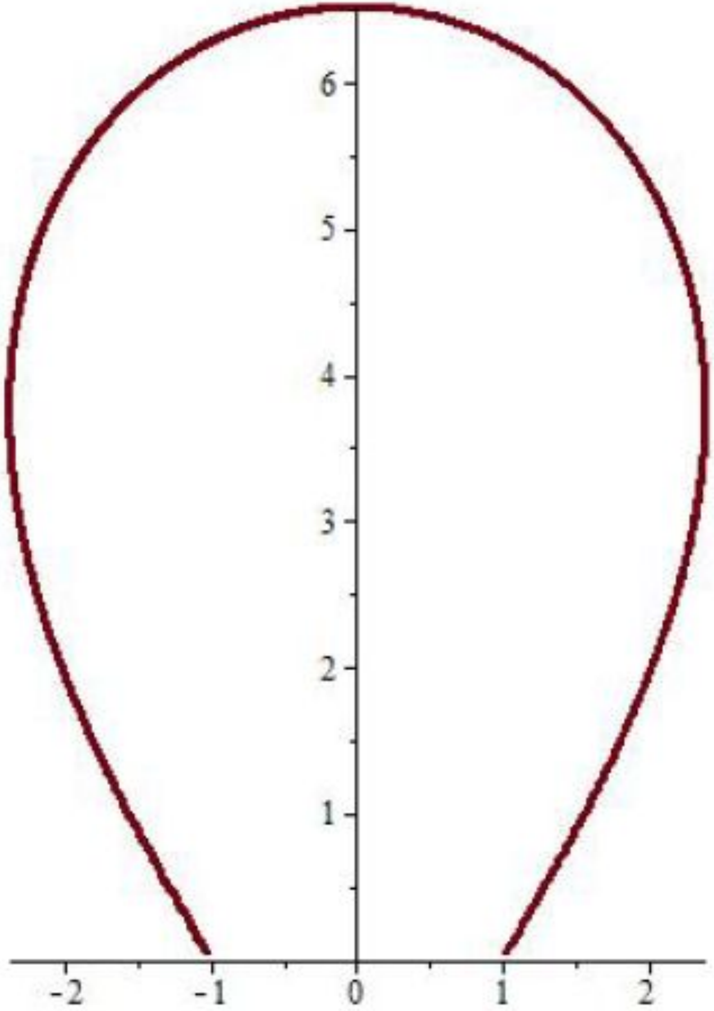
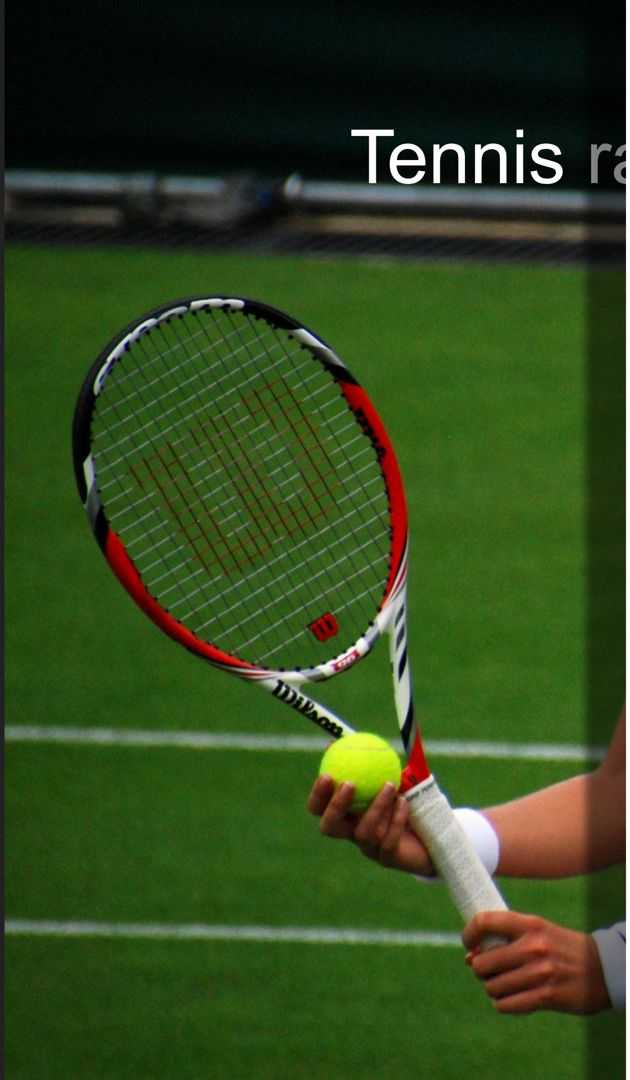
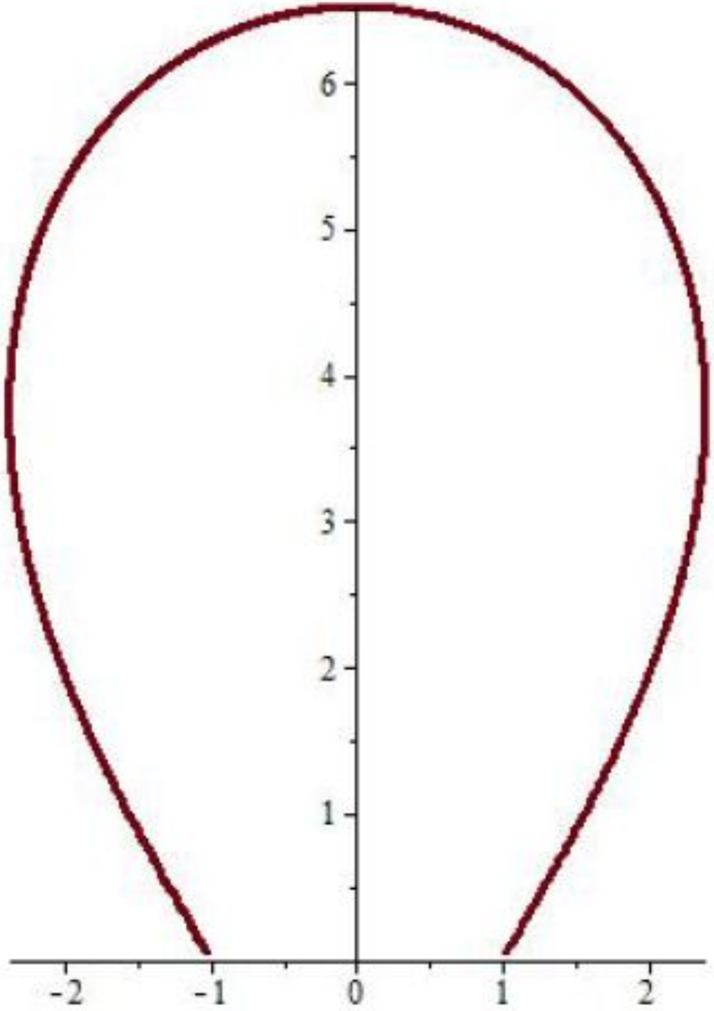
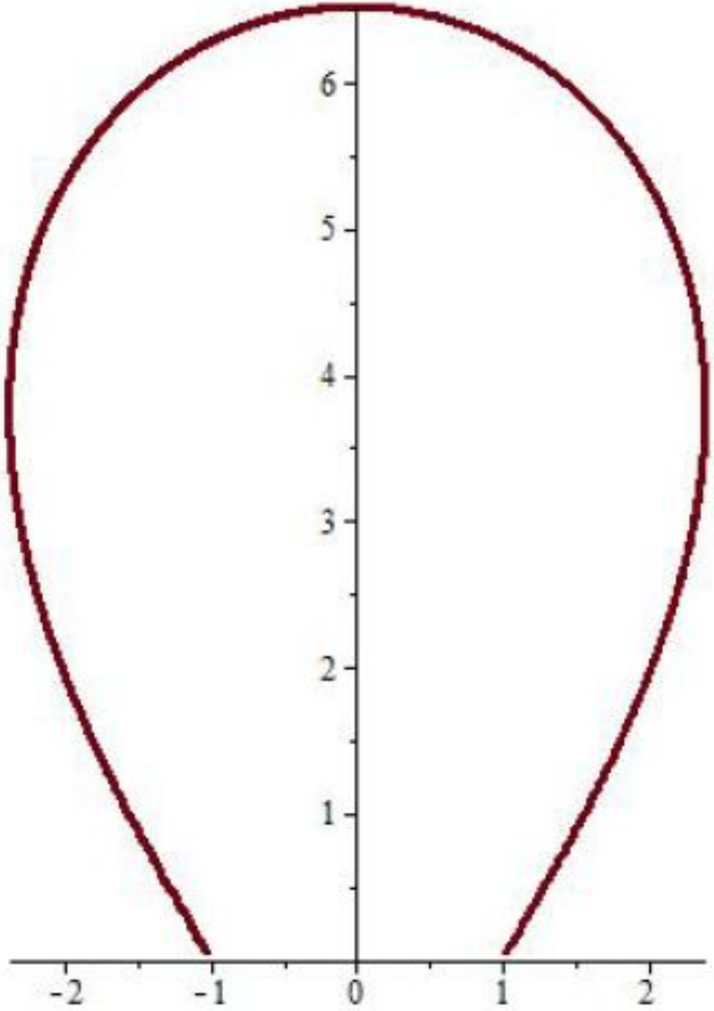


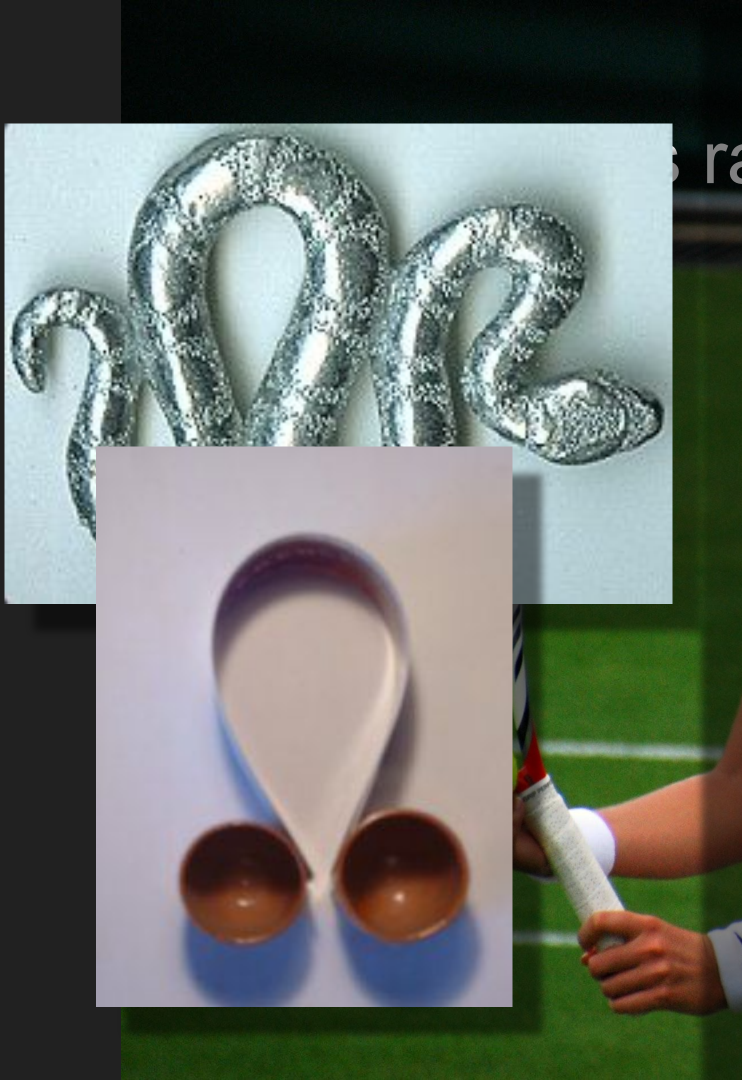
Tennis ra

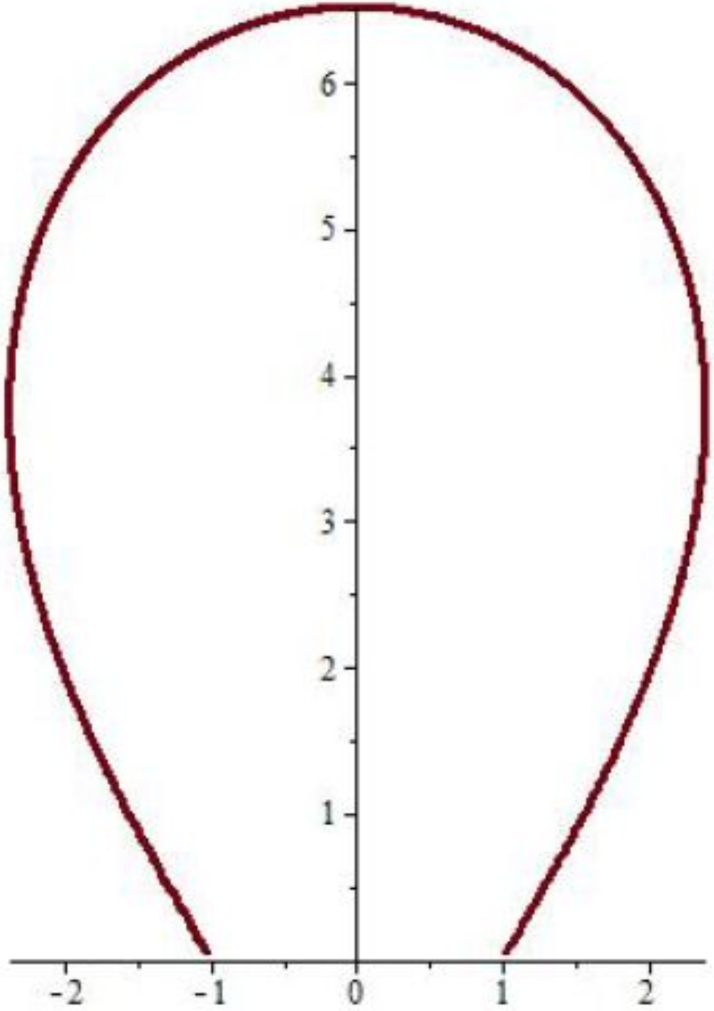




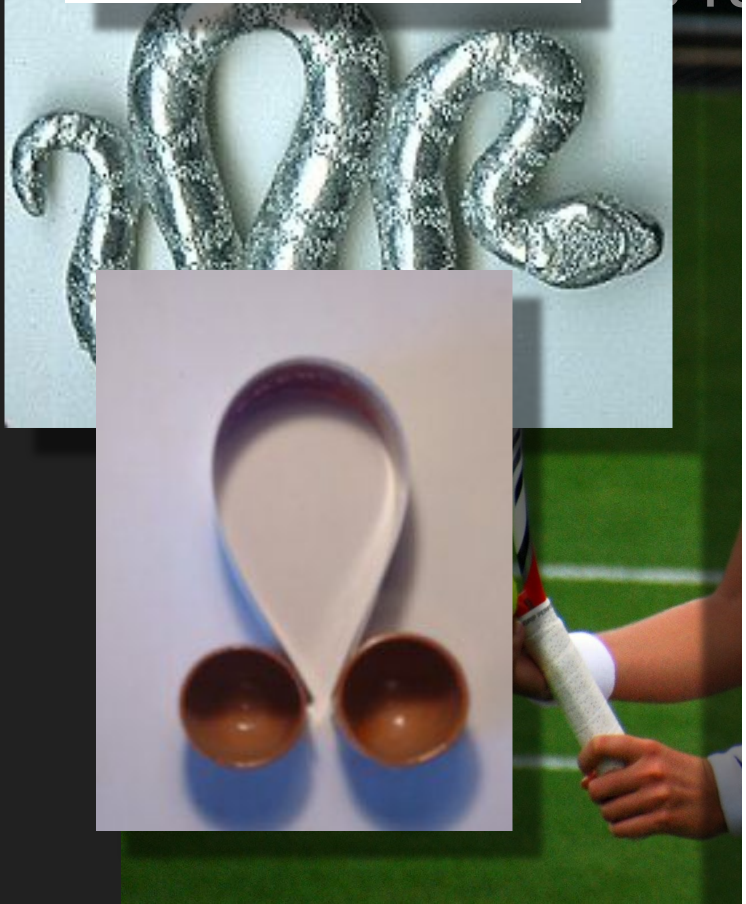
s ra

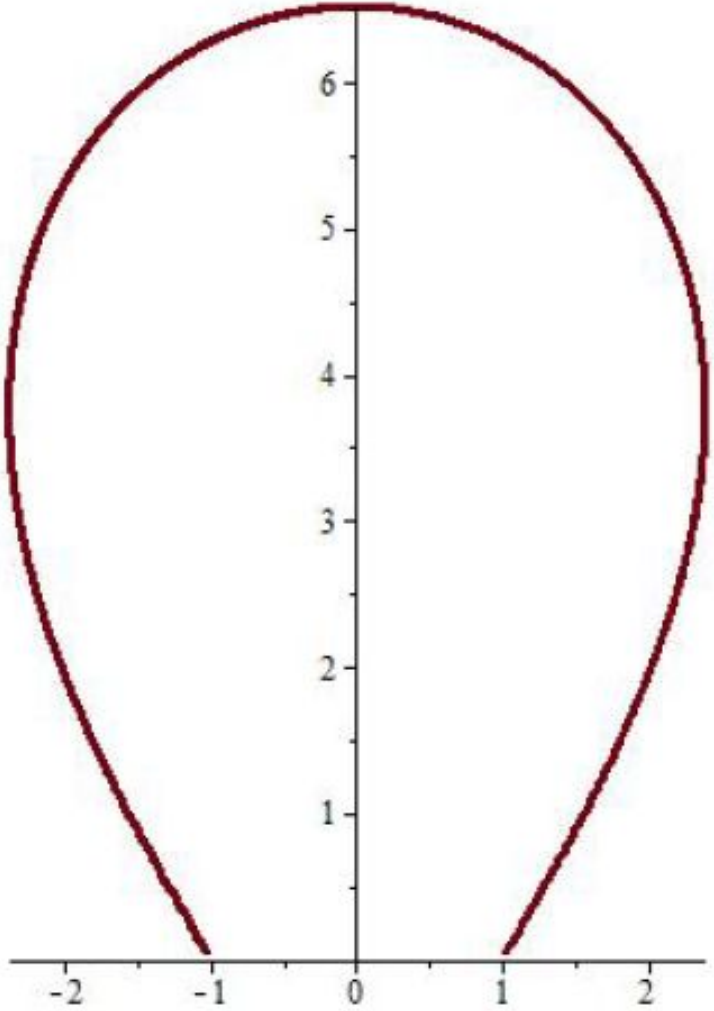





6
5
4
3
2
1
-2
-1
0
1
2

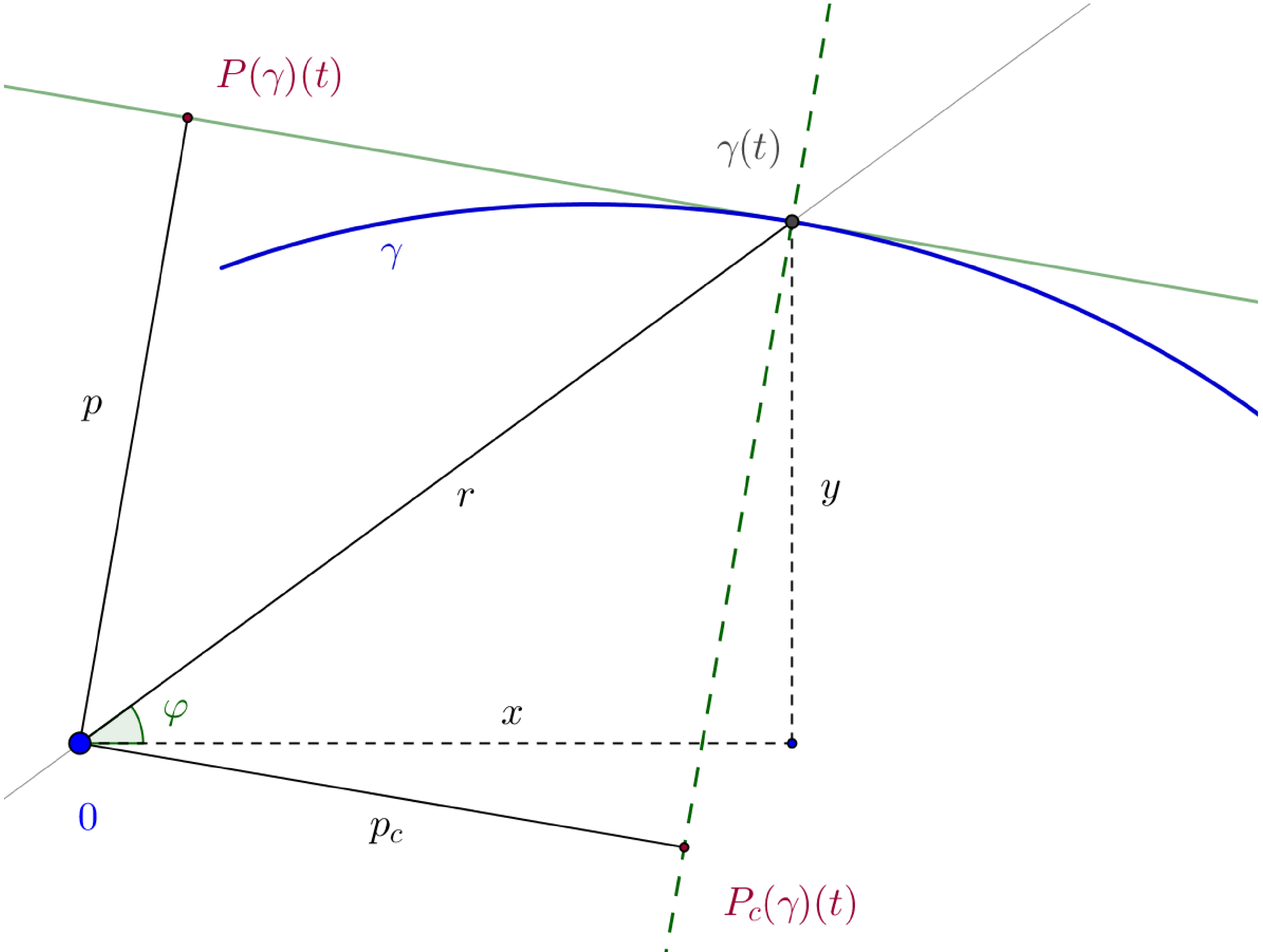
Elastic curve



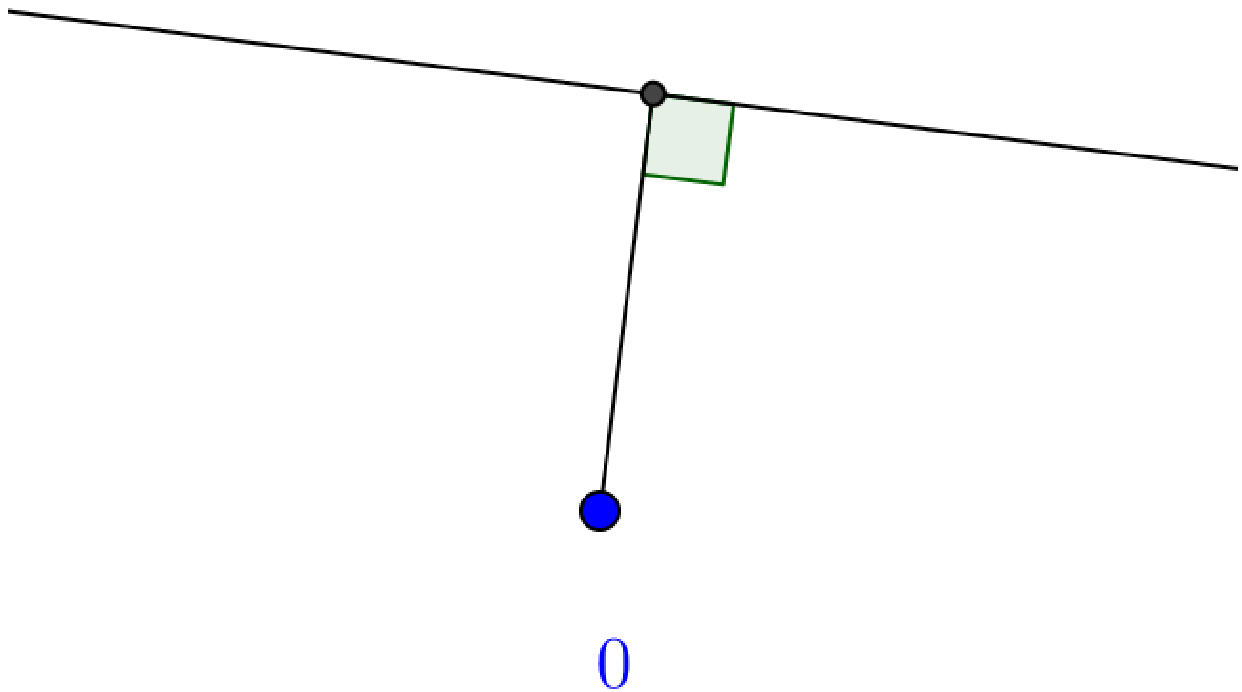



Pedal coordinates
Cartesian, polar coordinates

Coordinates

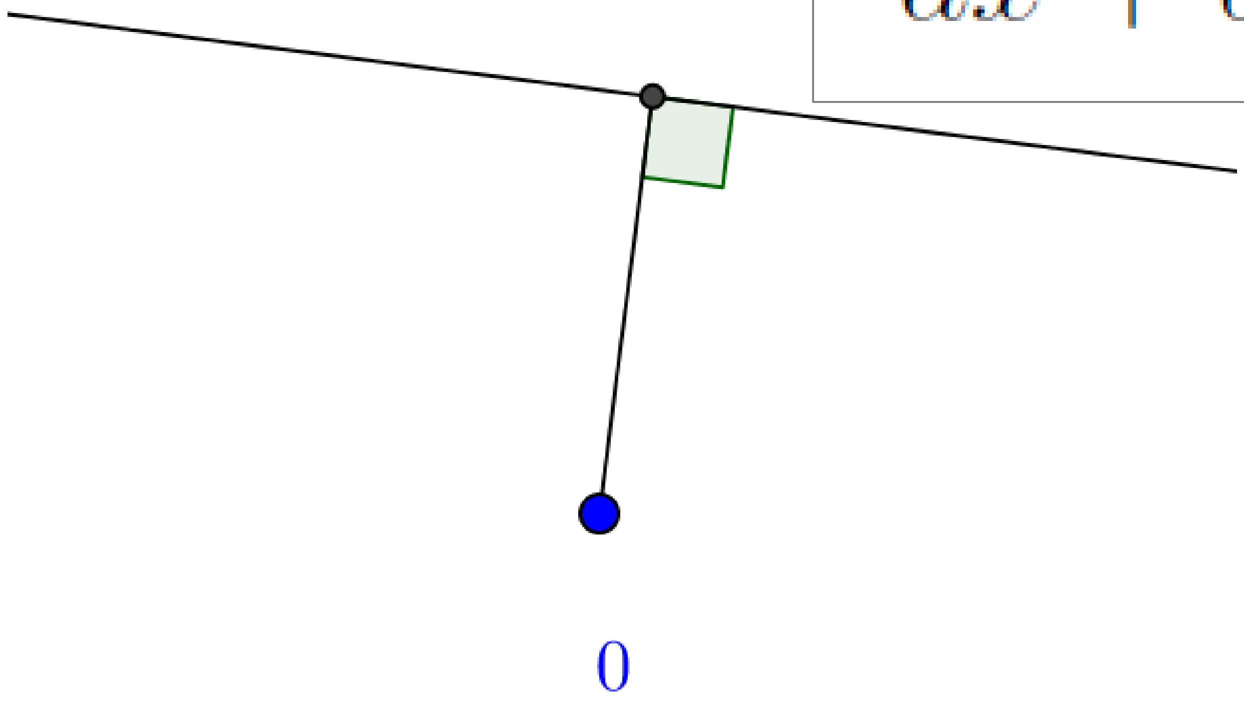


Line



Line

$$ax + by + c = 0.$$



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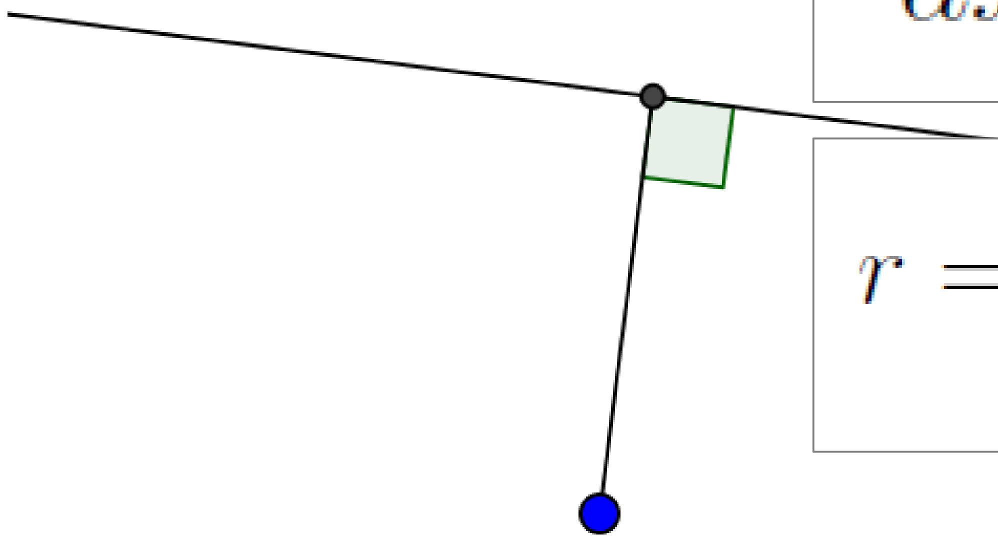
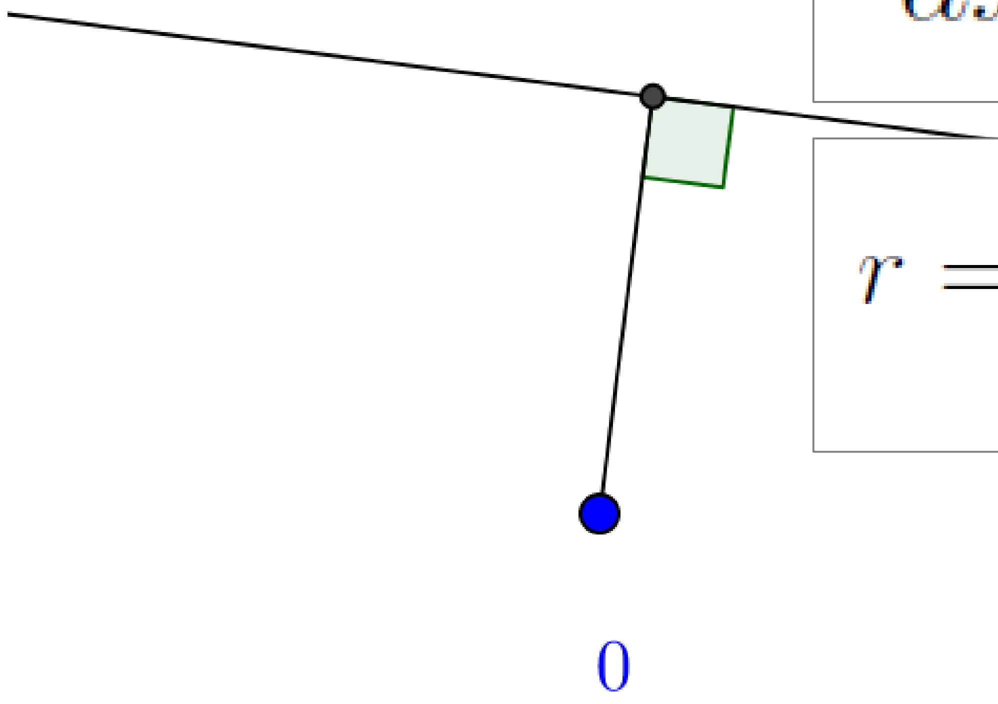


$$ax + by + c = 0.$$

$$r = -\frac{c}{a\cos\varphi + b\sin\varphi}.$$

Line

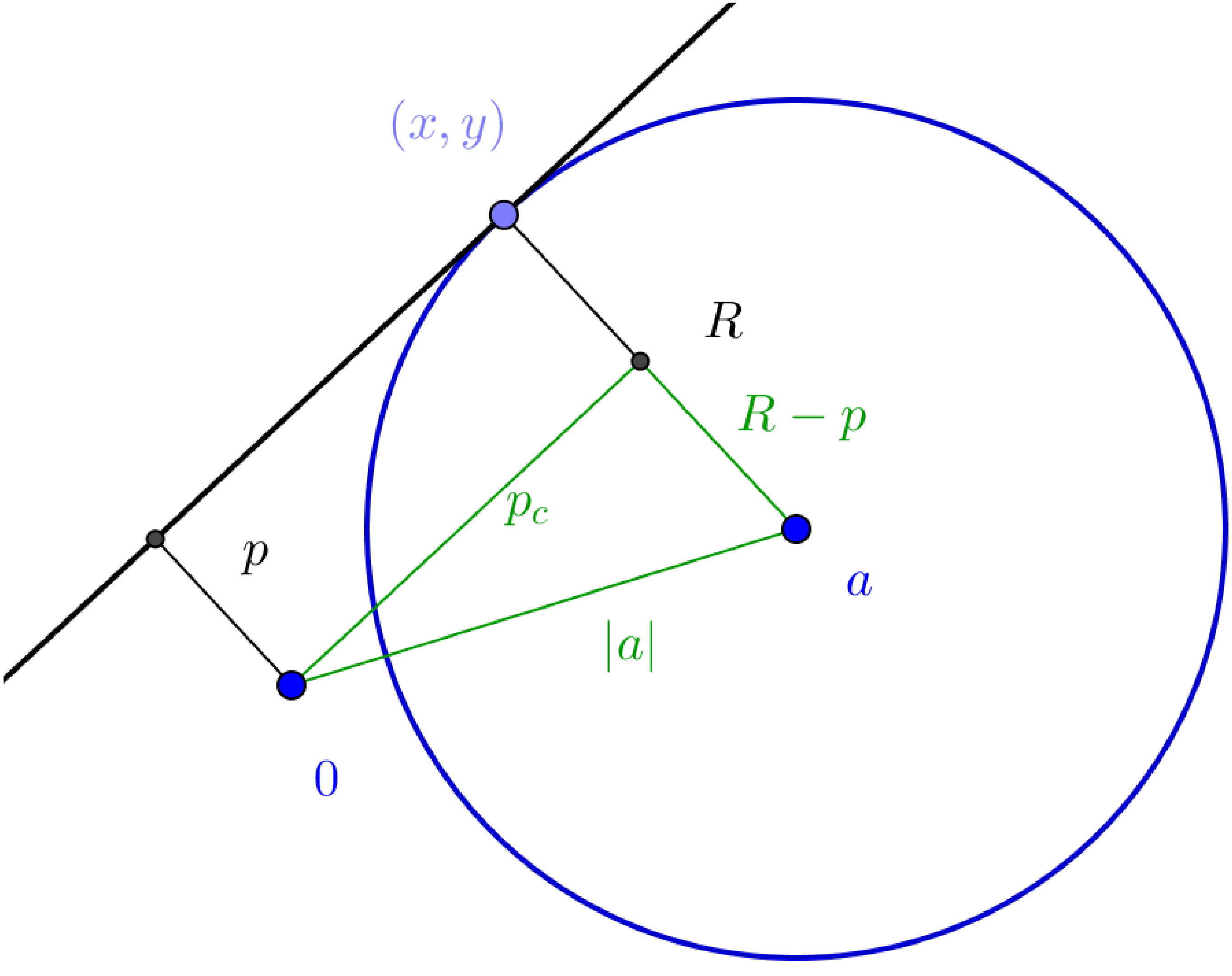


$$ax + by + c = 0.$$

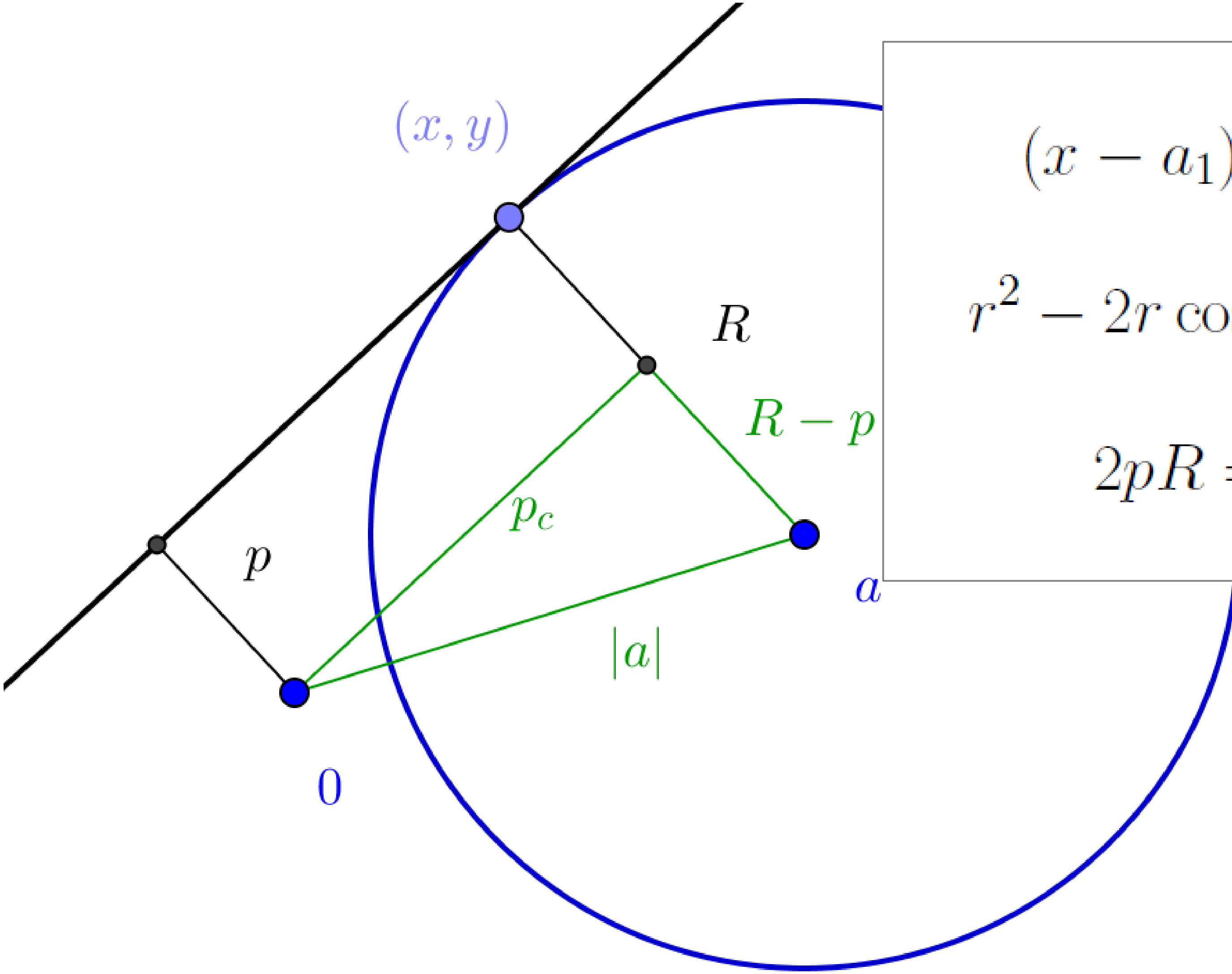
$$r = -\frac{c}{a\cos\varphi + b\sin\varphi}.$$

$$p = \alpha.$$

Circle



Circle

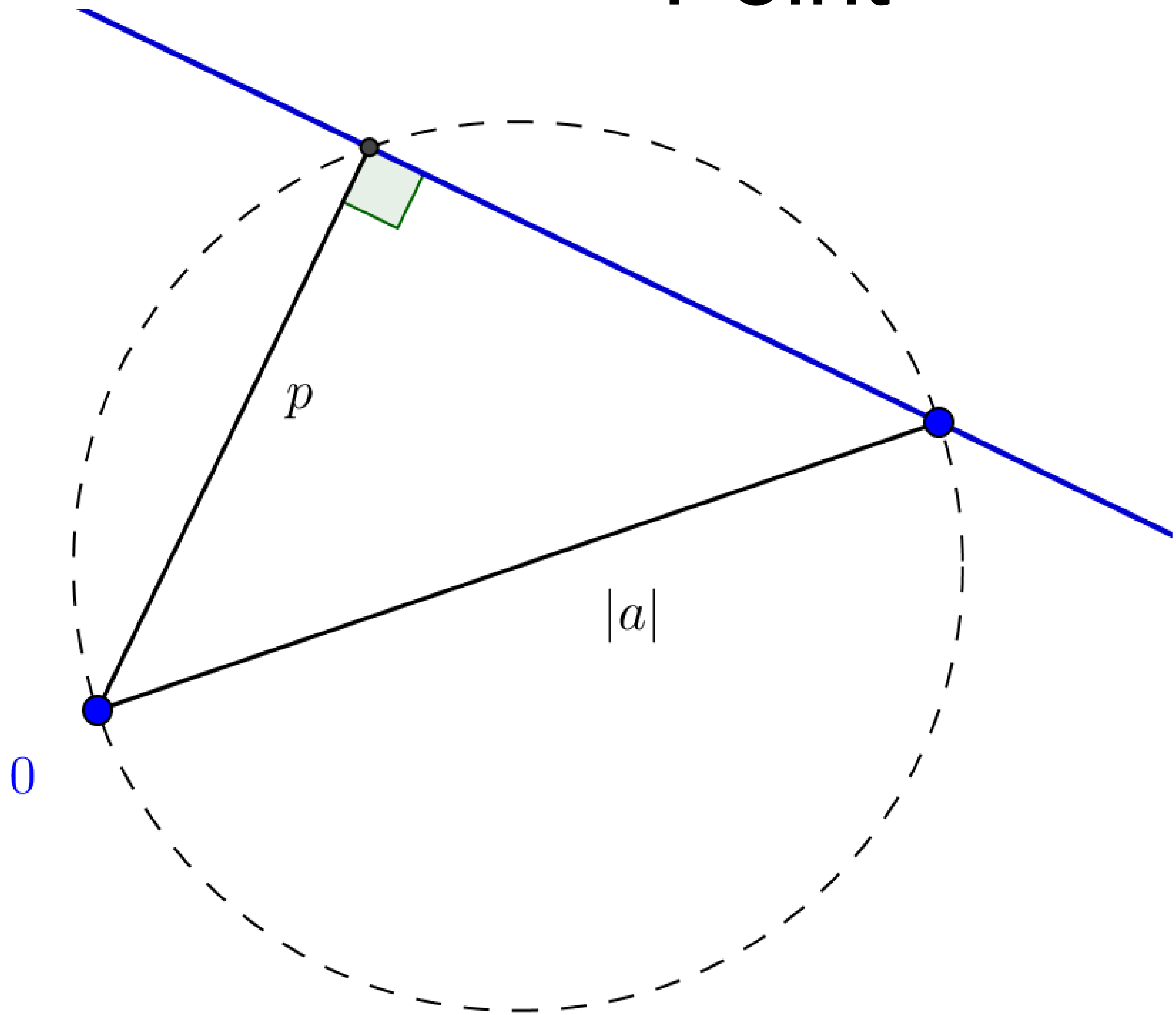


$$(x - a_1)^2 + (y - a_2)^2 = R^2.$$

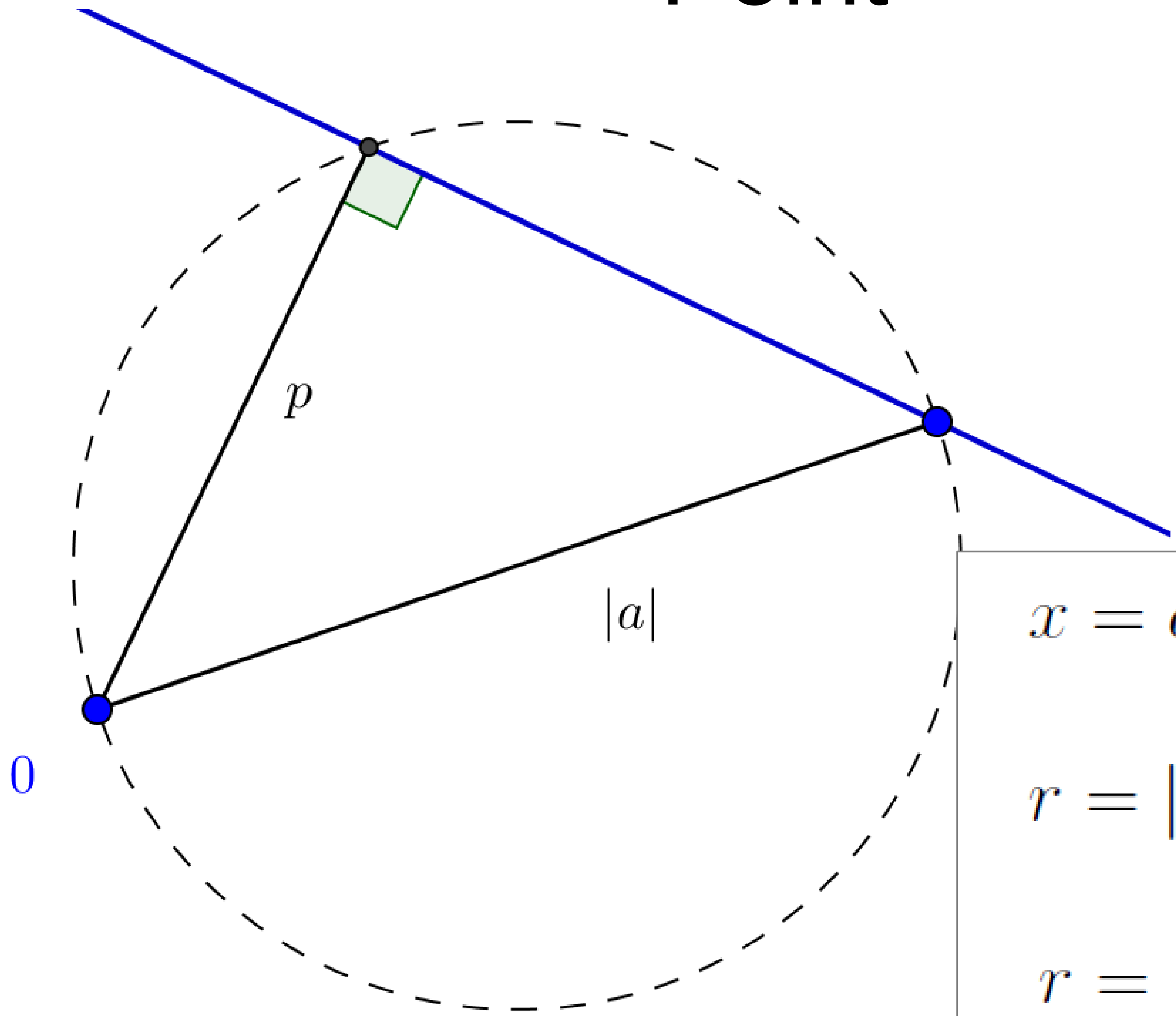
$$r^2 - 2r\cos(\varphi - \alpha) + |a|^2 = R^2.$$

$$2pR = r^2 + R^2 - |a|^2.$$

Point



Point

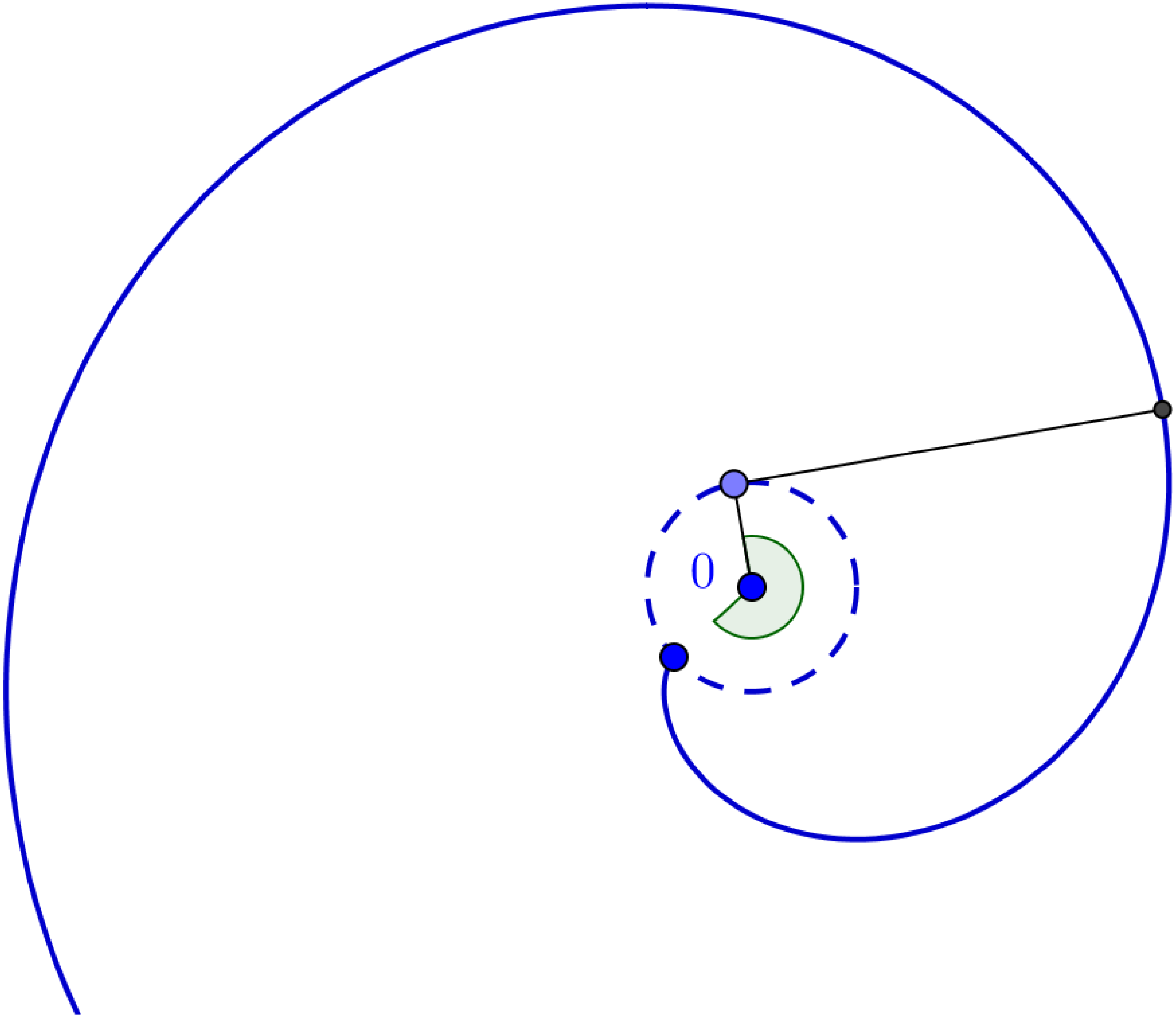


$$x = a_1, \qquad y = a_2.$$

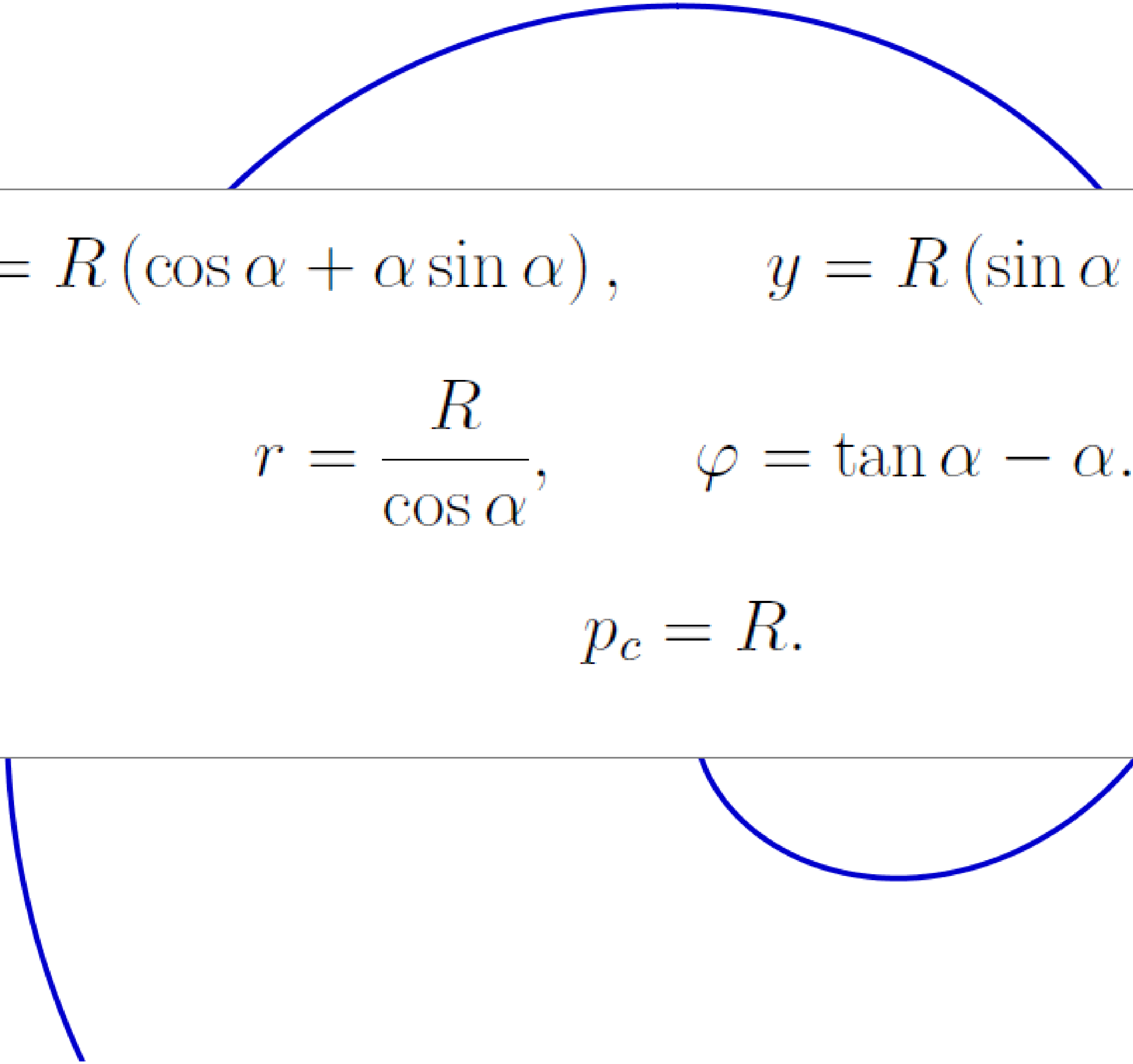
$$r = |a|, \qquad \varphi = \alpha.$$

$$r = |a|, \qquad p \leq r.$$

Involute of a circle



Involute of a circle

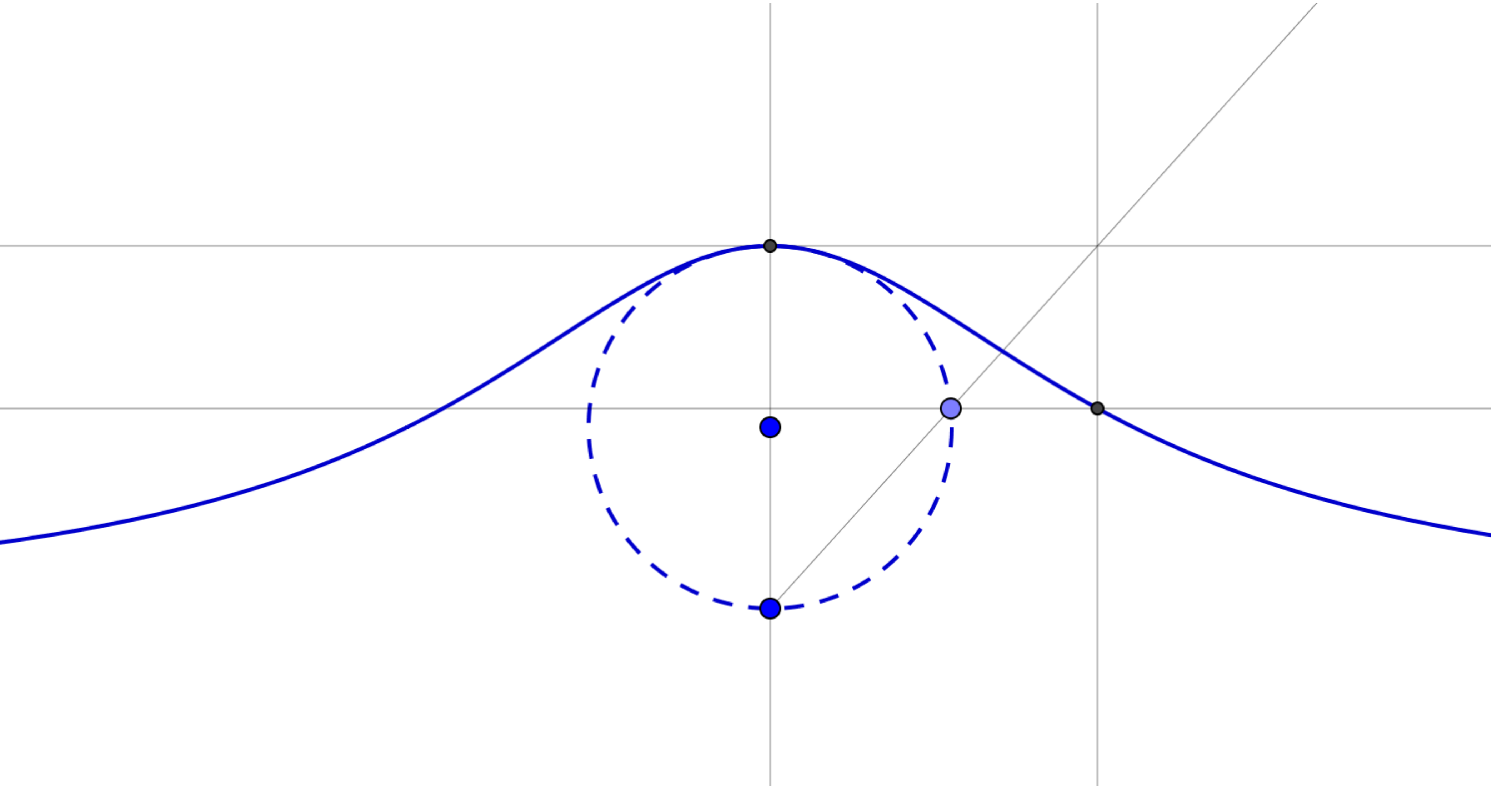


$$x = R\left(\cos\alpha + \alpha\sin\alpha\right), \qquad y = R\left(\sin\alpha - \alpha\cos\alpha\right).$$

$$r = \frac{R}{\cos\alpha}, \qquad \varphi = \tan\alpha - \alpha.$$

$$p_c = R.$$

Witch of Agnesi



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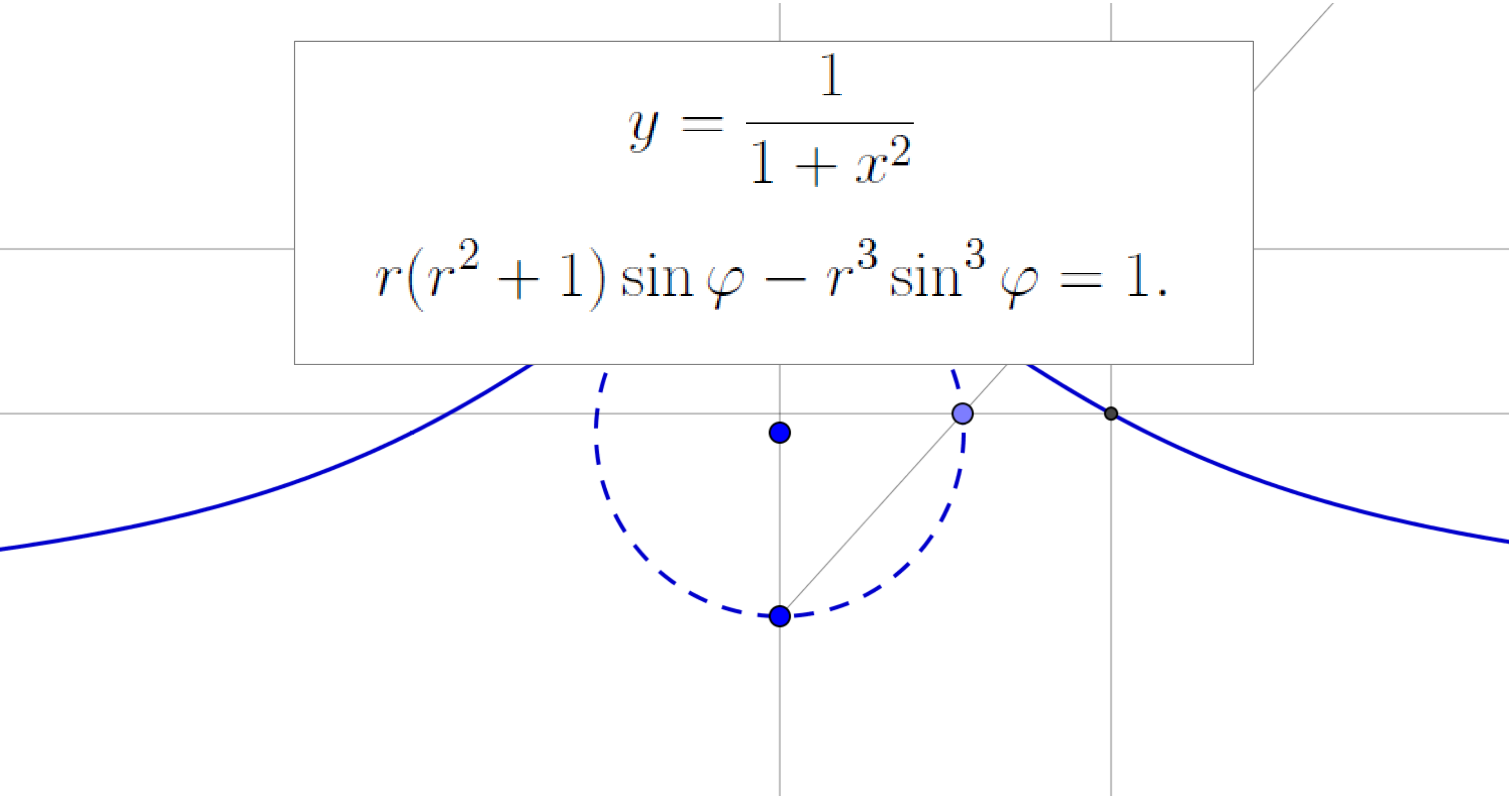
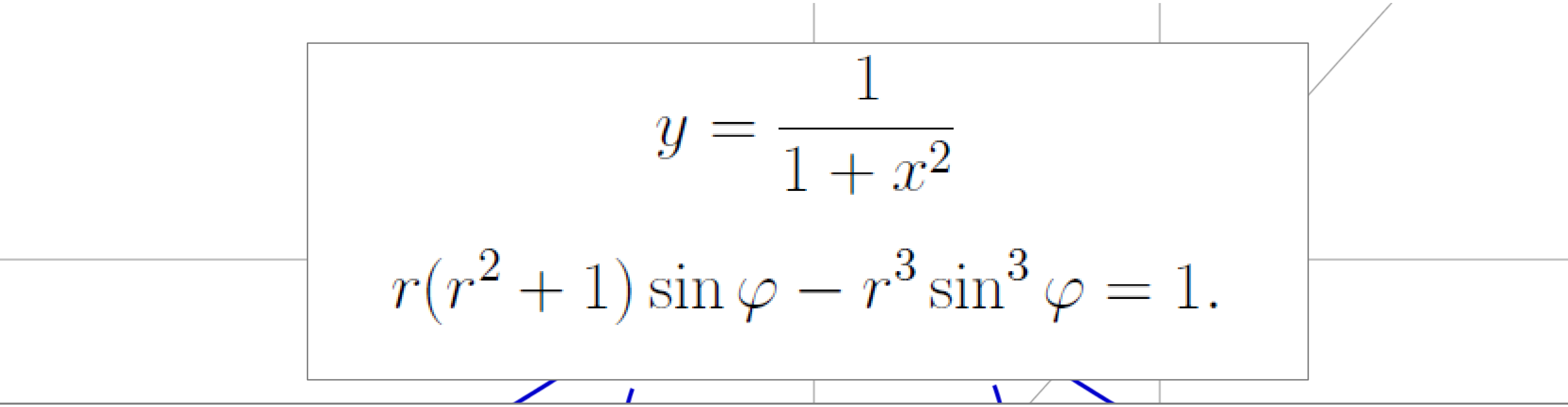


$$y = \frac{1}{1+x^2}$$

$$r(r^2+1)\sin\varphi - r^3\sin^3\varphi = 1.$$

Witch of Agnesi



$$y = \frac{1}{1+x^2}$$

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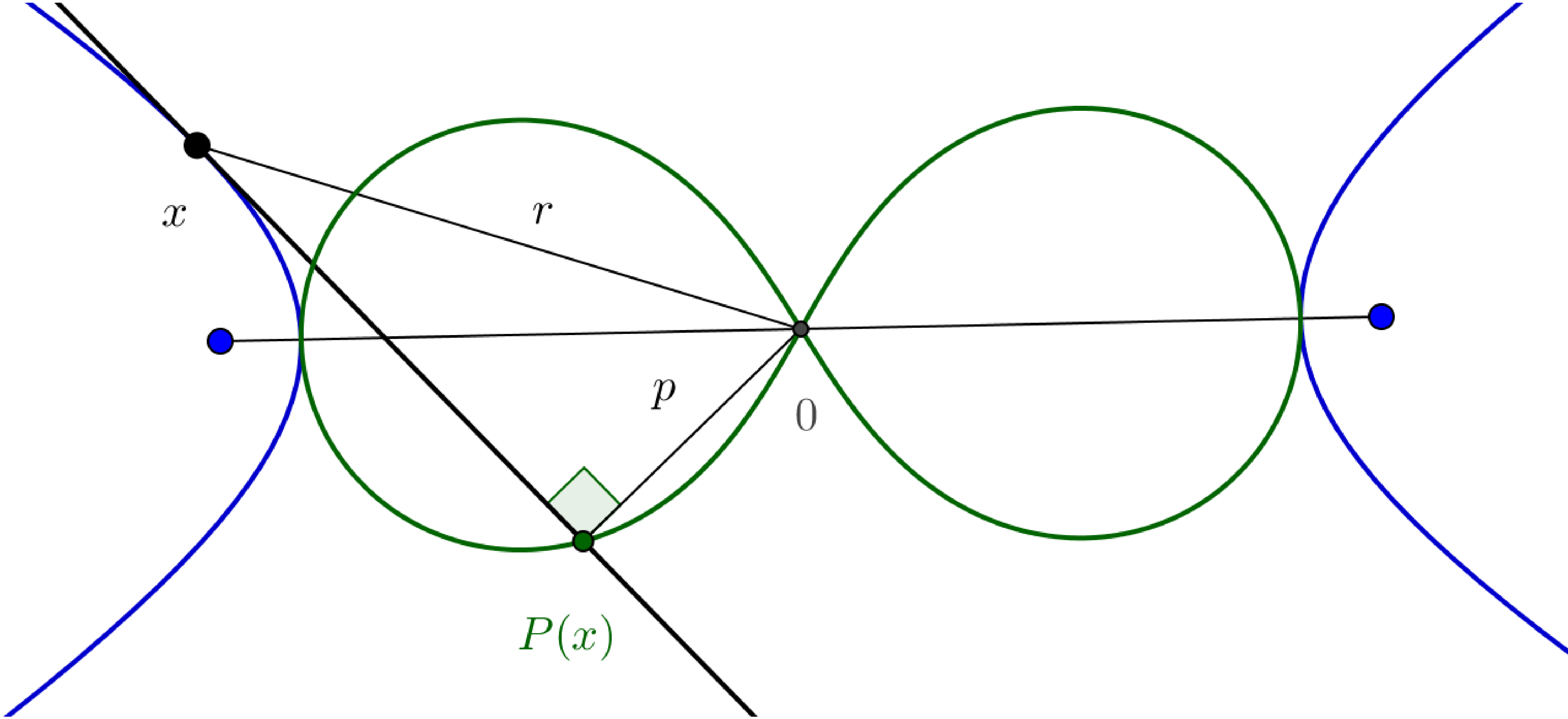
$$1/36\,\frac{C^2}{p^2} + 1/3\,\frac{p^2-2\,r^2+2}{p^2} + 4\,\frac{r^4+4\,p^2-2\,r^2+1}{p^2C^2} + 192\,\frac{r^2\left(r^2+1\right)^2(p-r)(p+r)}{p^2C^3B} - 8\,\frac{\left(r^2+1\right)^2(p-r)(p+r)}{p^2CB}$$

$$-1152\,\frac{\left(r^2+1\right)^3(r-1)(r+1)(p-r)(p+r)}{p^2C^5B} - 576\,\frac{r^2\left(r^2+1\right)^4(p-r)(p+r)}{p^2C^4B^2} - 82944\,\frac{r^2\left(r^2+1\right)^6(p-r)(p+r)}{p^2C^8B^2}$$

$$+13824\,\frac{r^2\left(r^2+1\right)^5(p-r)(p+r)}{p^2C^6B^2} = 0,$$

$$C := \sqrt[3]{12B-108}, \qquad B := \sqrt{-12r^6-36r^4-36r^2+69}.$$

Transformations of curves



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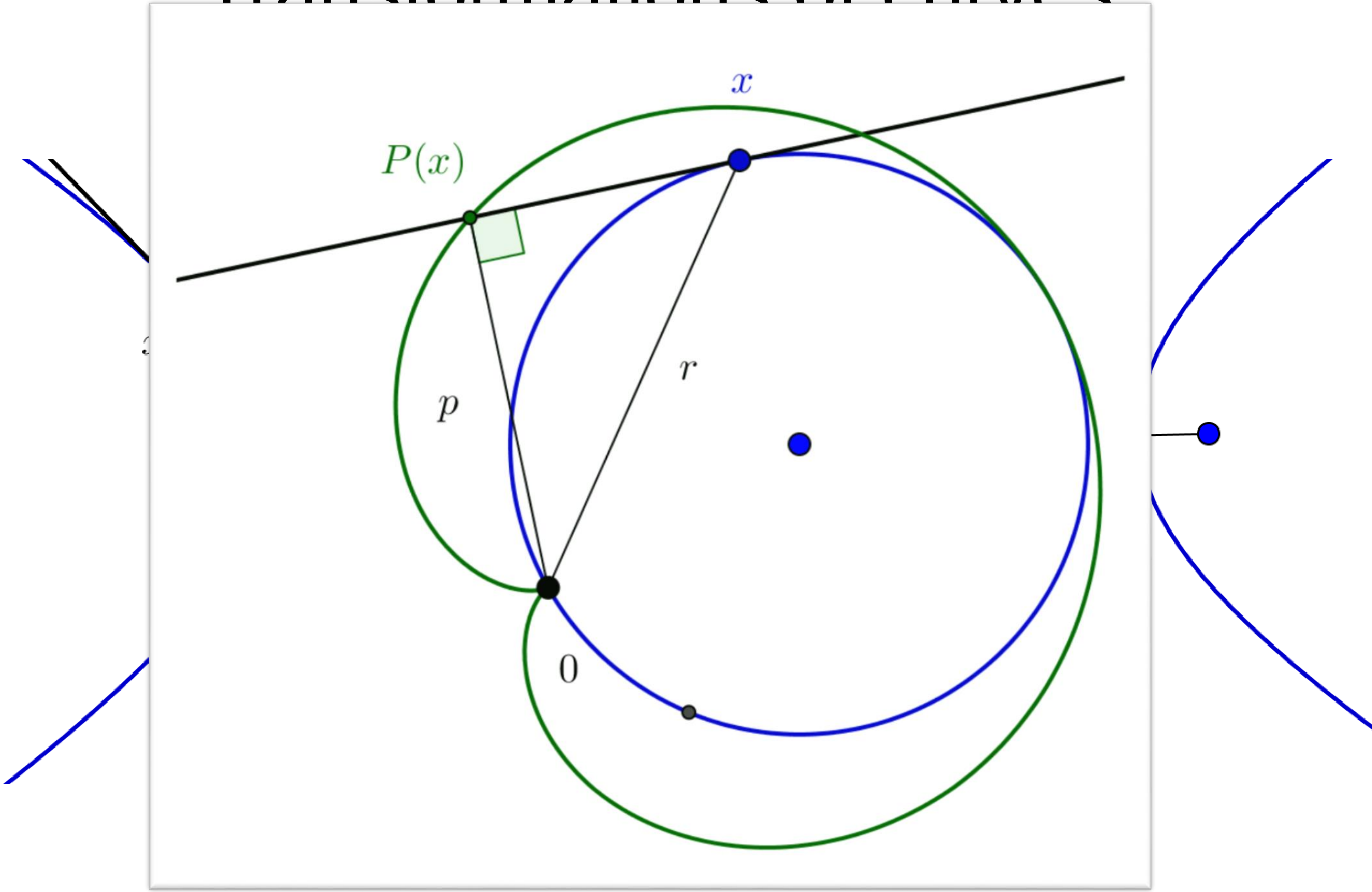
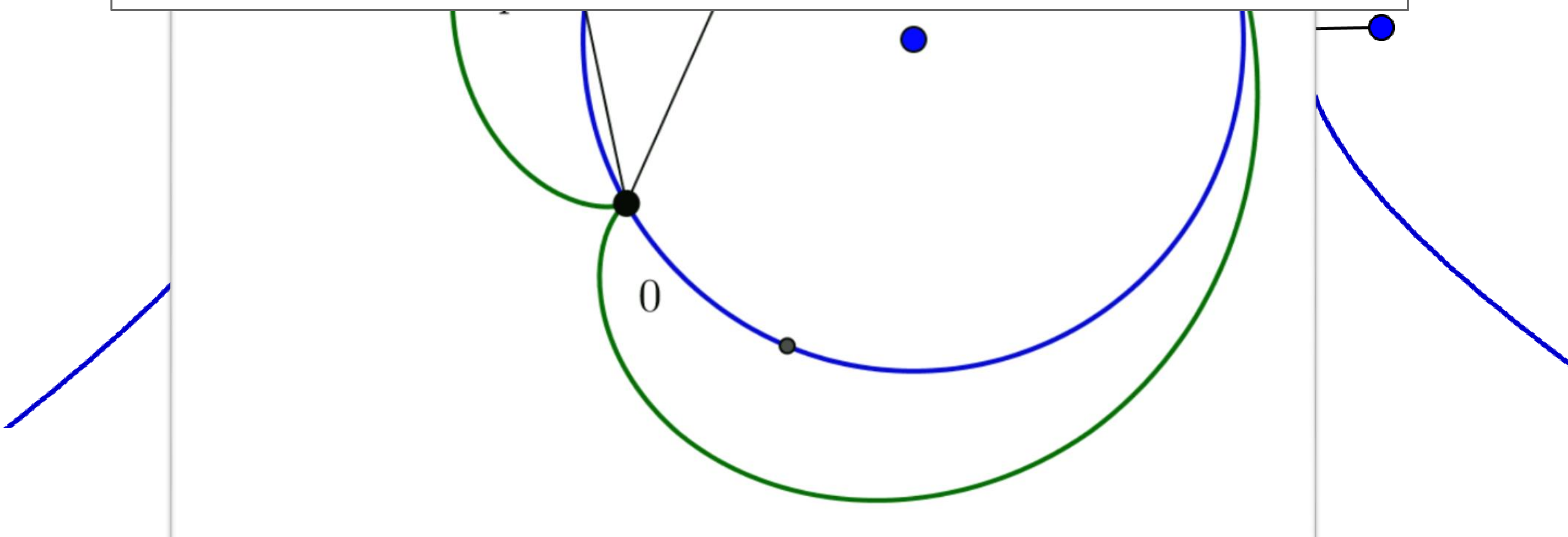


Transformations of curves

$$x \rightarrow x - \frac{\dot{x}x + \dot{y}y}{x^2 + y^2}\dot{x},$$

$$y \rightarrow y - \frac{\dot{x}x + \dot{y}y}{x^2 + y^2}\dot{y}.$$



Transformations of curves

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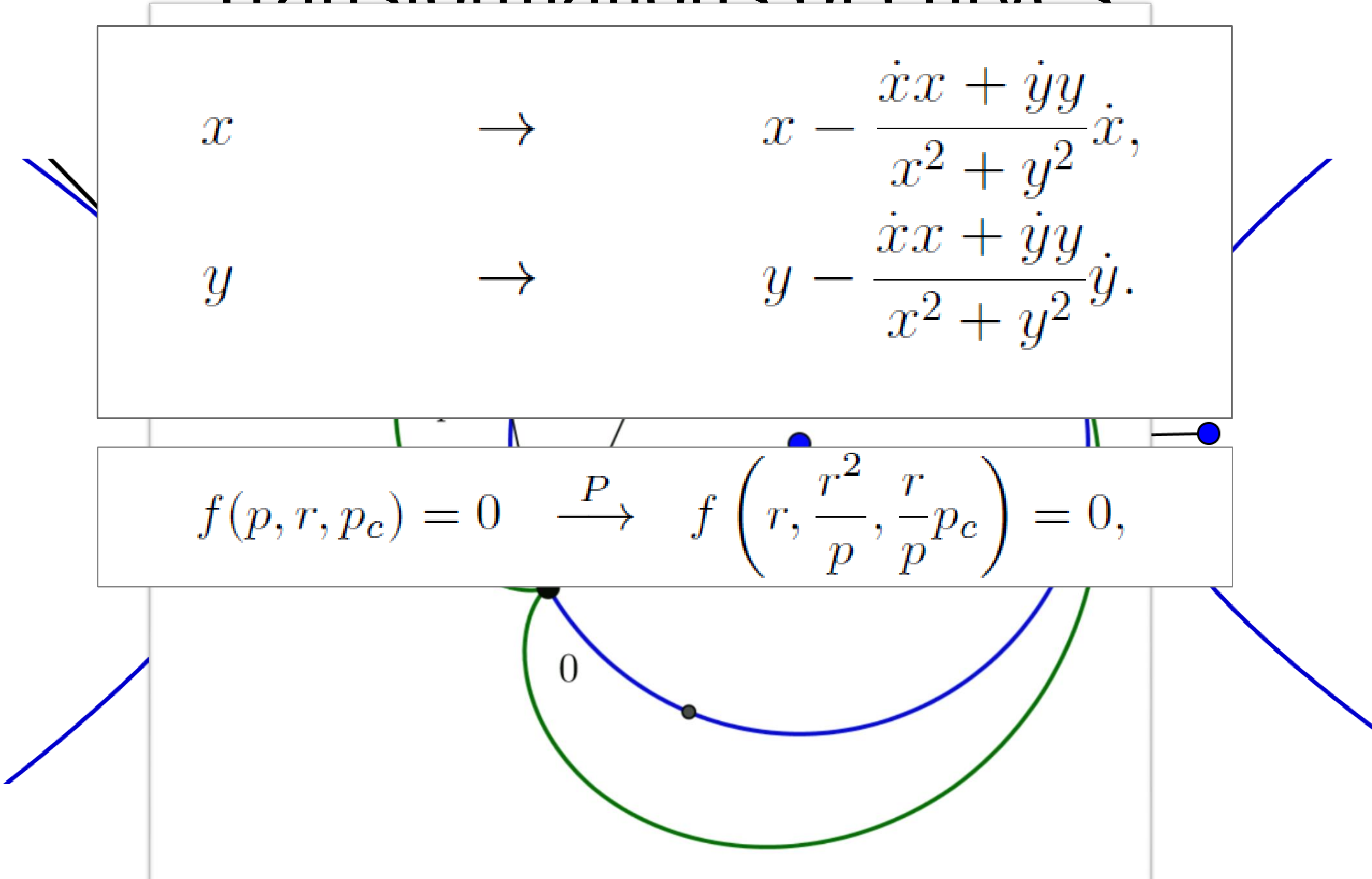
$$y \rightarrow y - \frac{\dot{x}x + \dot{y}y}{x^2 + y^2}\dot{y}.$$

$$f(p, r, p_c) = 0 \quad \xrightarrow{P} \quad f\left(r, \frac{r^2}{p}, \frac{r}{p}p_c\right) = 0,$$

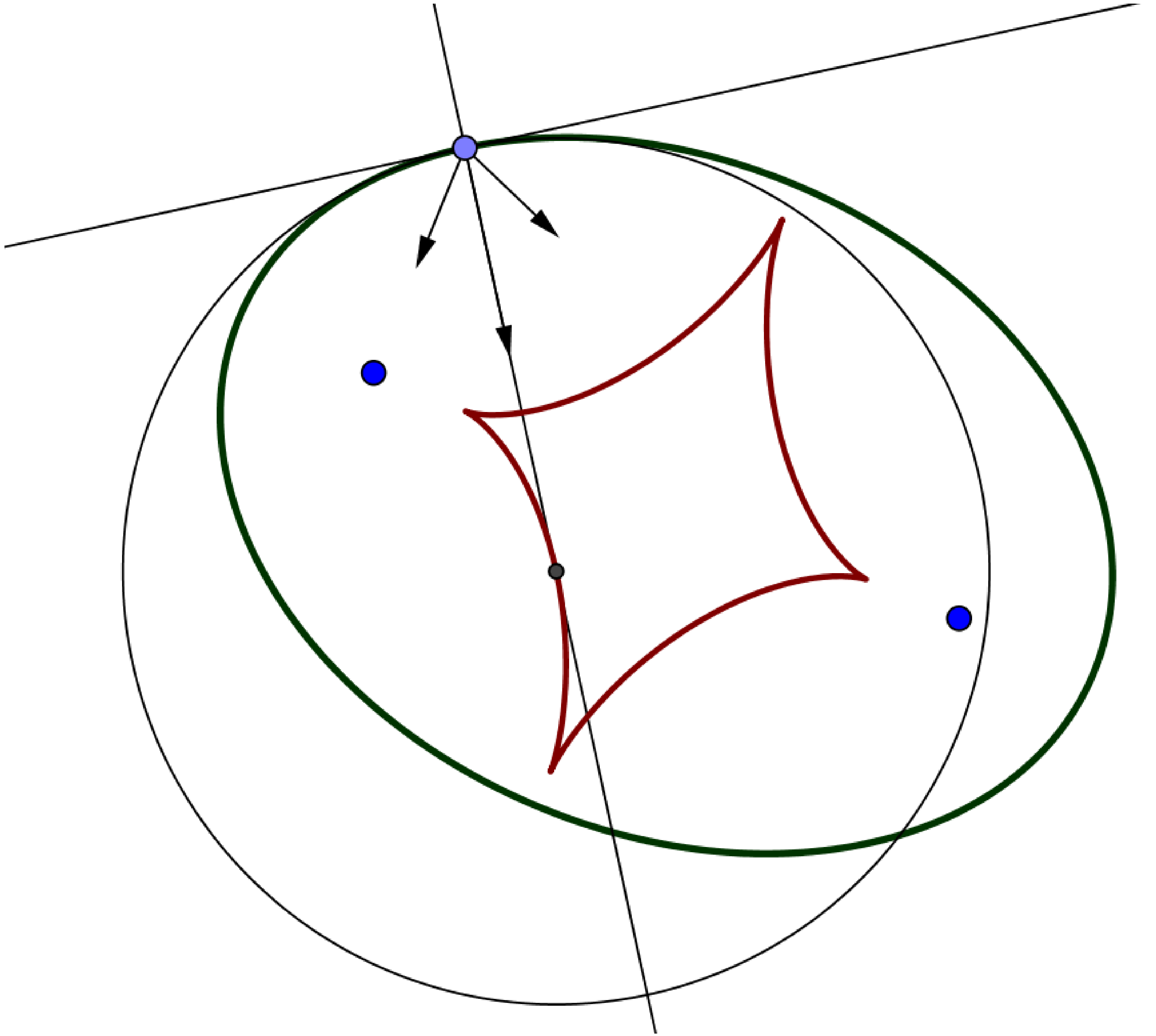


Transformations of curves

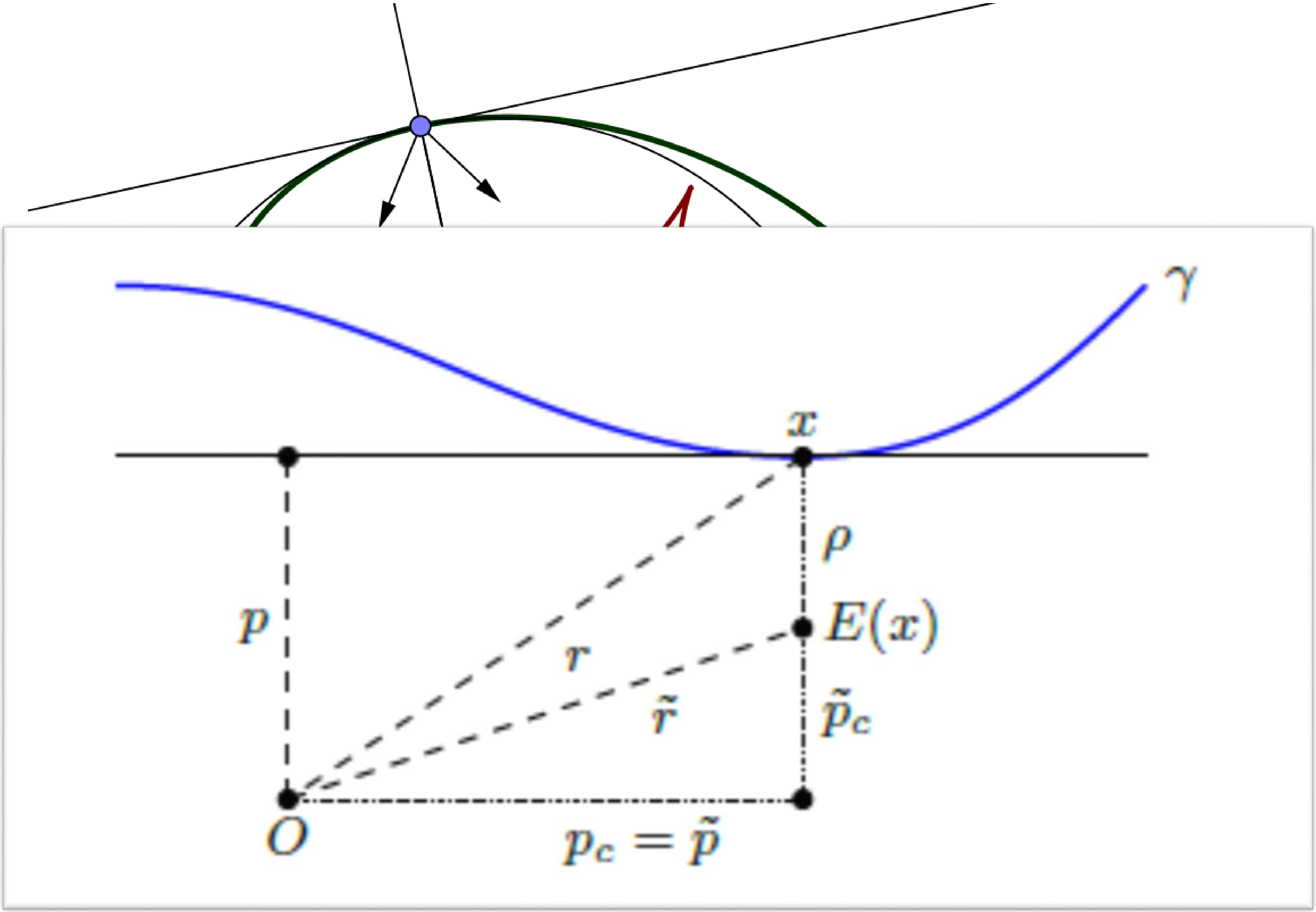
$$x \quad \to \quad x - \frac{\dot{x}x + \dot{y}y}{\cdots}\dot{x},$$

$$
\begin{aligned}
f(p, r, p_c) = 0 &\xrightarrow{S_\alpha} f(\alpha p, \alpha r, \alpha p_c) = 0,\\
f(p, r, p_c) = 0 &\xrightarrow{P} f\left(r, \frac{r^2}{p}, \frac{r}{p}p_c\right) = 0,\\
f(p, r, p_c) = 0 &\xrightarrow{I_R} f\left(\frac{Rp}{r^2}, \frac{R}{r}, \frac{R}{r^2}p_c\right) = 0,\\
f\left(p, r^2, p_c\right) = 0 &\xrightarrow{E_c} f\left(p - c, r^2 - 2pc + c^2, p_c\right) = 0,\\
f\left(\frac{1}{p^2}, r; \frac{r^2}{p^2}\right) = 0 &\xrightarrow{J_\alpha} f\left(r^{2-2\alpha}\left(\frac{\alpha^2}{p^2} + \frac{1-\alpha^2}{r^2}\right), r^\alpha; \alpha^2\frac{r^2}{p^2} + \alpha^2 - 1\right) = 0,\\
f\left(\frac{1}{p^2}, r\right) = 0 &\xrightarrow{H_k} f\left(\frac{k^2}{p^2} - \frac{k^2-1}{r^2}, r\right) = 0,\\
\frac{\left(L - G(r^2)\right)^2}{p^2} = F(r^2) + c &\xrightarrow{R_\omega} \frac{\left(L - G(r^2) + \omega r^2\right)^2}{p^2} = F(r^2) + \omega^2 r^2 - 2G\omega + c + 2\omega L,
\end{aligned}
$$

Evolute



Evolute



Evolute

Proposition 2. *The evolute $E(\gamma)$ of a curve γ which satisfies*

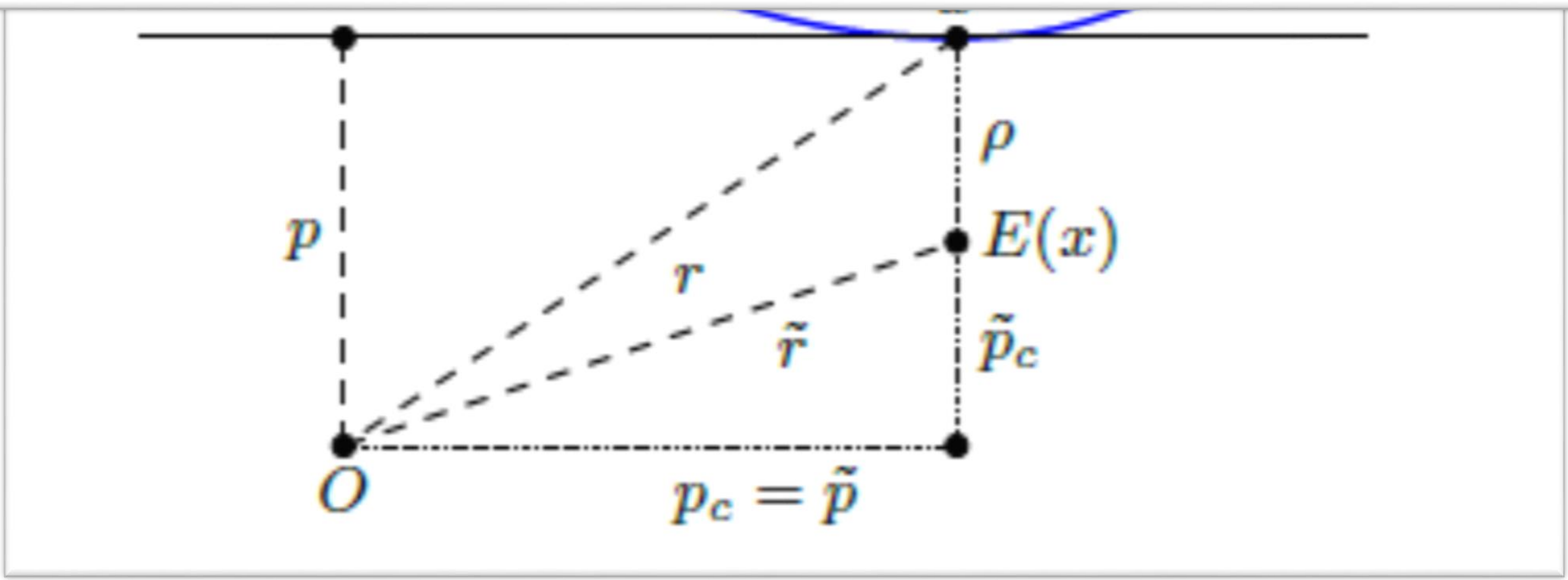
$$f\left(p_c, p_c p_c', (p_c p_c')' p_c, \ldots, (p_c \partial_p)^n p\right) = 0,$$

where $n > 1$, satisfies

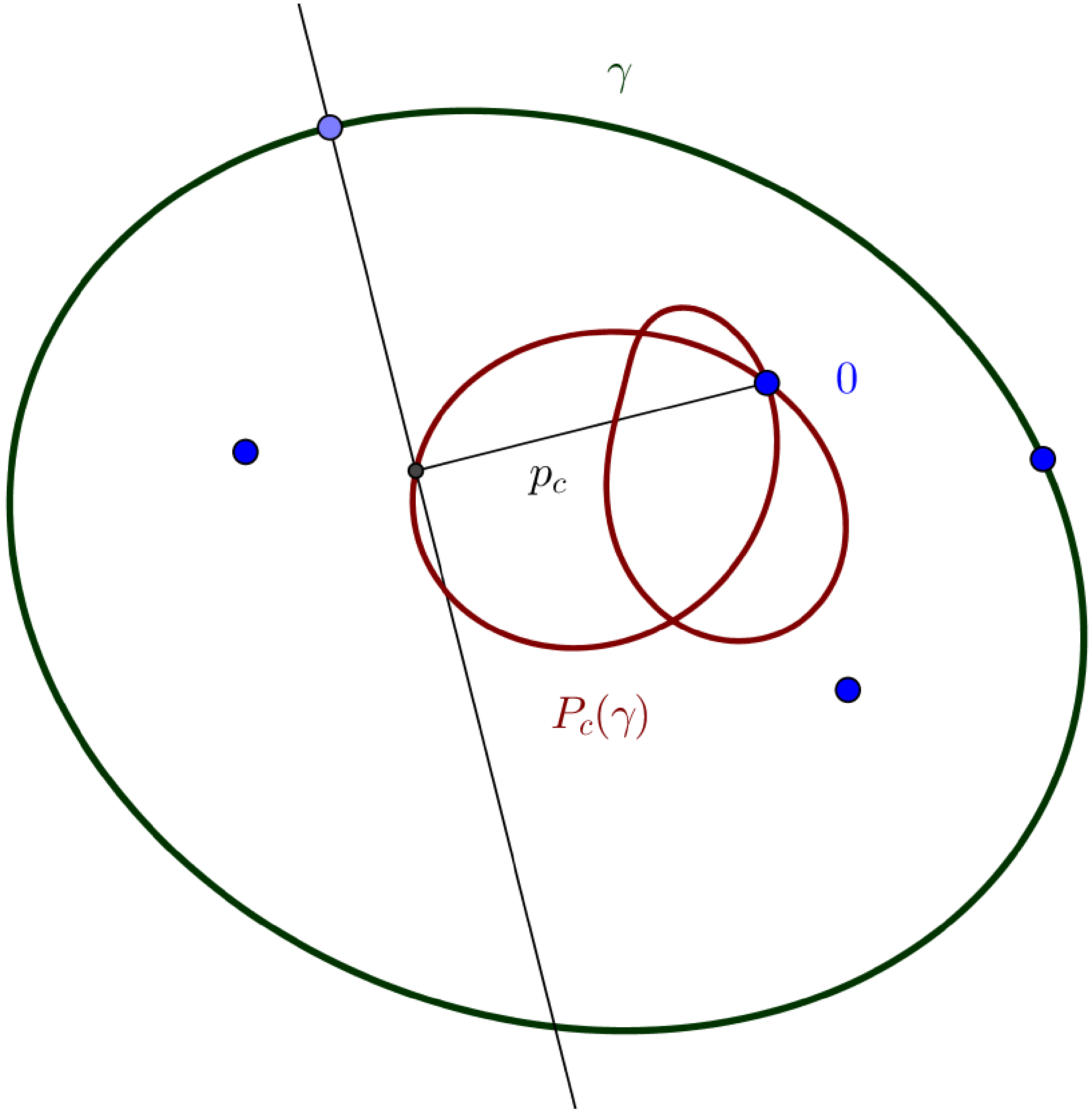
$$f\left(p, p_c, p_c p_c', (p_c p_c')' p_c, \ldots, (p_c \partial_p)^{n-1} p\right) = 0.$$

In other words

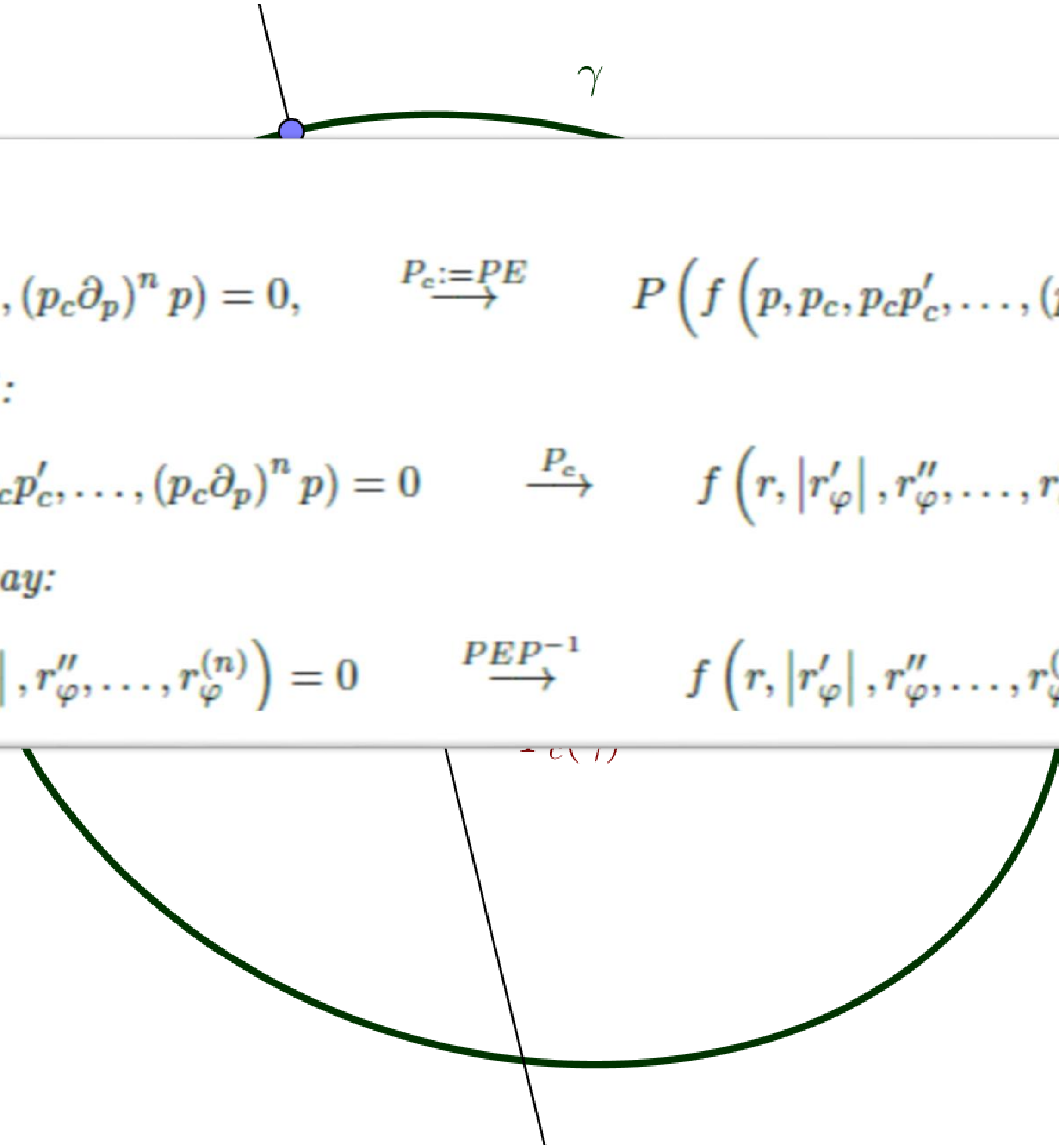
$$f\left(p_c, p_c p_c', \ldots, (p_c \partial_p)^n p\right) = 0, \quad \xrightarrow{E} \quad f\left(p, p_c, p_c p_c', \ldots, (p_c \partial_p)^{n-1} p\right) = 0.$$



Contrapedal



Contrapedal



Corollary 2.

$$f\left(p_c, p_c p_c', \ldots, (p_c \partial_p)^n p\right) = 0, \quad \xrightarrow{P_c := PE} \quad P\left(f\left(p, p_c, p_c p_c', \ldots, (p_c \partial_p)^{n-1} p\right) = 0\right),$$

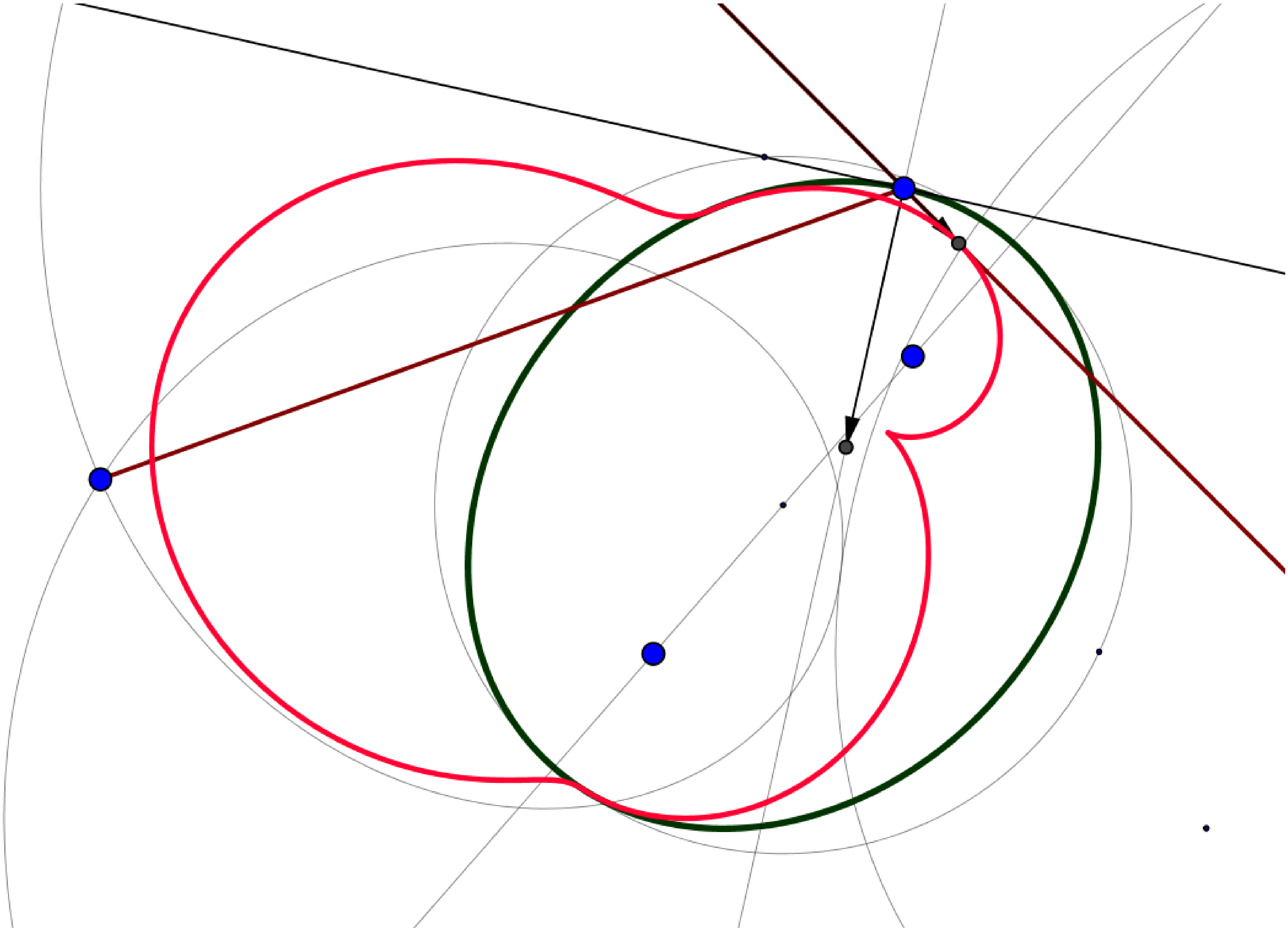
or using Proposition 1:

$$f\left(p_c, p_c p_c', \ldots, (p_c \partial_p)^n p\right) = 0 \quad \xrightarrow{P_c} \quad f\left(r, \left|r_\varphi'\right|, r_\varphi'', \ldots, r_\varphi^{(n-1)}\right) = 0.$$

Equivalently, we can say:

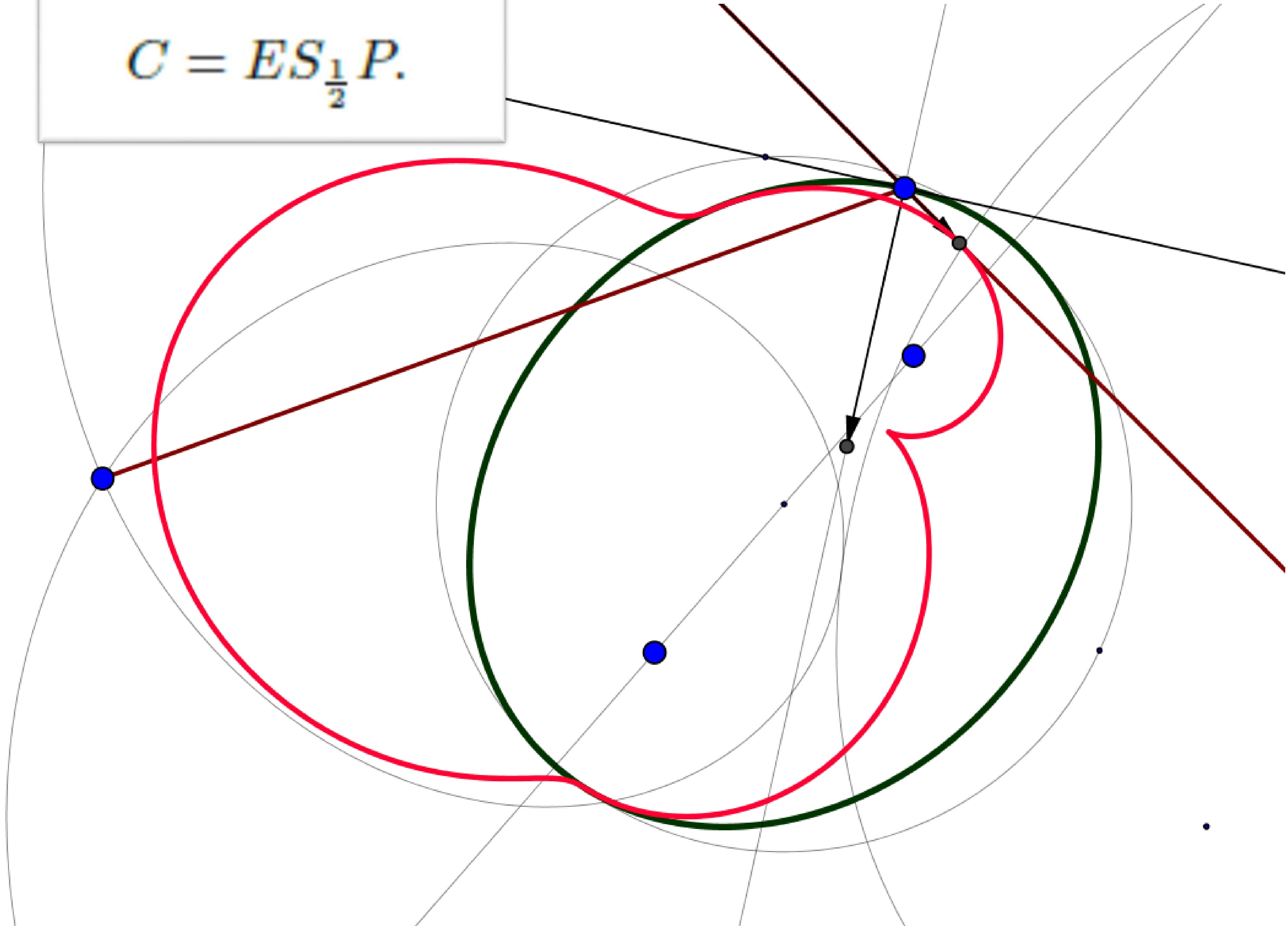
$$f\left(\left|r_\varphi'\right|, r_\varphi'', \ldots, r_\varphi^{(n)}\right) = 0 \quad \xrightarrow{PEP^{-1}} \quad f\left(r, \left|r_\varphi'\right|, r_\varphi'', \ldots, r_\varphi^{(n-1)}\right) = 0.$$

Catacaustic

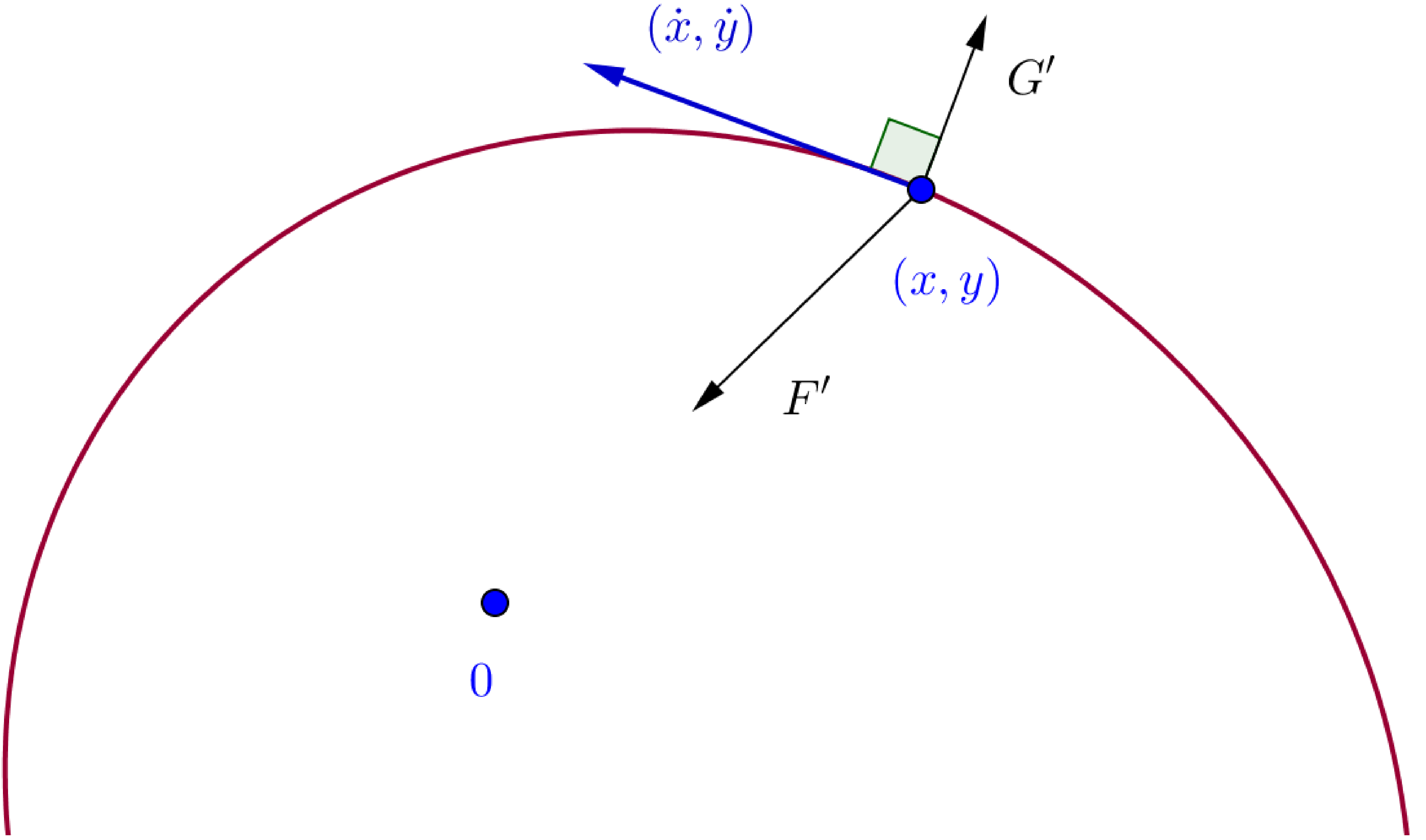


Catacaustic

$$C = ES_{\frac{1}{2}}P.$$

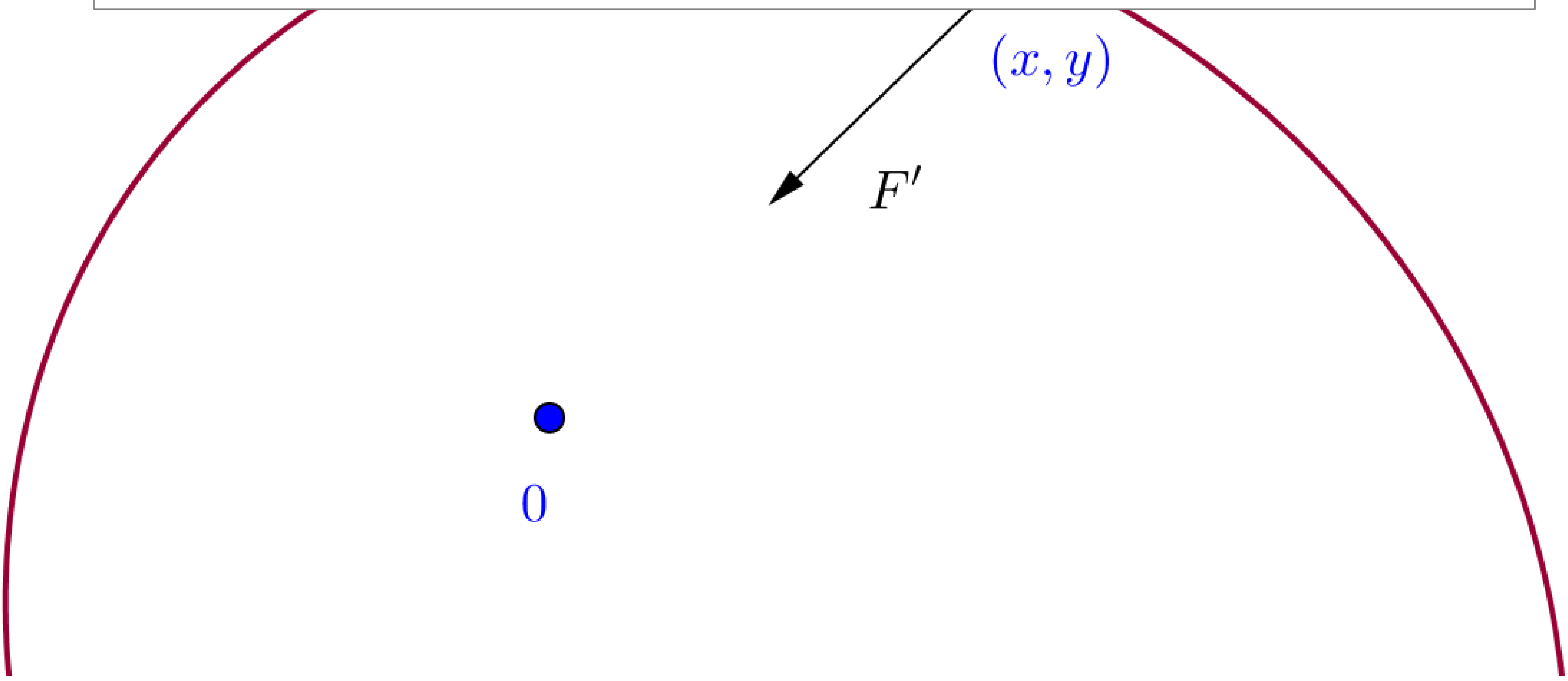


Force problems



Force problems

$$\ddot{x} = F'\left(x^2+y^2\right)x - 2G'\left(x^2+y^2\right)\dot{y},$$
$$\ddot{y} = F'\left(x^2+y^2\right)y + 2G'\left(x^2+y^2\right)\dot{x}.$$



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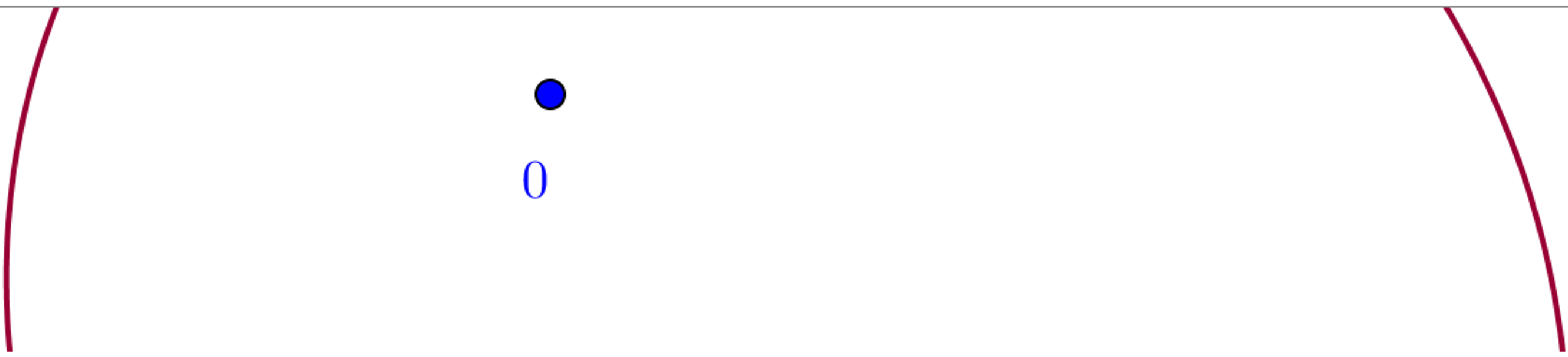
$$\ddot{y} = F'\left(x^2+y^2\right)y + 2G'\left(x^2+y^2\right)\dot{x}.$$

$$rr'' - 2r'^2 + 2G'(r^2)r'^2 - r^2 + \frac{2r^4G'(r^2)}{(G(r^2)+L)} = \frac{F'(r^2)r^6}{(G+L)^2}$$



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$$\frac{\left(L - G(r^2)\right)^2}{p^2} = F(r^2) + c.$$

Force problems generalized

$$\mathrm{x} := (x, y), \qquad \mathrm{x}^{\perp} := (-y, x), \qquad \mathrm{x} \cdot \mathrm{y}^{\perp} = -\mathrm{x}^{\perp} \cdot \mathrm{y}, \qquad \left(\mathrm{x}^{\perp}\right)^{\perp} = -\mathrm{x}.$$

$$p := \frac{\mathrm{x}^{\perp} \cdot \dot{\mathrm{x}}}{|\dot{\mathrm{x}}|}, \qquad p_c := \frac{\mathrm{x} \cdot \dot{\mathrm{x}}}{|\dot{\mathrm{x}}|}, \qquad \kappa := \frac{\dot{\mathrm{x}}^{\perp} \cdot \ddot{\mathrm{x}}}{|\dot{\mathrm{x}}|^3}$$

$$\ddot{\mathrm{x}} = \frac{\partial_t \left(|\dot{\mathrm{x}}|\right)}{p_c}\mathrm{x} + \frac{\partial_t \left(|\dot{\mathrm{x}}|\, p\right)}{p_c \, |\dot{\mathrm{x}}|}\dot{\mathrm{x}}^{\perp}, \qquad \mathrm{x}^{\perp} = \frac{p}{p_c}\mathrm{x} + \frac{r^2}{p_c \, |\dot{\mathrm{x}}|}\dot{\mathrm{x}}^{\perp}, \qquad \dot{\mathrm{x}} = \frac{|\dot{\mathrm{x}}|}{p_c}\mathrm{x} + \frac{p}{p_c}\dot{\mathrm{x}}^{\perp}.$$

Force problems generalized

$$\ddot{\mathrm{x}} = f\mathrm{x} + g\dot{\mathrm{x}}^{\perp}, \qquad \frac{\left(\int gr\mathrm{d}r\right)^2}{p^2} = 2\int fr\mathrm{d}r,$$

$$\mathrm{x} := (x, y), \qquad \mathrm{x}^{\perp} := (-y, x), \qquad \mathrm{x}\cdot\mathrm{y}^{\perp} = -\mathrm{x}^{\perp}\cdot\mathrm{y}, \qquad \left(\mathrm{x}^{\perp}\right)^{\perp} = -\mathrm{x}.$$

$$p := \frac{\mathrm{x}^{\perp}\cdot\dot{\mathrm{x}}}{|\dot{\mathrm{x}}|}, \qquad p_c := \frac{\mathrm{x}\cdot\dot{\mathrm{x}}}{|\dot{\mathrm{x}}|}, \qquad \kappa := \frac{\dot{\mathrm{x}}^{\perp}\cdot\ddot{\mathrm{x}}}{|\dot{\mathrm{x}}|^3}$$

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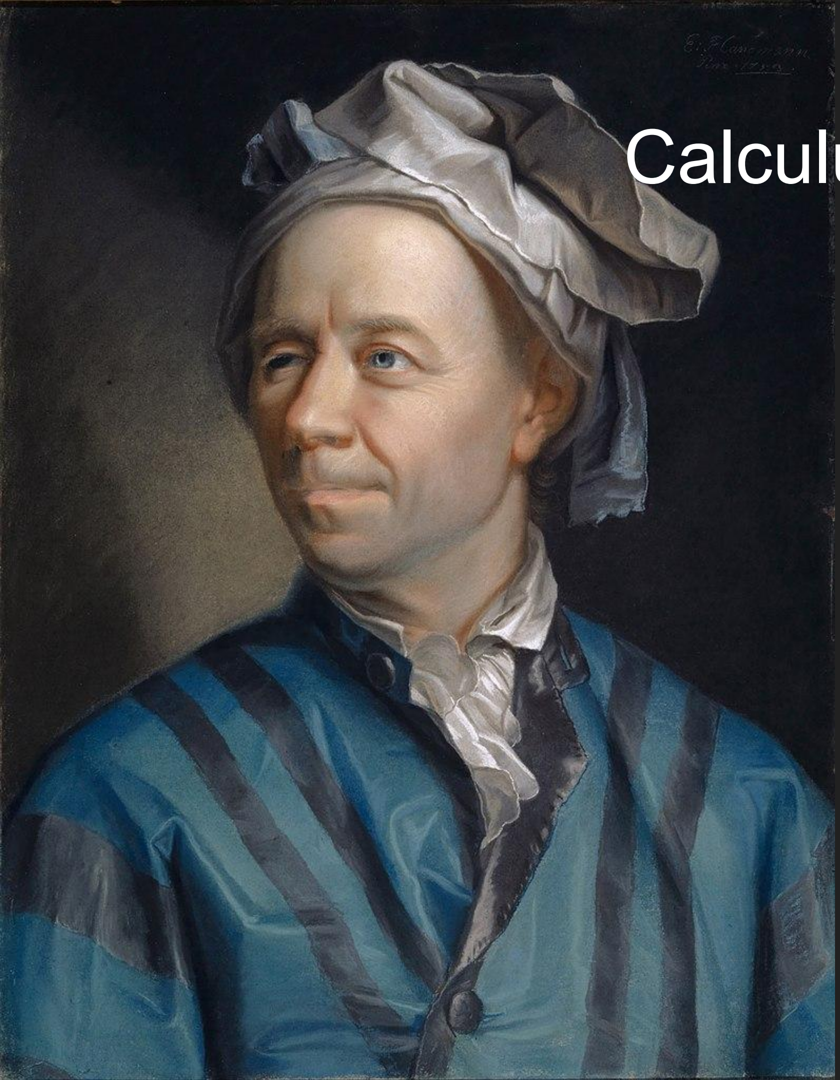
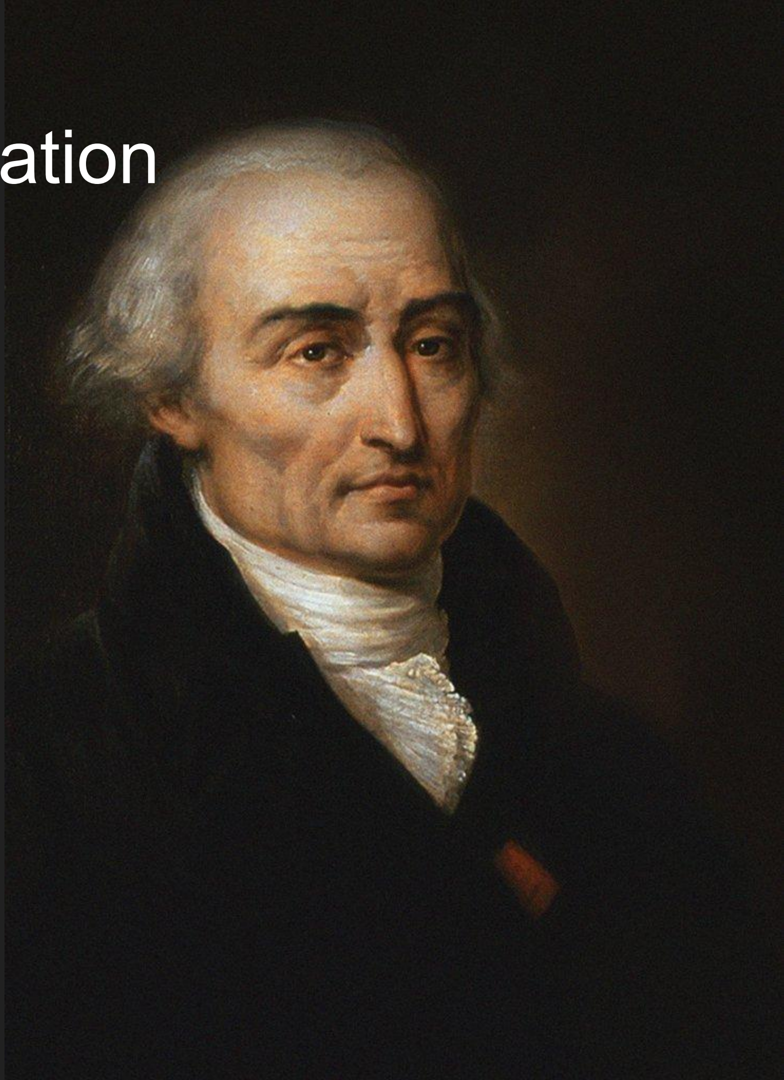
$$\ddot{\mathrm{x}} = f\frac{\mathrm{x}}{|\dot{\mathrm{x}}|^{\alpha}} + g\frac{\dot{\mathrm{x}}^{\perp}}{|\dot{\mathrm{x}}|^{\beta}}, \qquad \left(\frac{(1+\beta)\int g p^{\beta} r \mathrm{d}r}{p^{1+\beta}}\right)^{2+\alpha} = \left((2+\alpha)\int f r \mathrm{d}r\right)^{1+\beta}, \qquad \begin{matrix}\alpha \neq -2\\ \beta \neq -1\end{matrix}$$

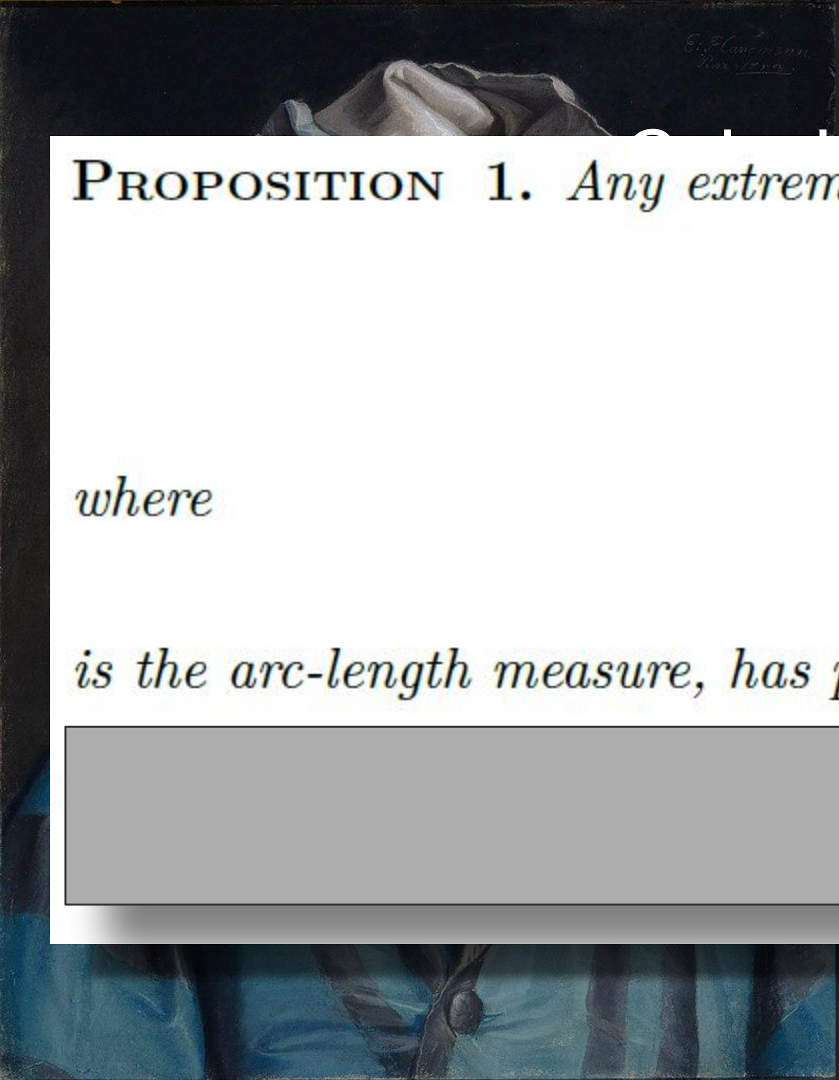
Calculus of variation





Proposition 1. *Any extremal curve of the functional:*

$$\mathcal{L}[r] := \int_{s_0}^{s_1} f(r)\mathrm{d}s,$$



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Brachistochone

$$dt = \frac{ds}{|\dot{\mathbf{x}}|},$$

chistochone

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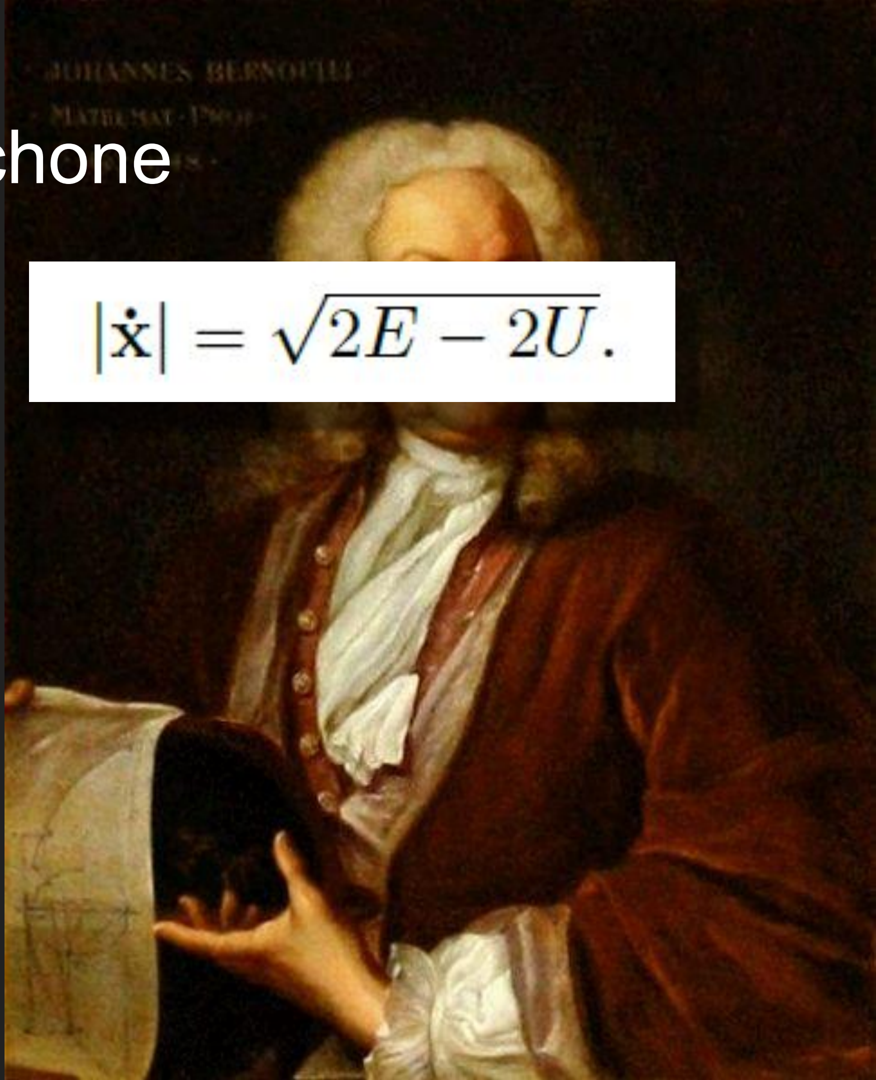

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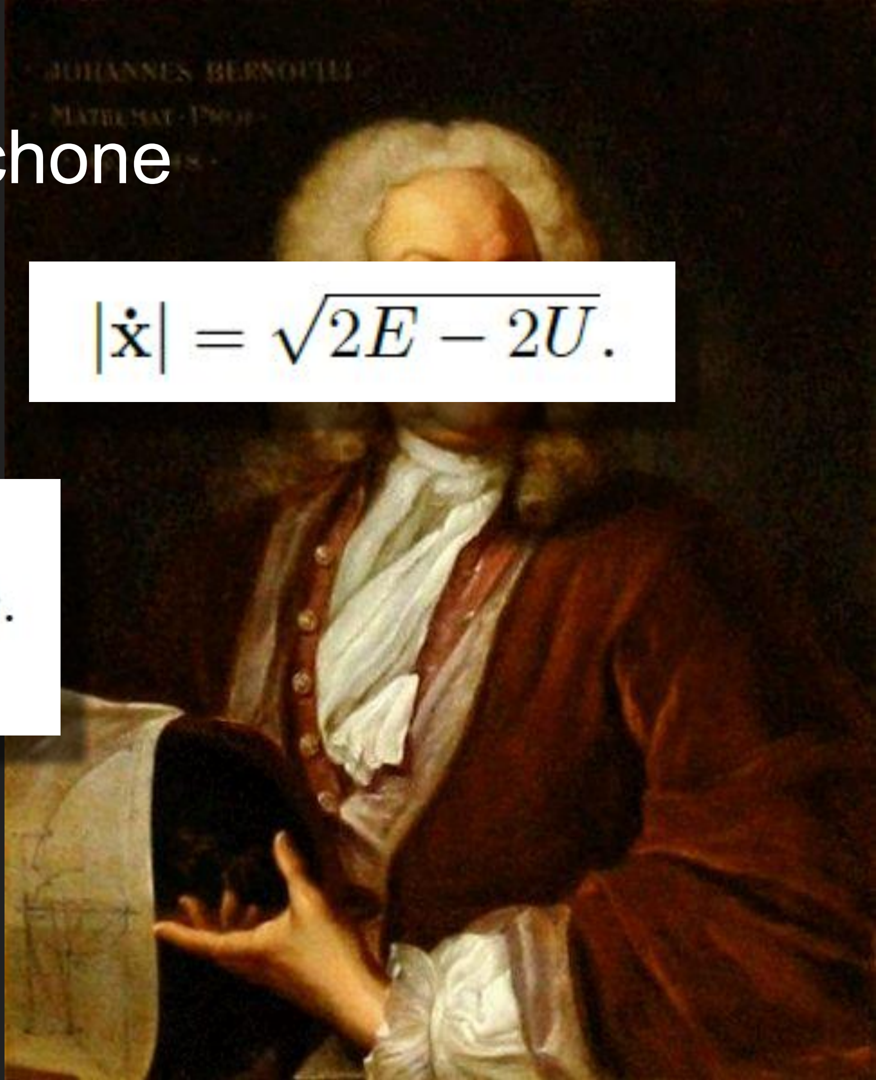

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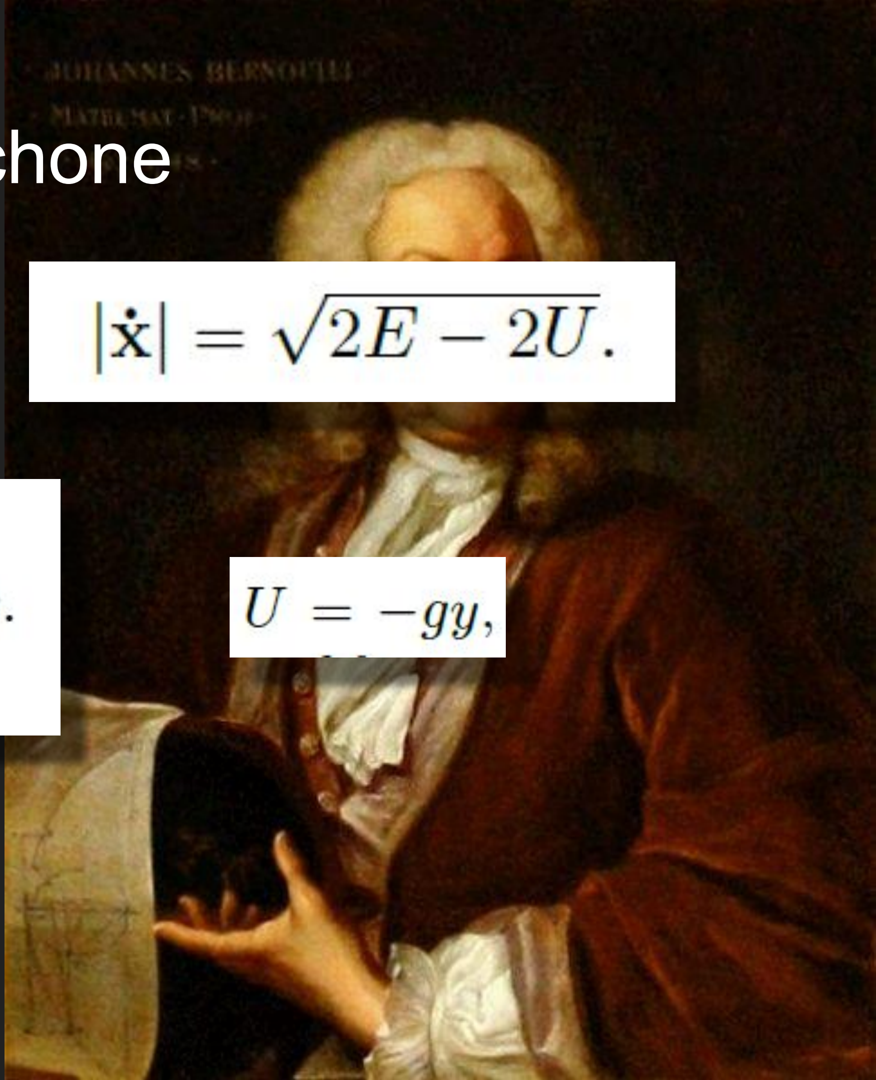

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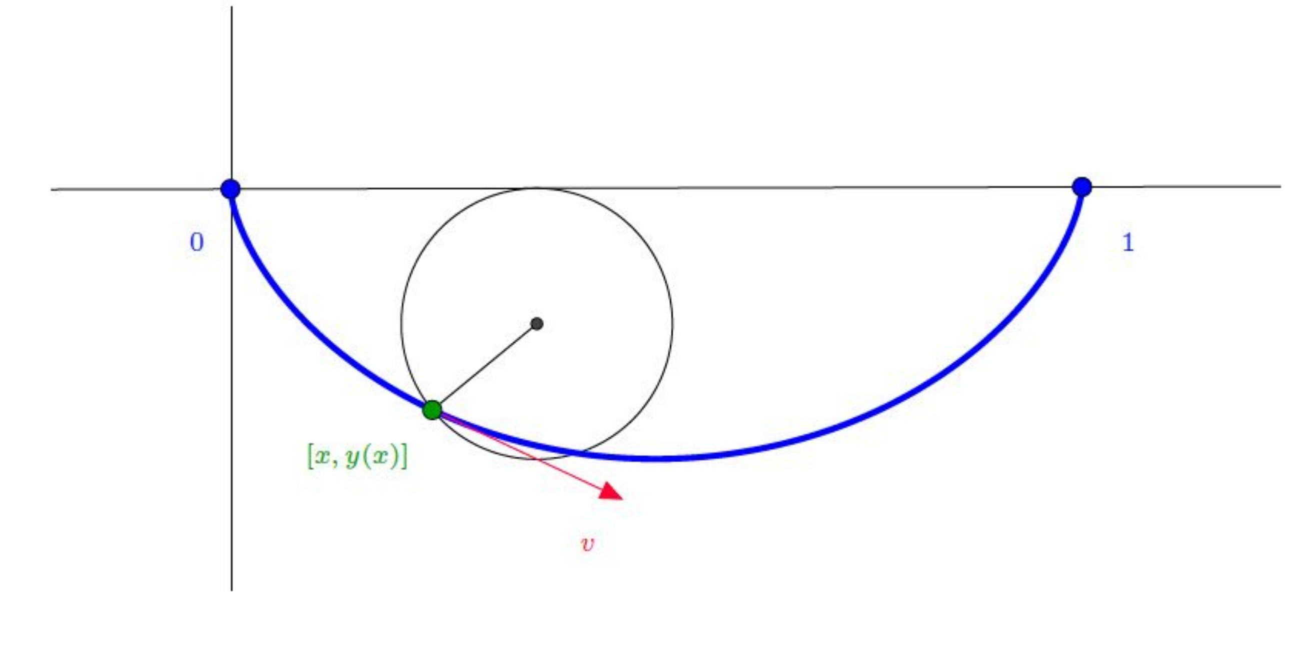

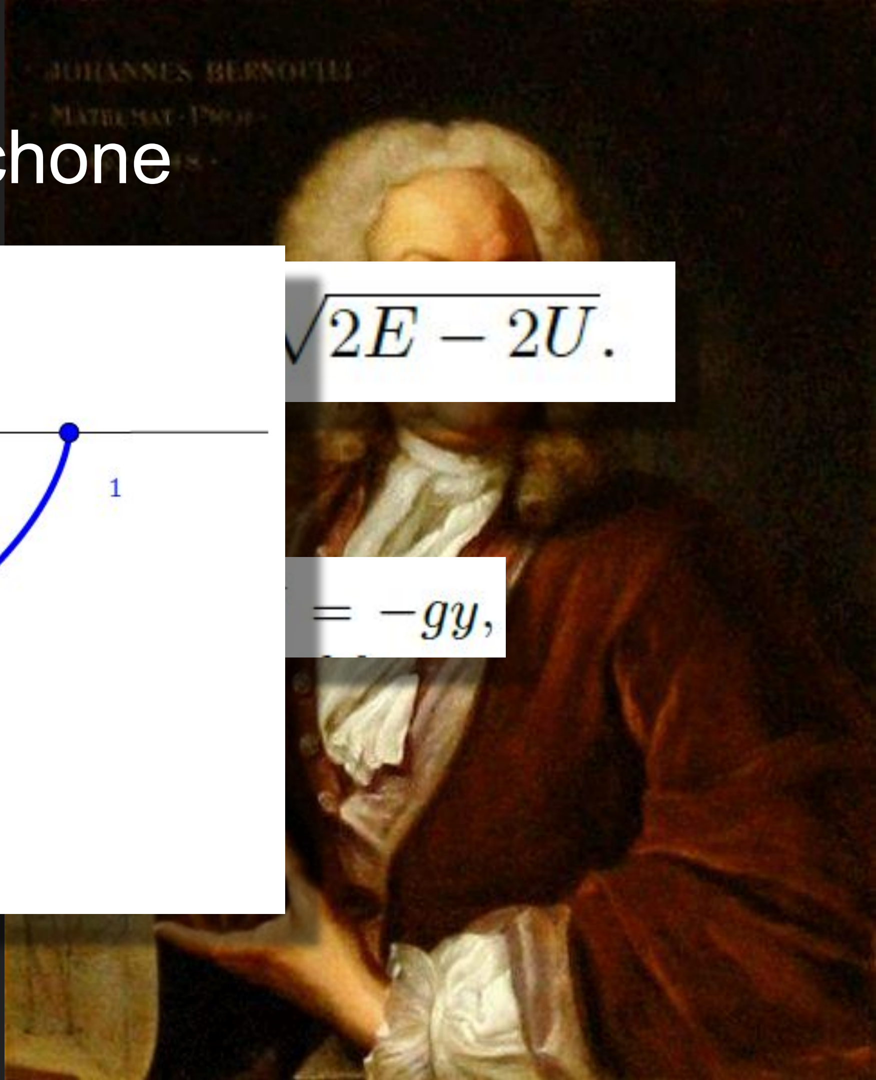

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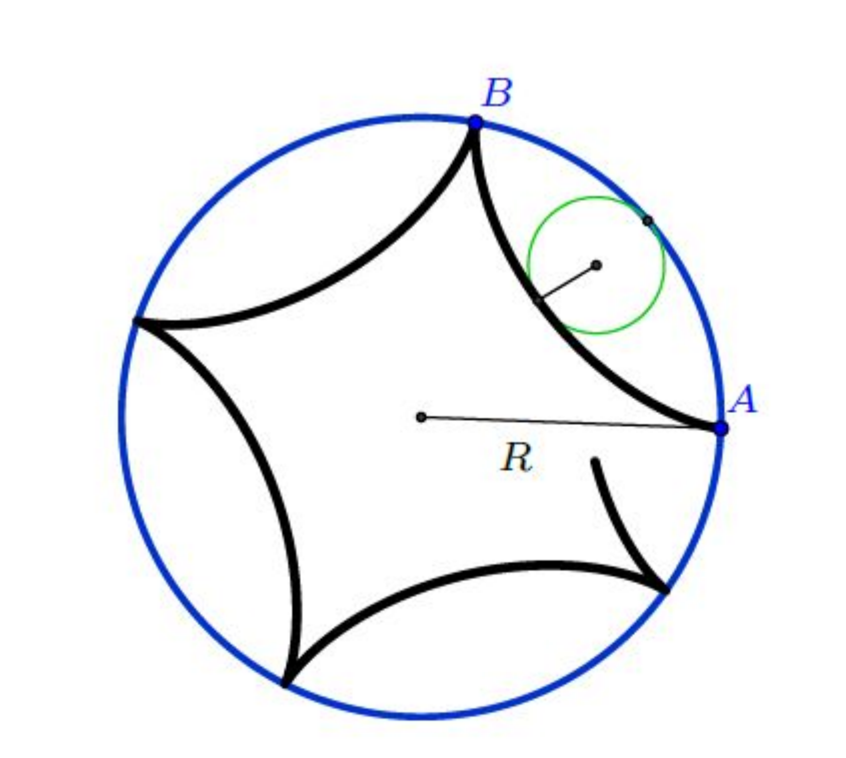

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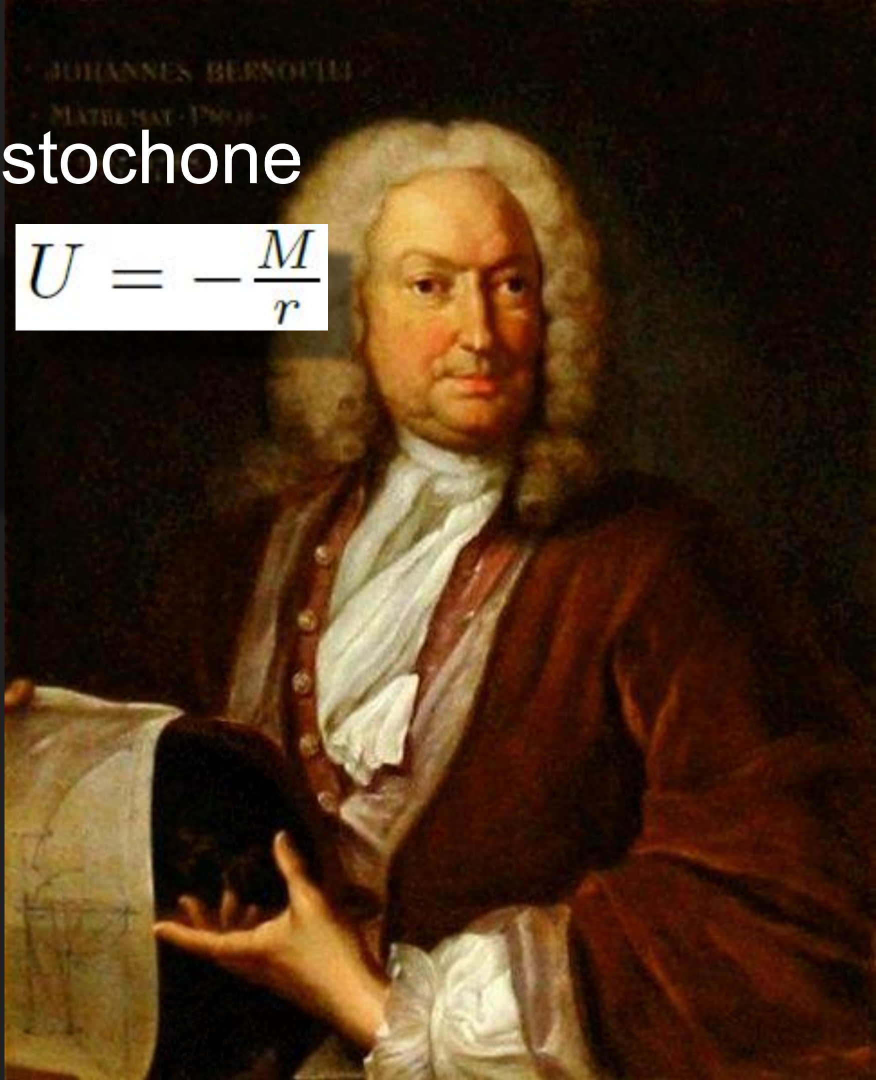

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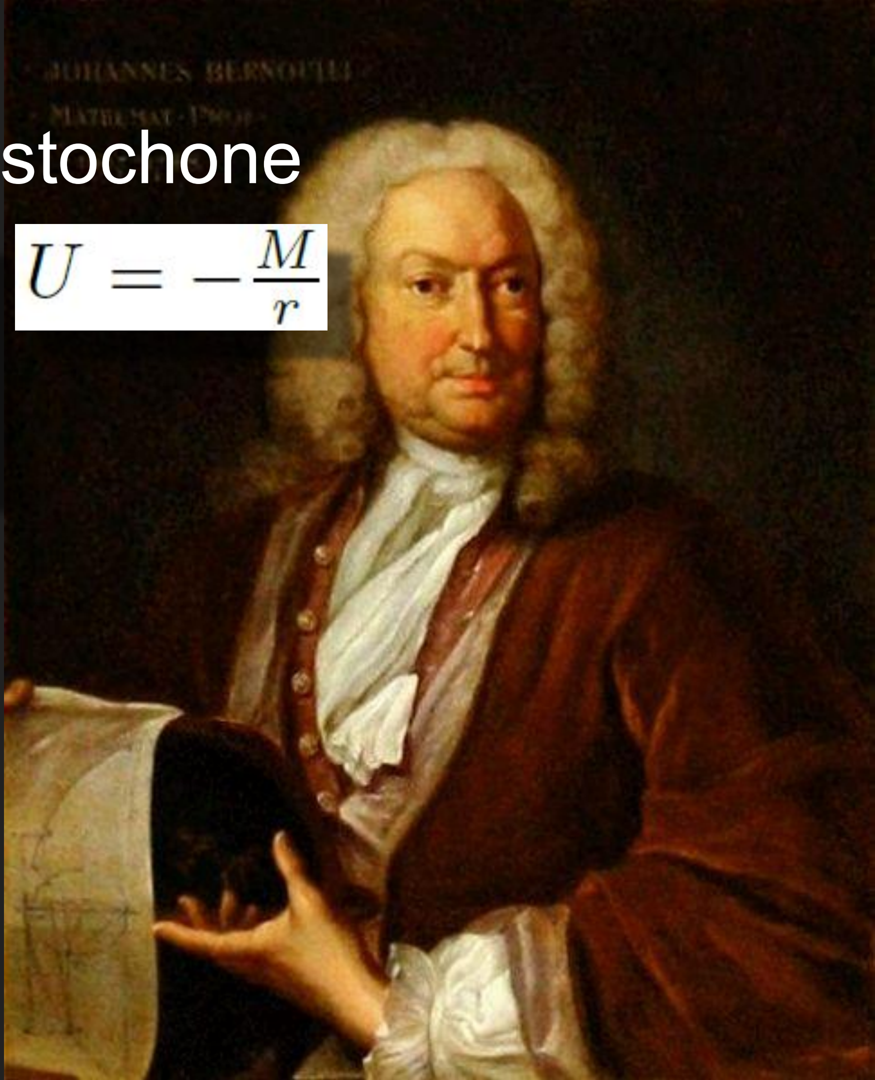

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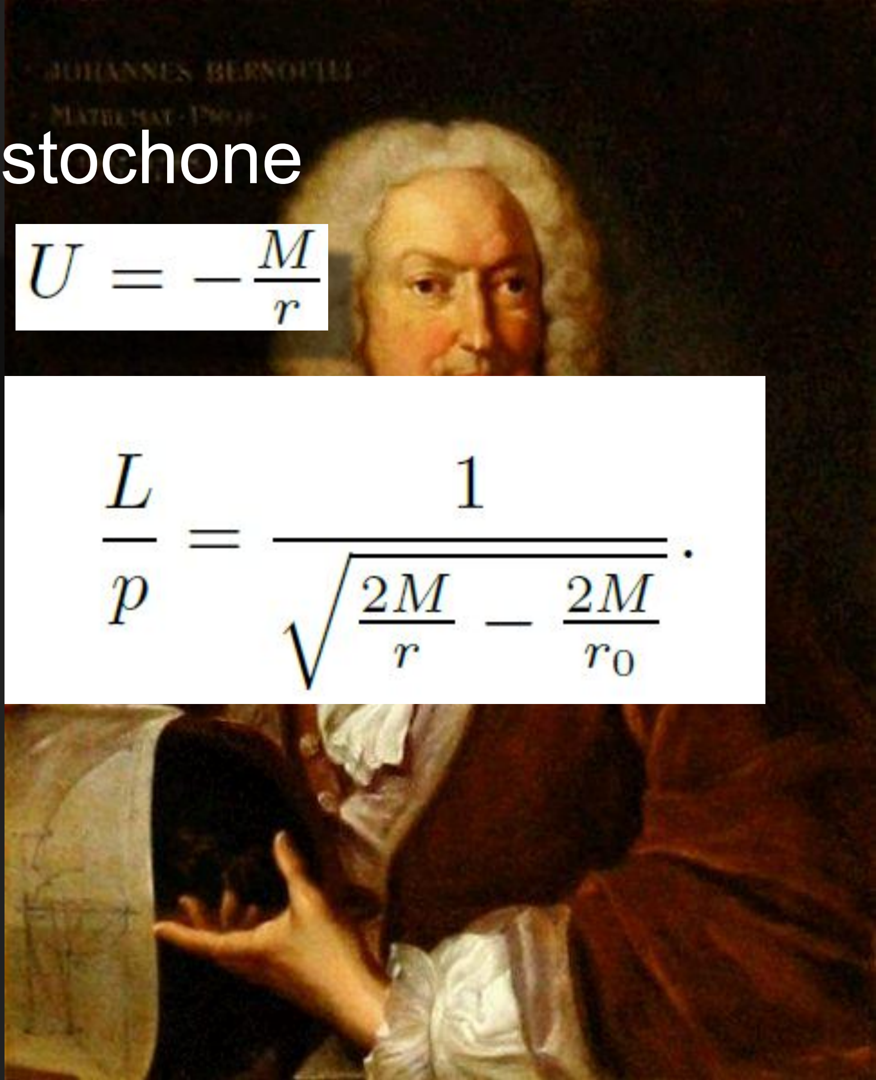

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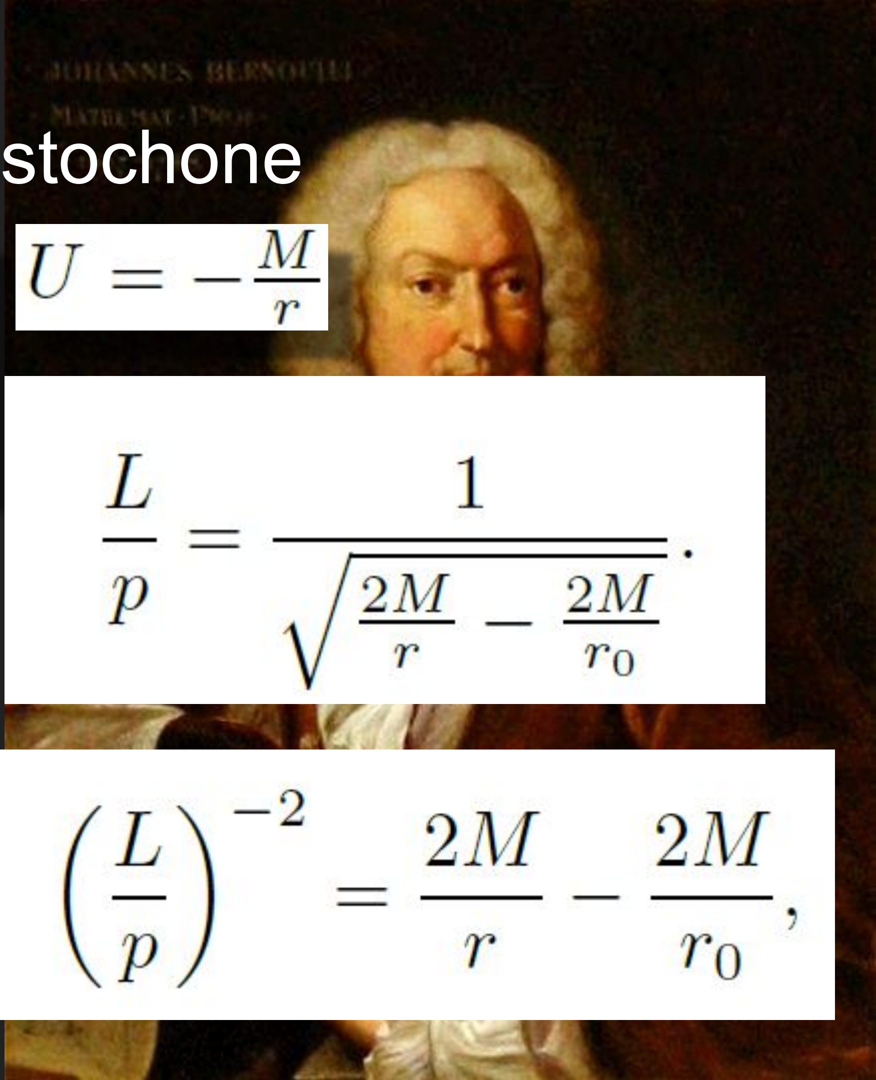

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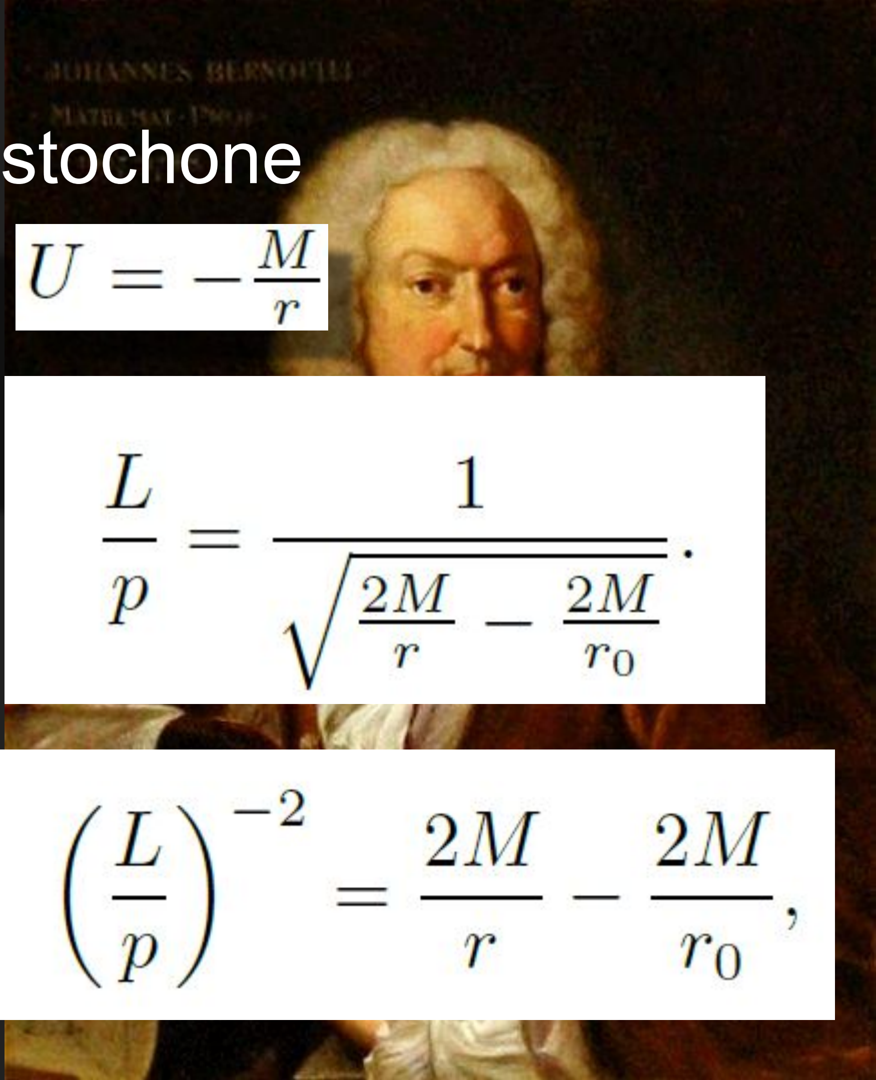

Catenary

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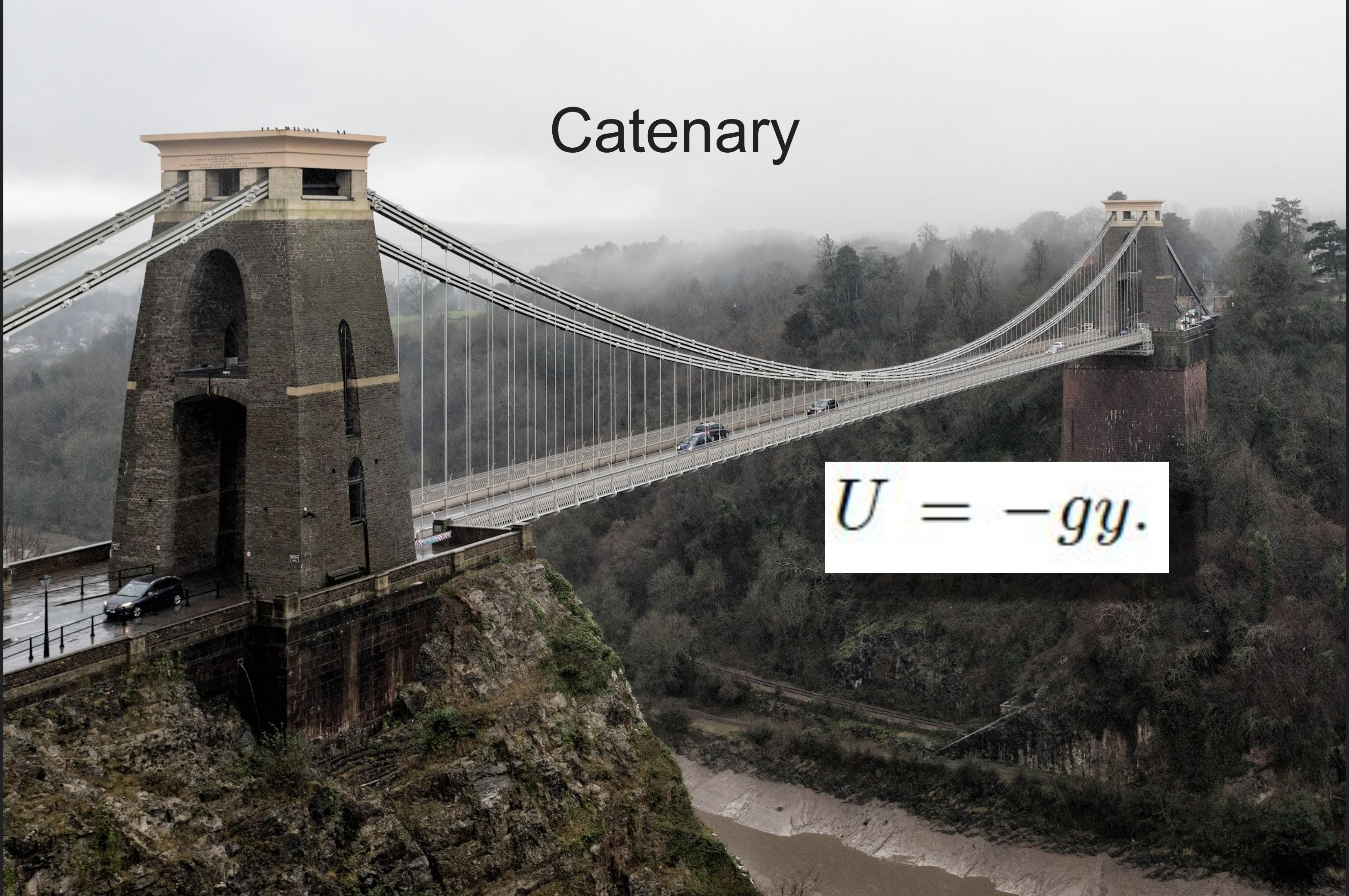

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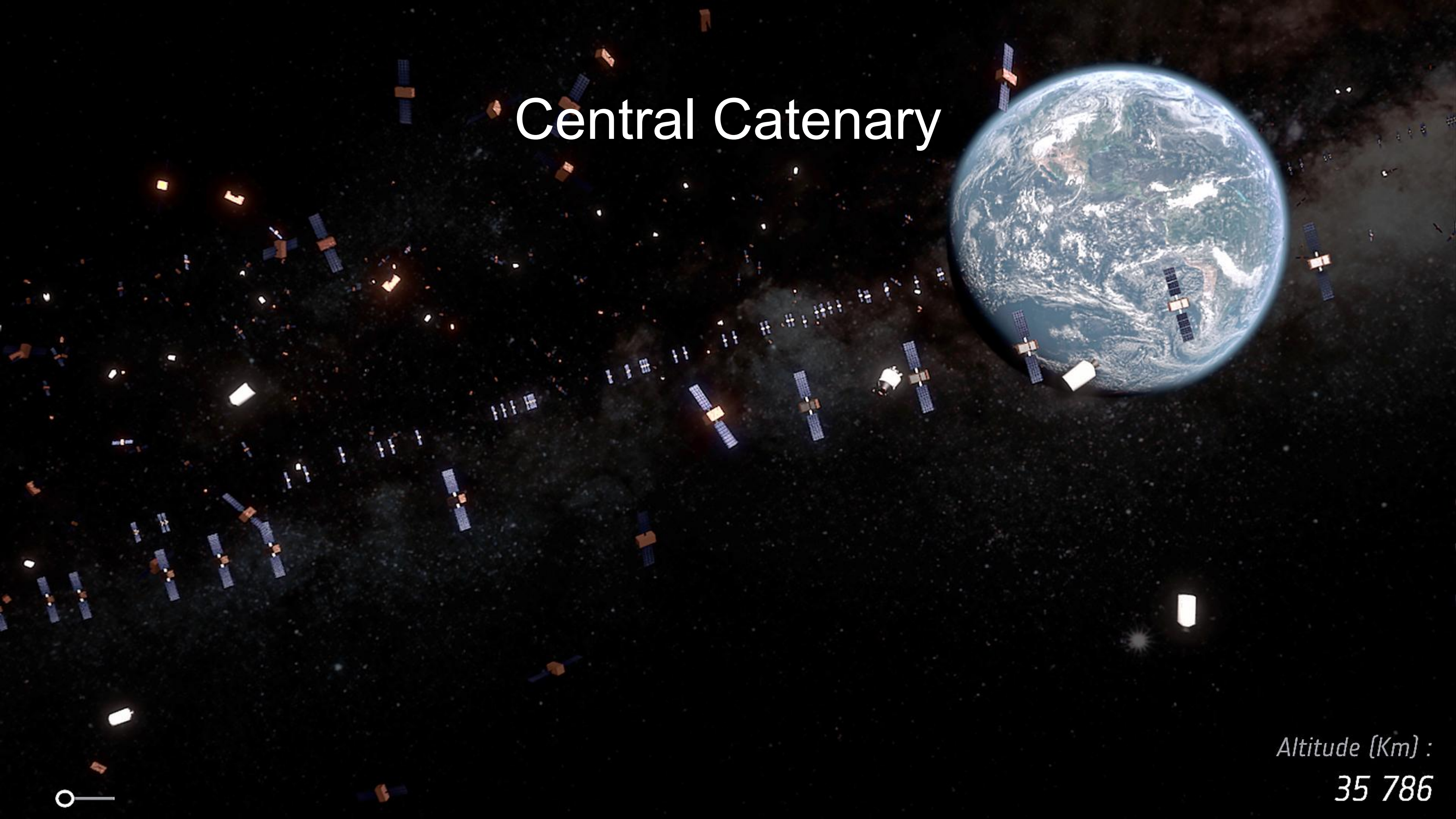
Central Catenary



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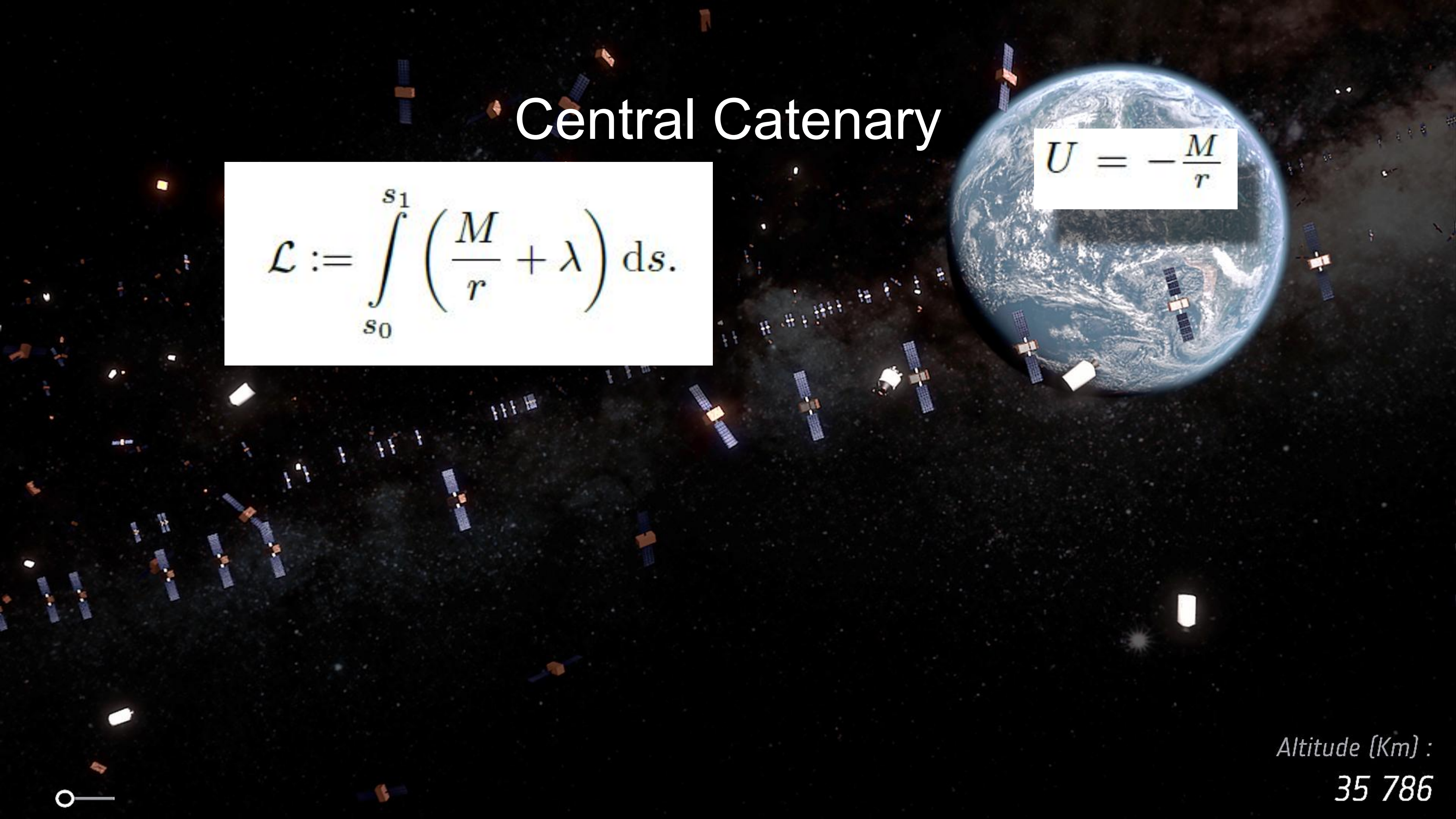

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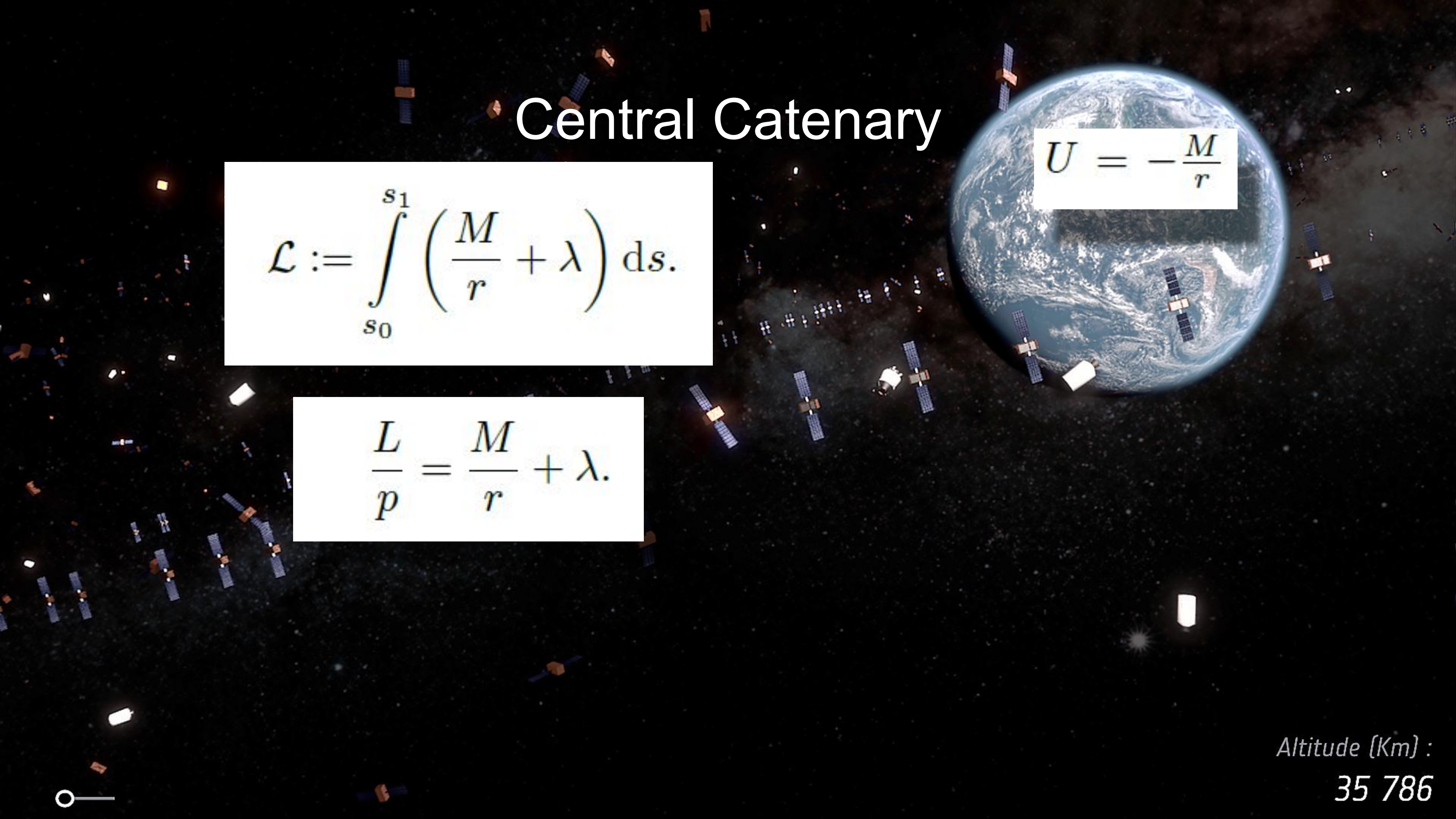

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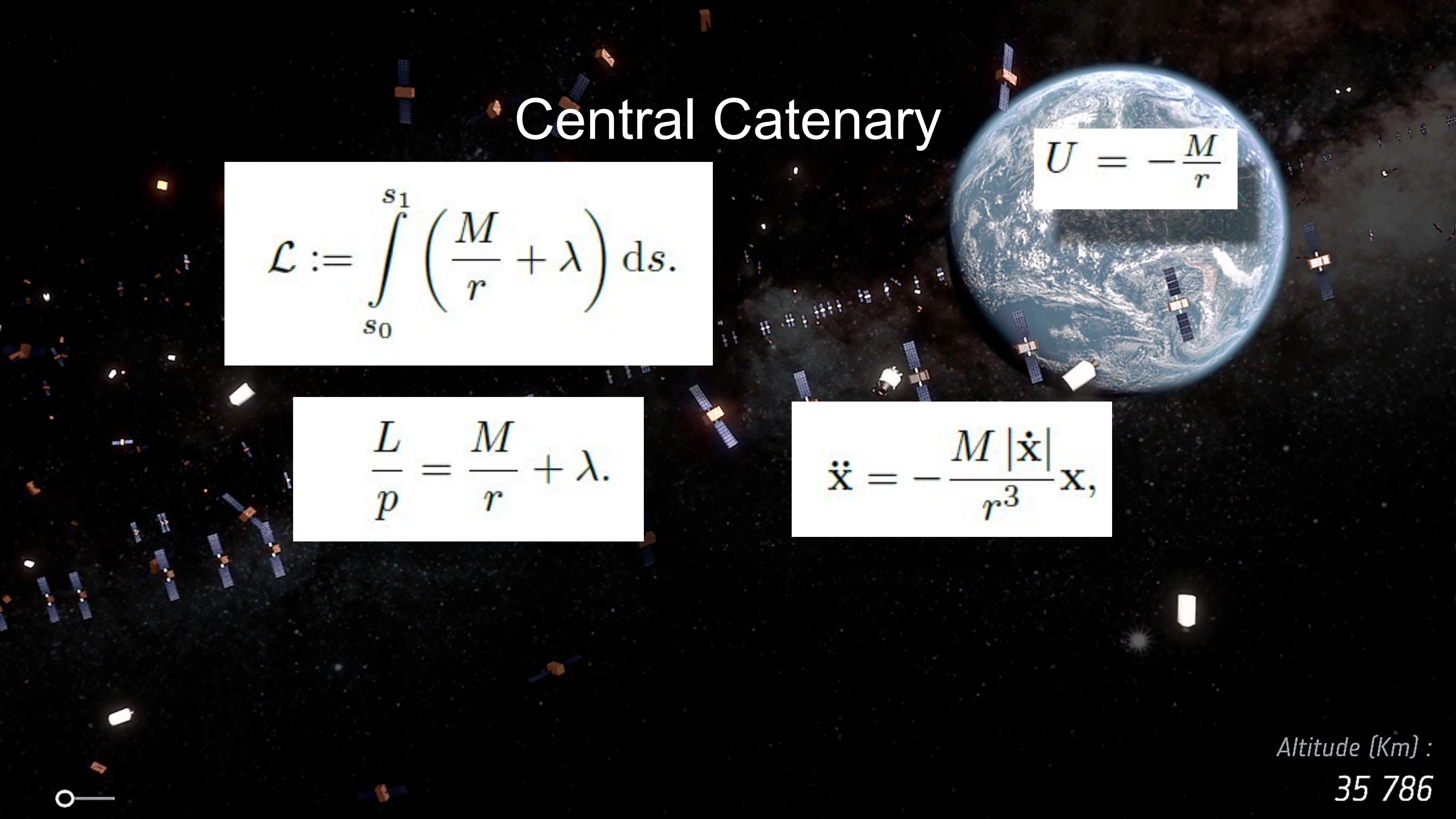

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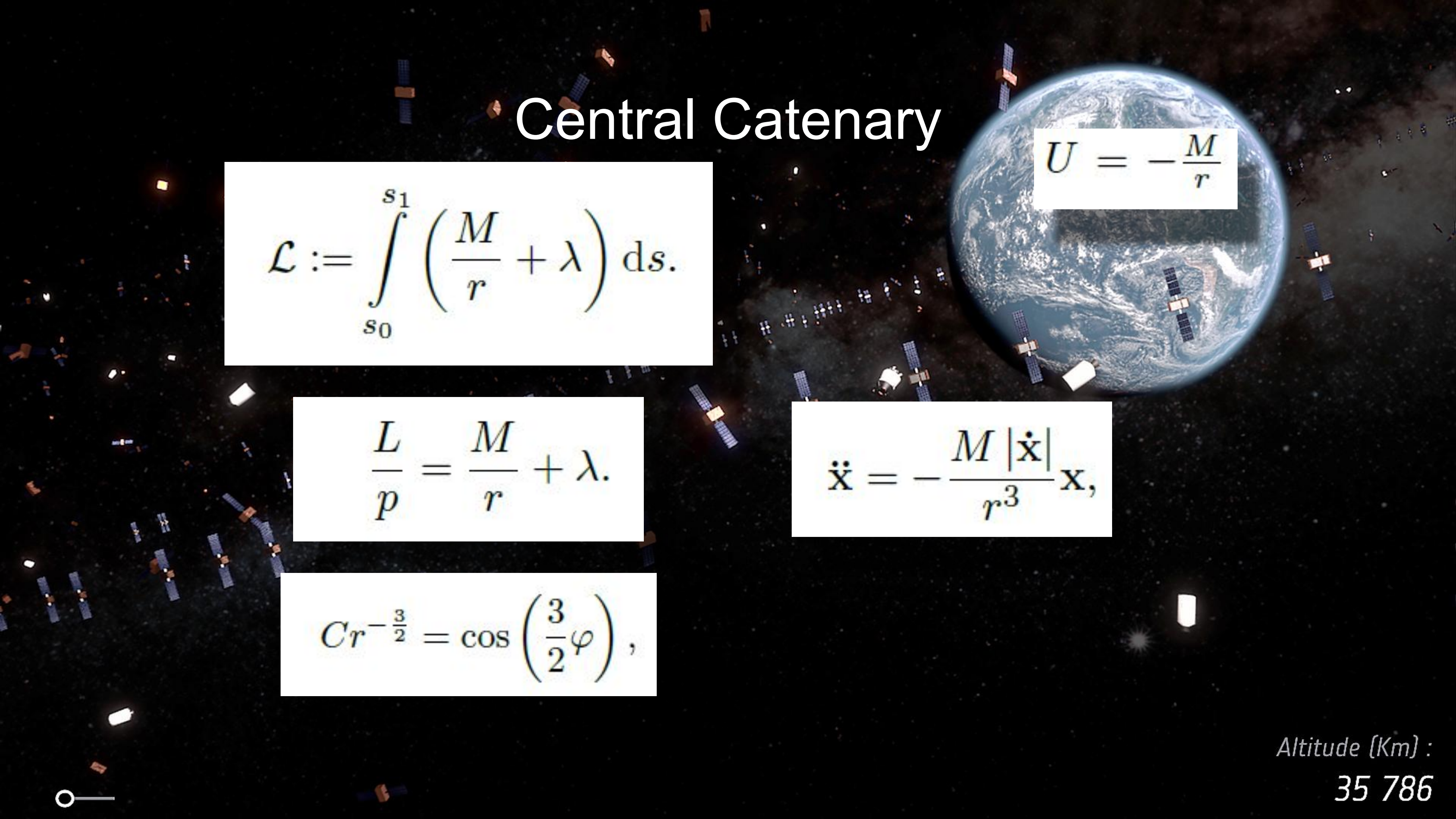

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for $L^2 \geq M^2$ Central catenary is $\sqrt{1 - \frac{M^2}{L^2}}$-harmonic of a hyperbola!

Altitude (Km) :

35 786

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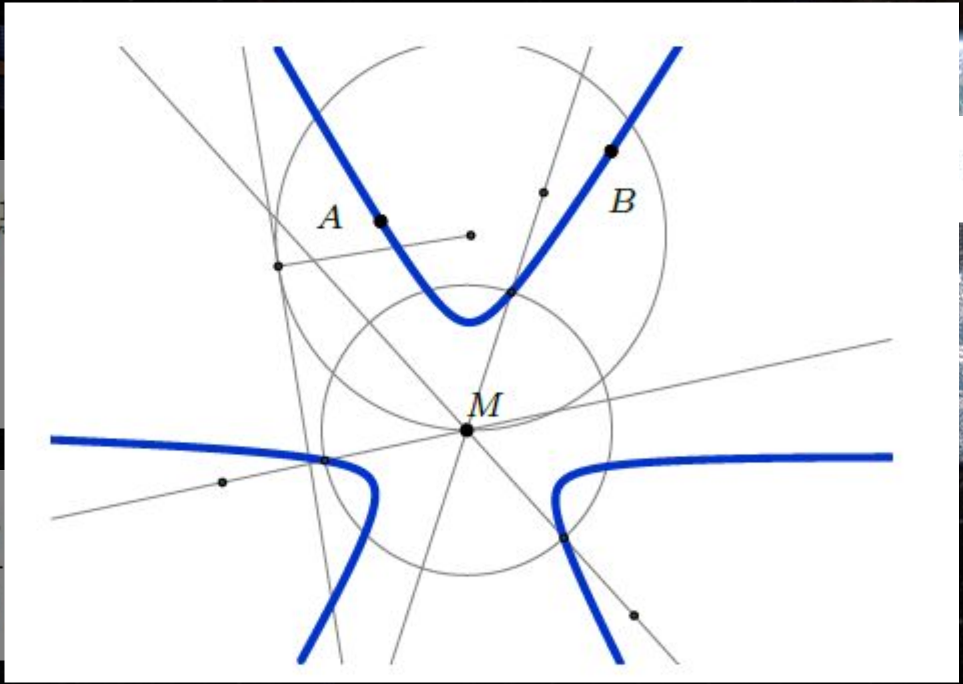

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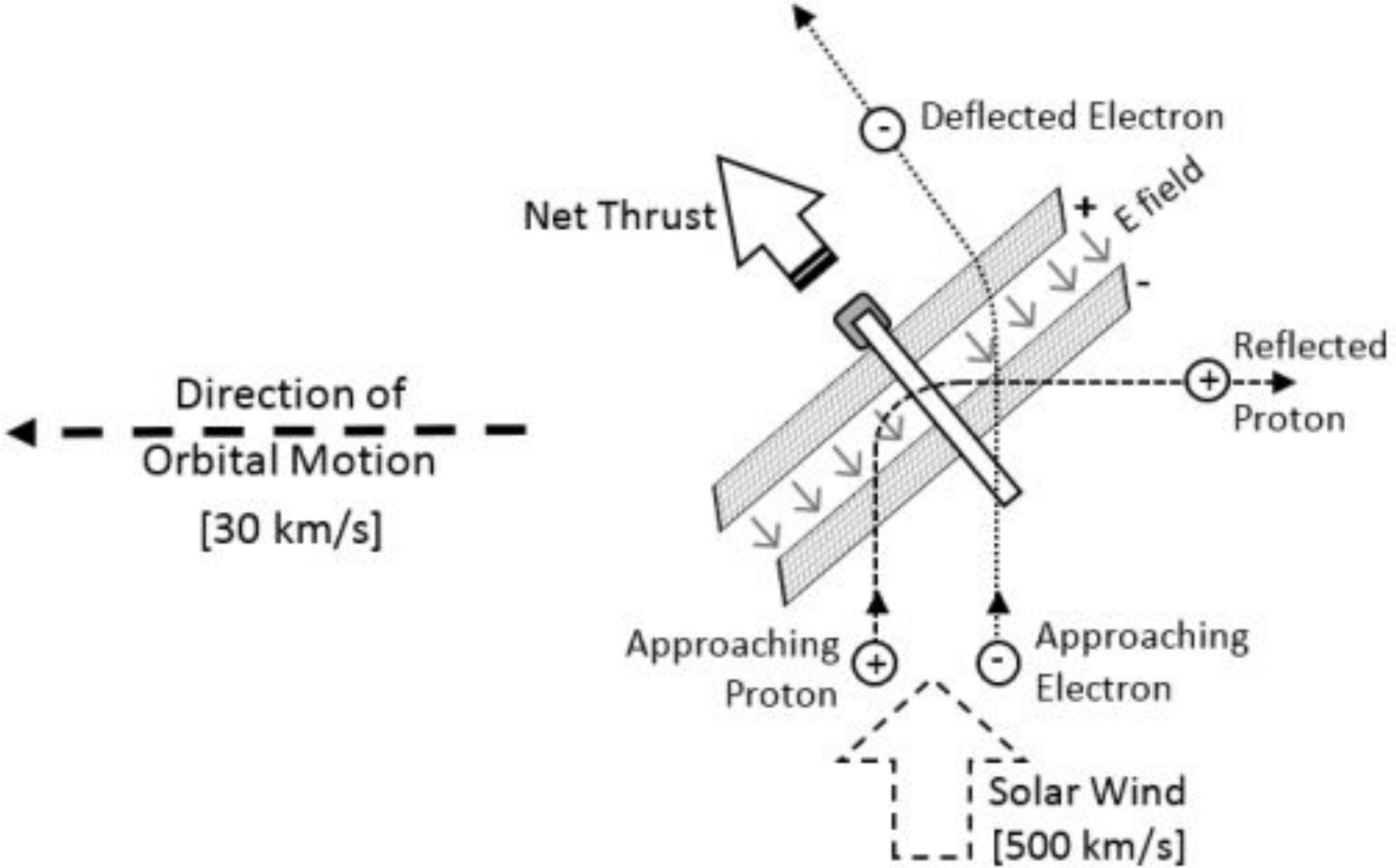
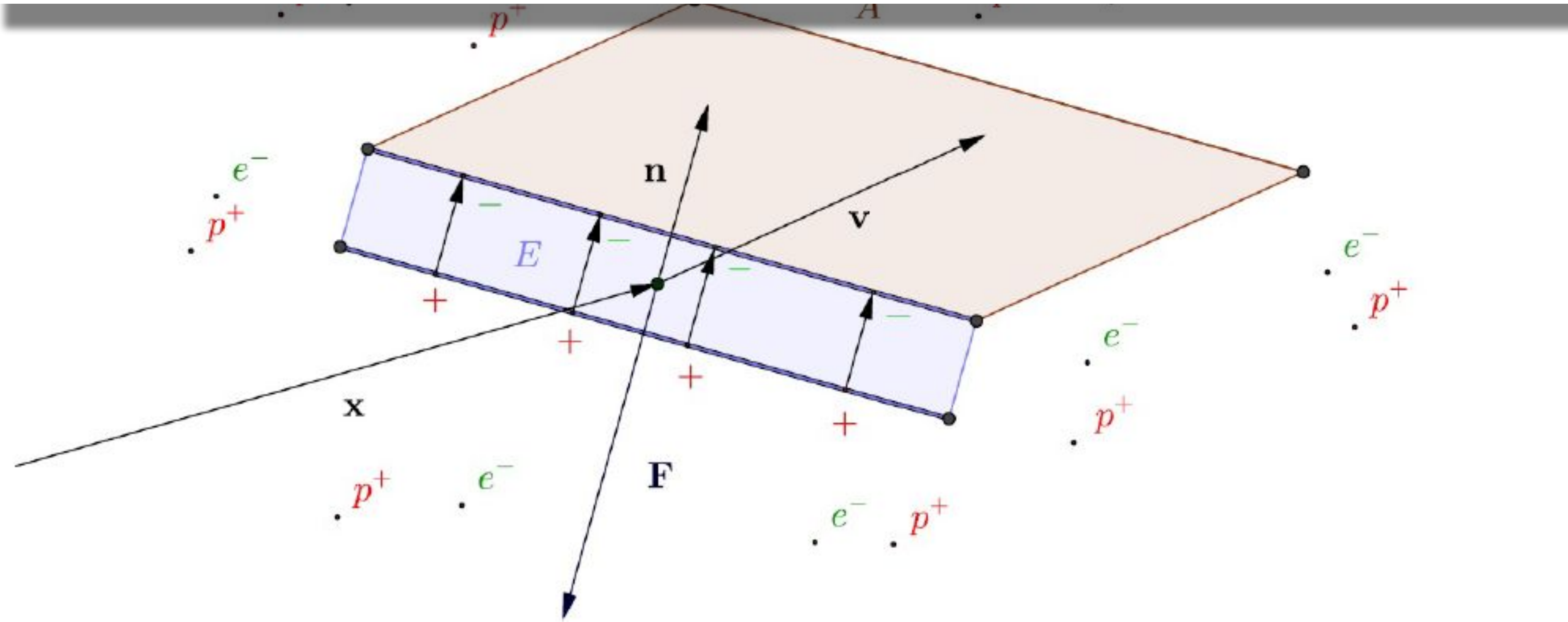
Dipole drive by Robert Zubrin



Deflected Electron
E field
+
-
Net Thrust
Reflected
Proton
Direction of
Orbital Motion
[30 km/s]
Approaching
Proton
Approaching
Electron
Solar Wind
[500 km/s]

Orbits of Dipole drive pointing in direction perpendicular to motion.



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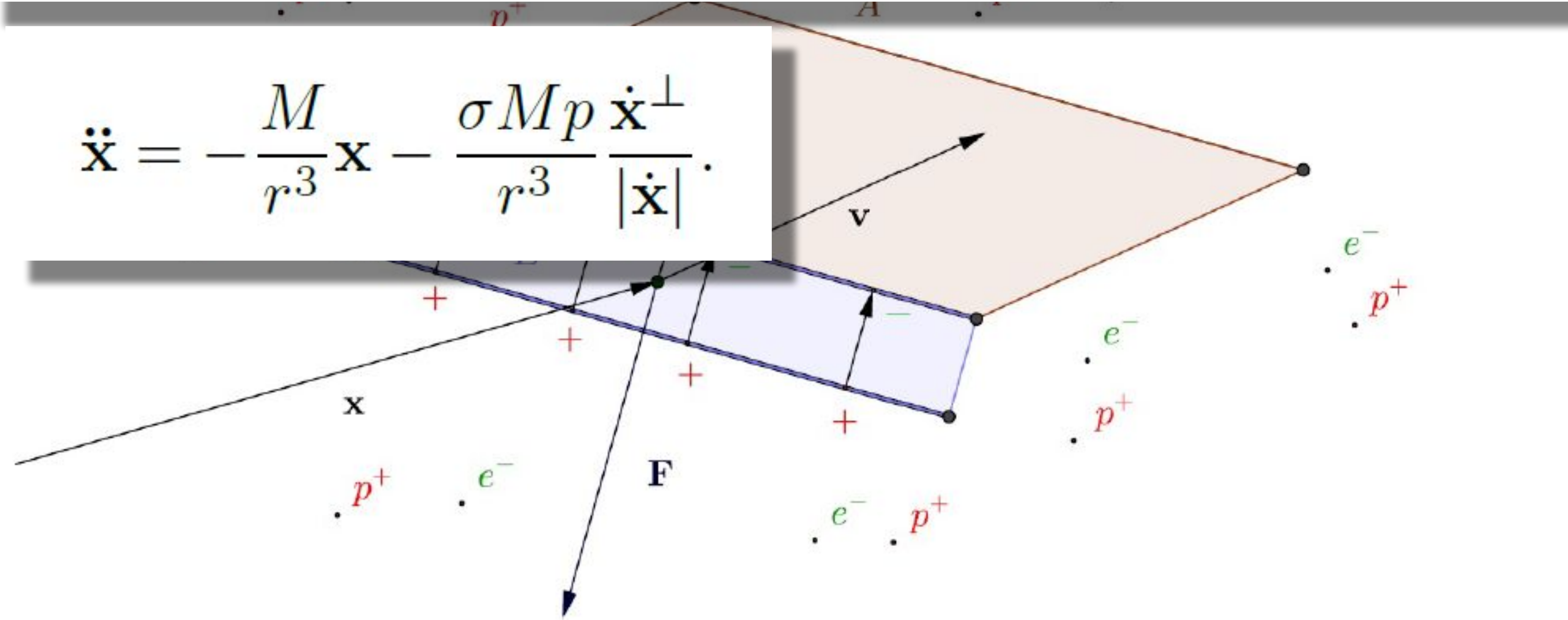
$$\ddot{\mathbf{x}} = -\frac{M}{r^3}\mathbf{x} - \frac{\sigma M p}{r^3}\frac{\dot{\mathbf{x}}^{\perp}}{|\dot{\mathbf{x}}|}.$$



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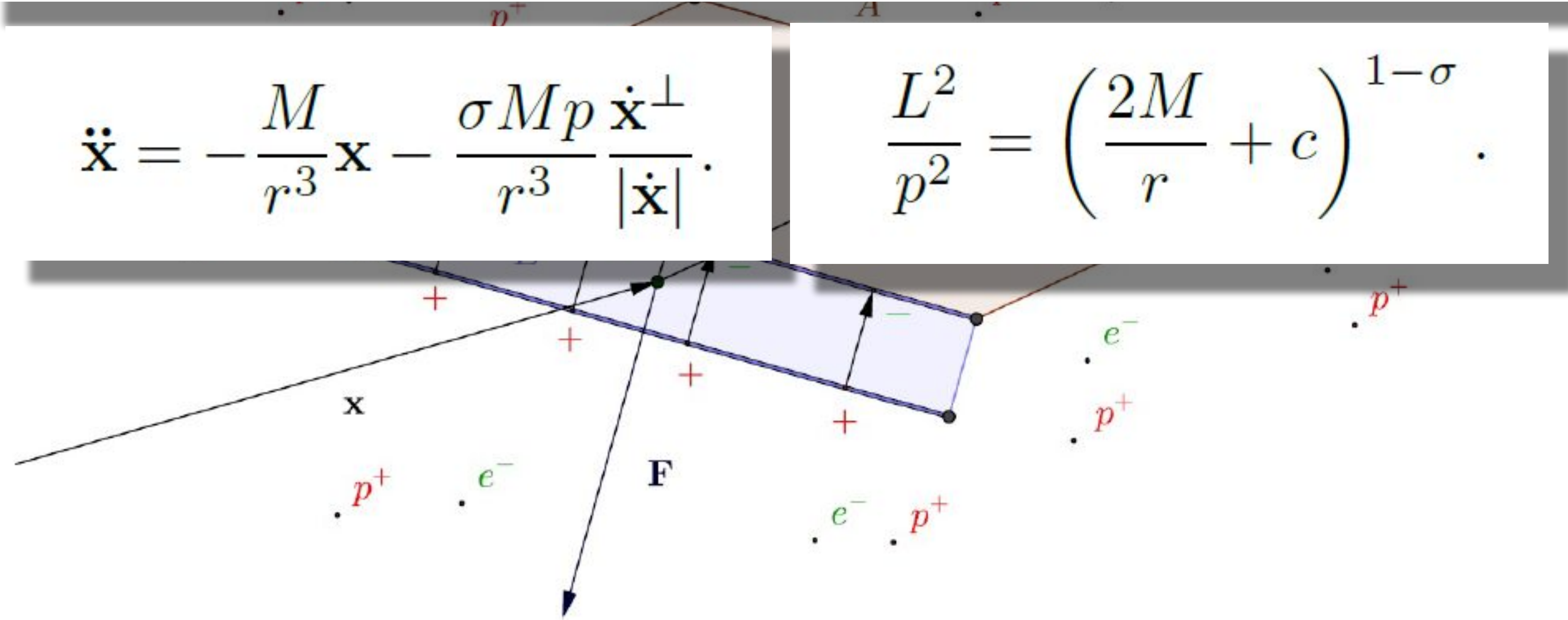
$$\frac{L^2}{p^2} = \left(\frac{2M}{r} + c\right)^{1-\sigma}.$$



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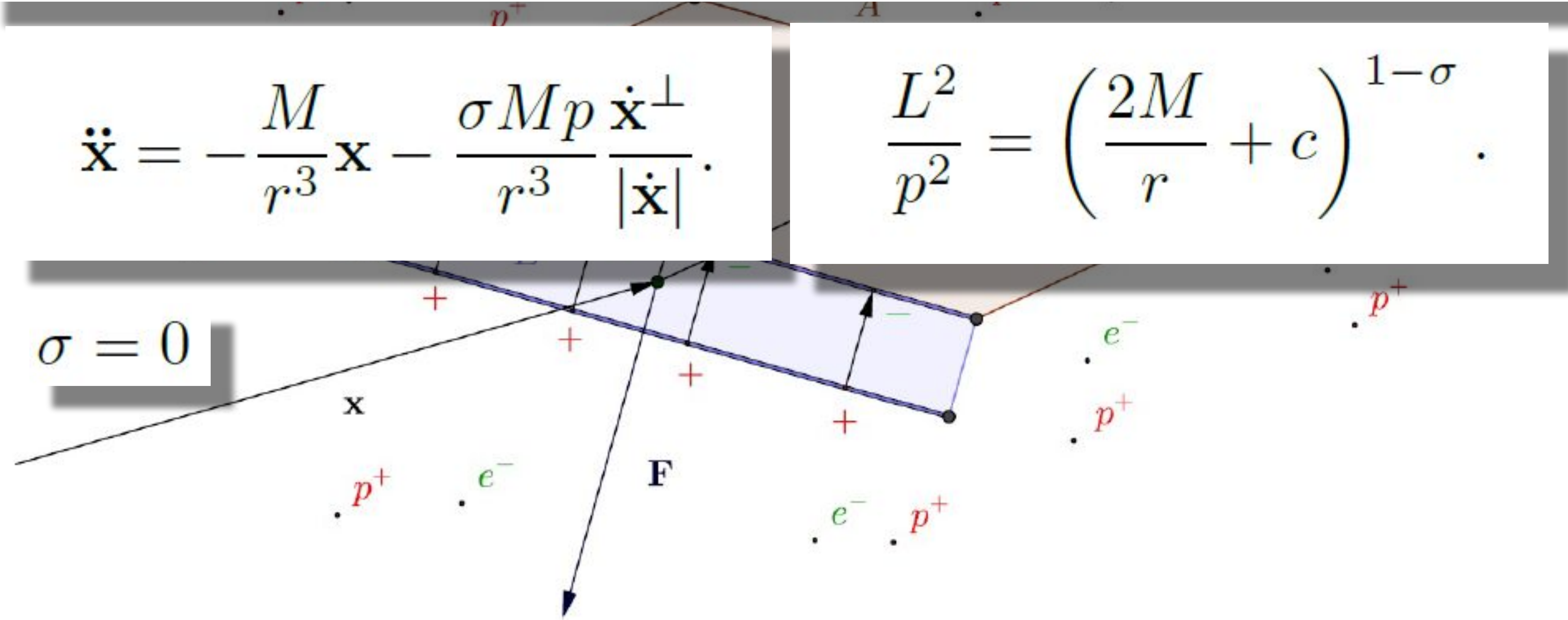

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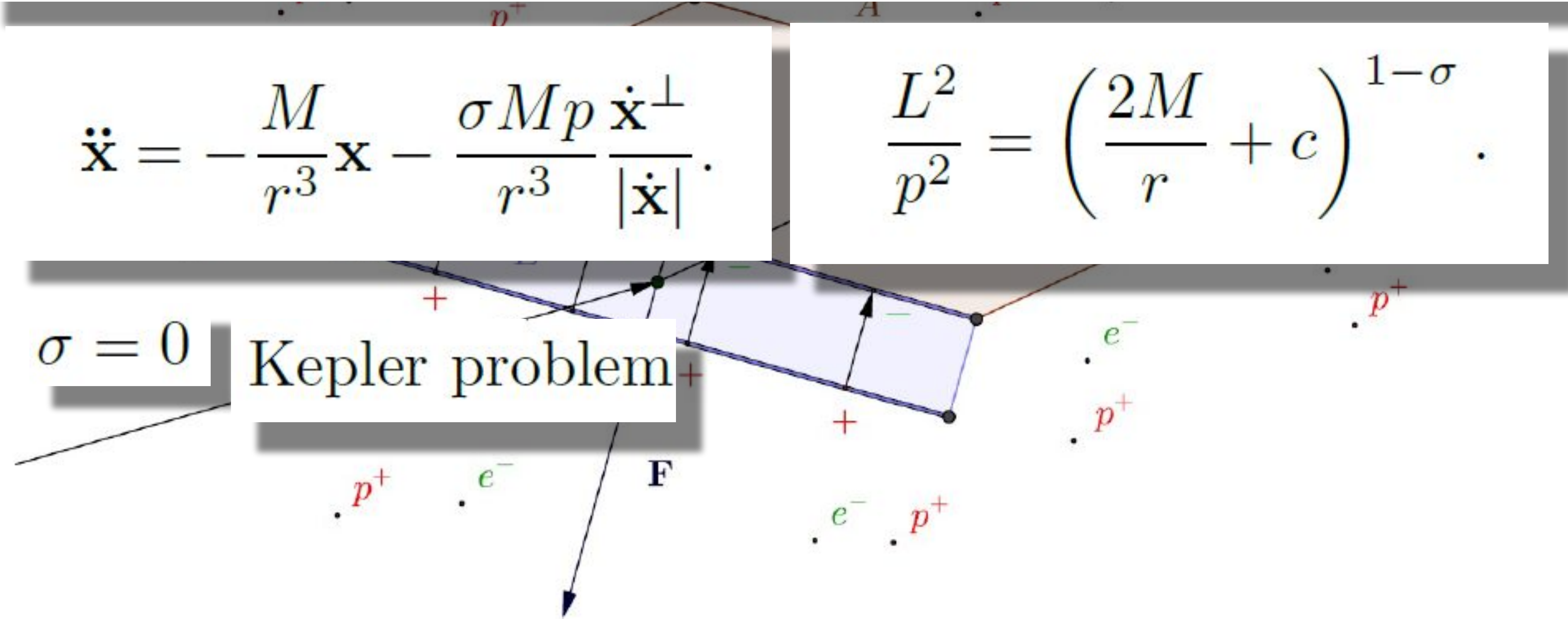
$\sigma = 0$ Kepler problem



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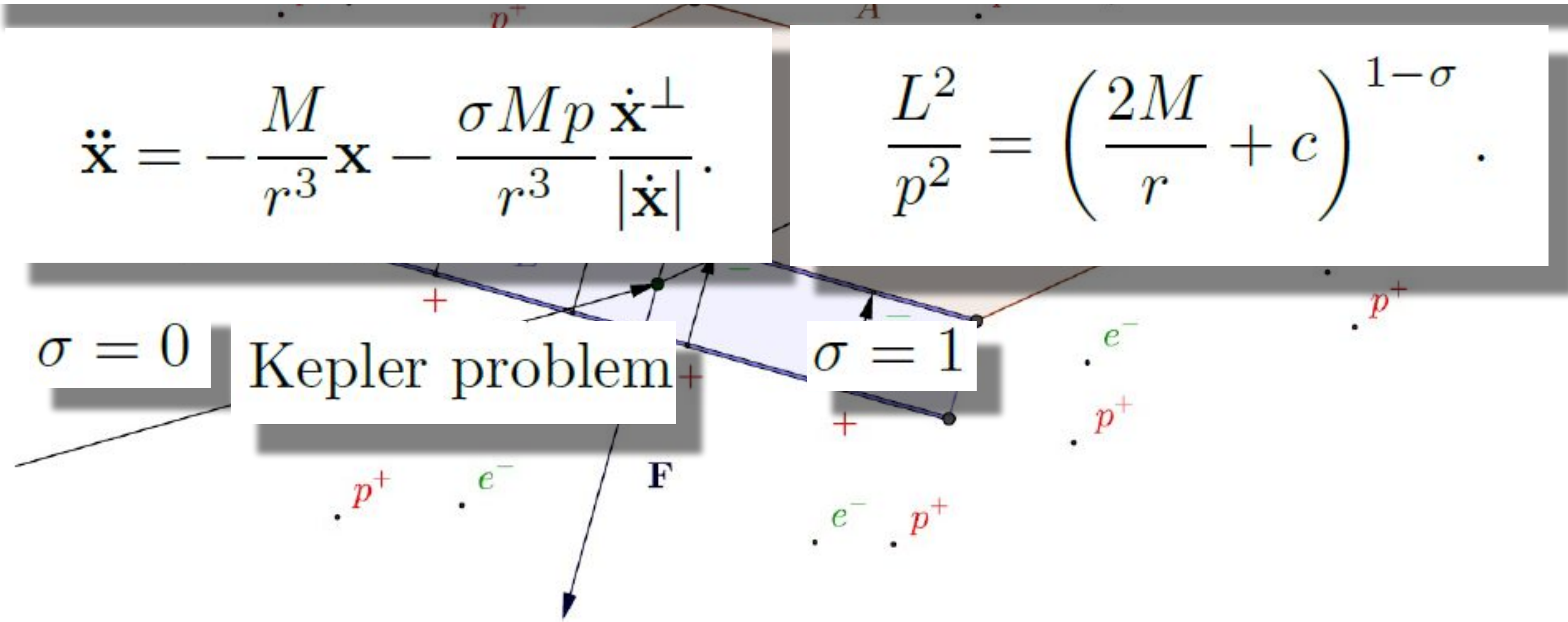

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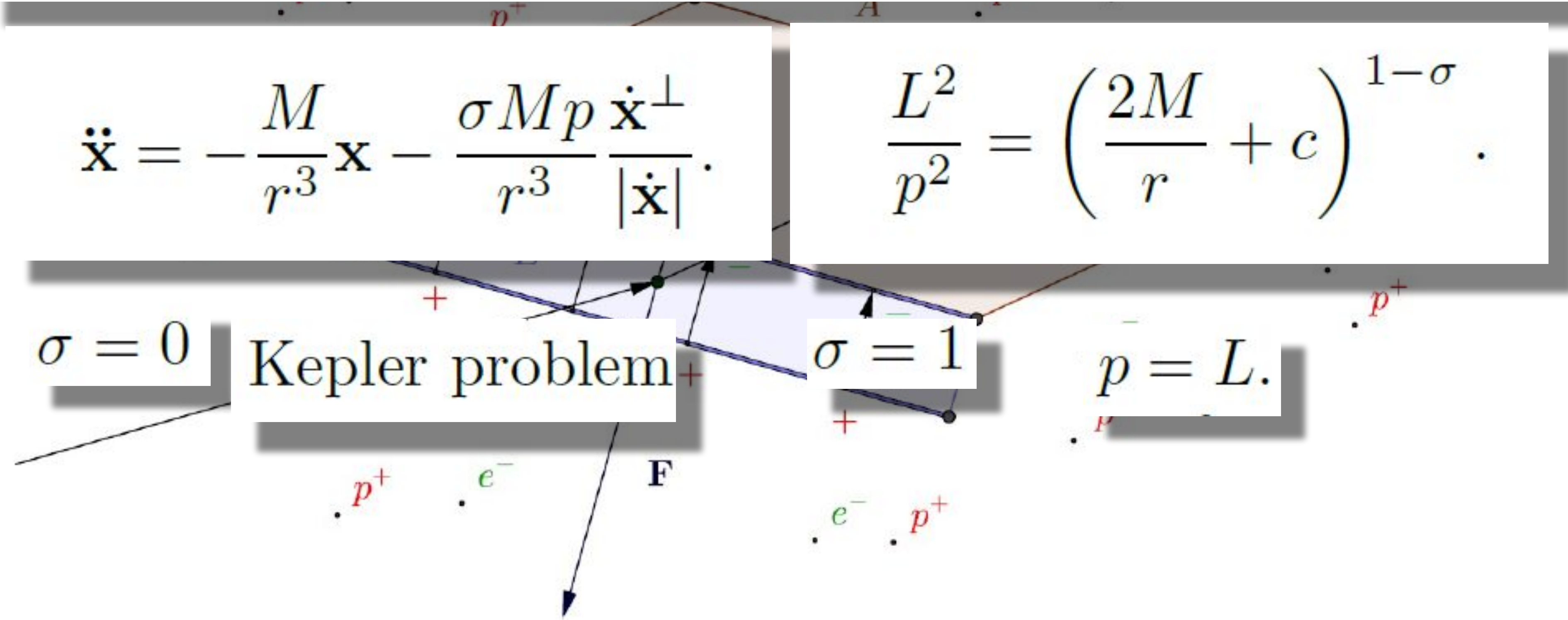

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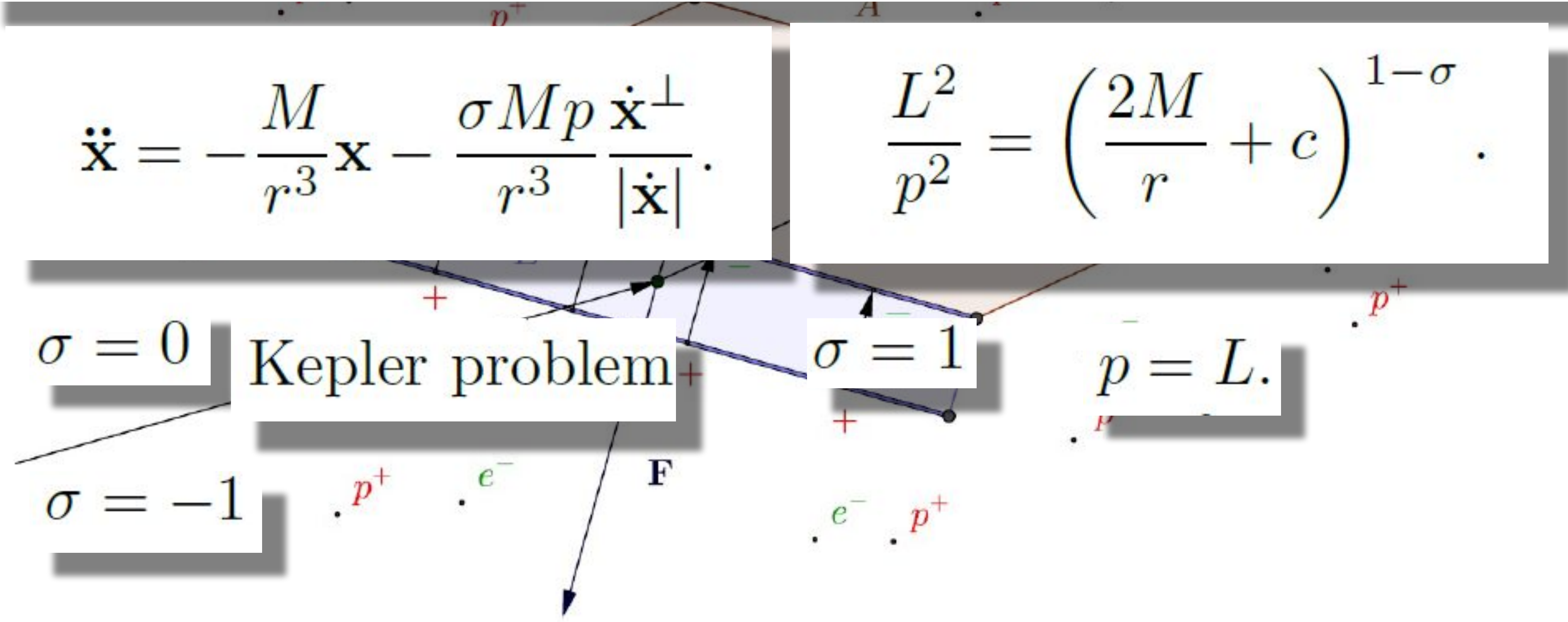
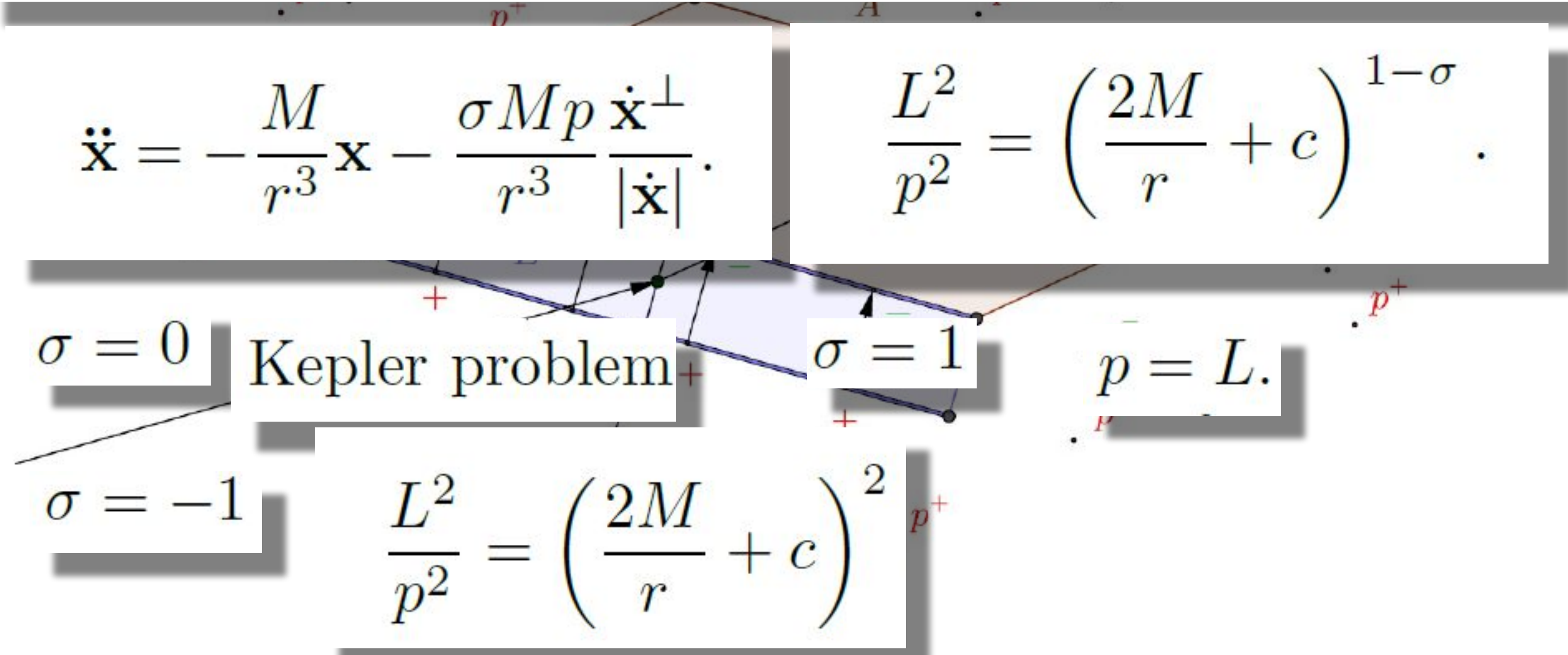


$\sigma = 0$ Kepler problem

$\sigma = 1$ $p = L.$

$\sigma = -1$

Orbits of Dipole drive pointing in direction perpendicular to motion.



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central Catenary

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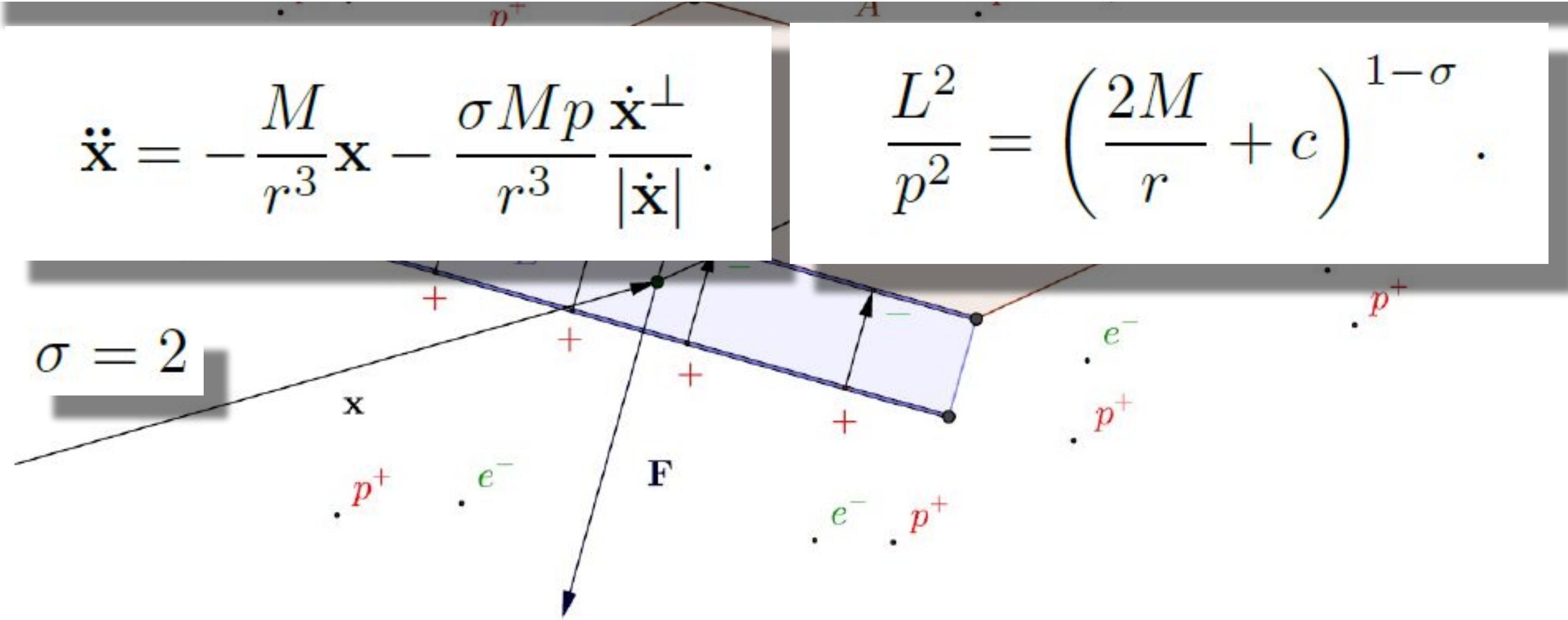
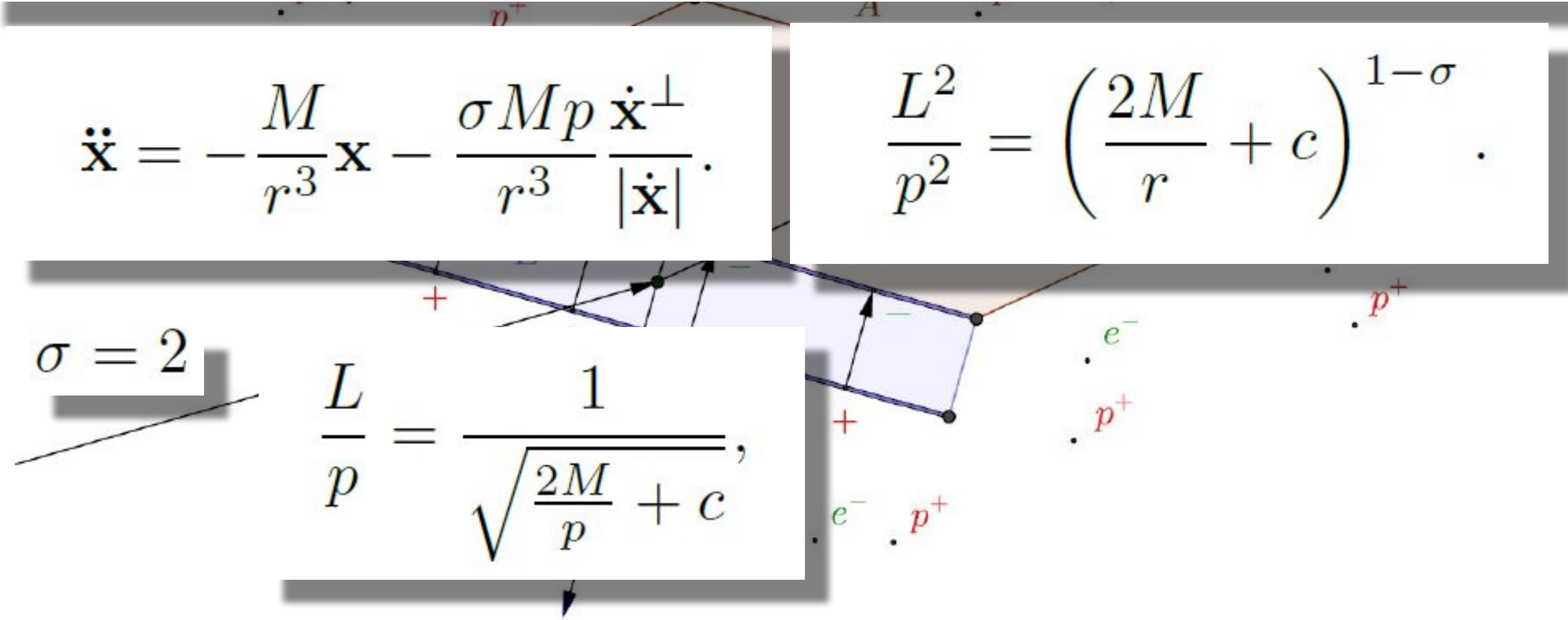
$\sigma = 2$



Orbits of Dipole drive pointing in direction perpendicular to motion.



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$$\frac{L^2}{p^2} = \left(\frac{2M}{r} + c\right)^{1-\sigma}.$$

$\sigma = 2$

$$\frac{L}{p} = \frac{1}{\sqrt{\frac{2M}{p} + c}},$$

Orbits of Dipole drive pointing in direction perpendicular to motion.

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central Brachistochrone

Orbits of Dipole drive pointing in direction perpendicular to motion.

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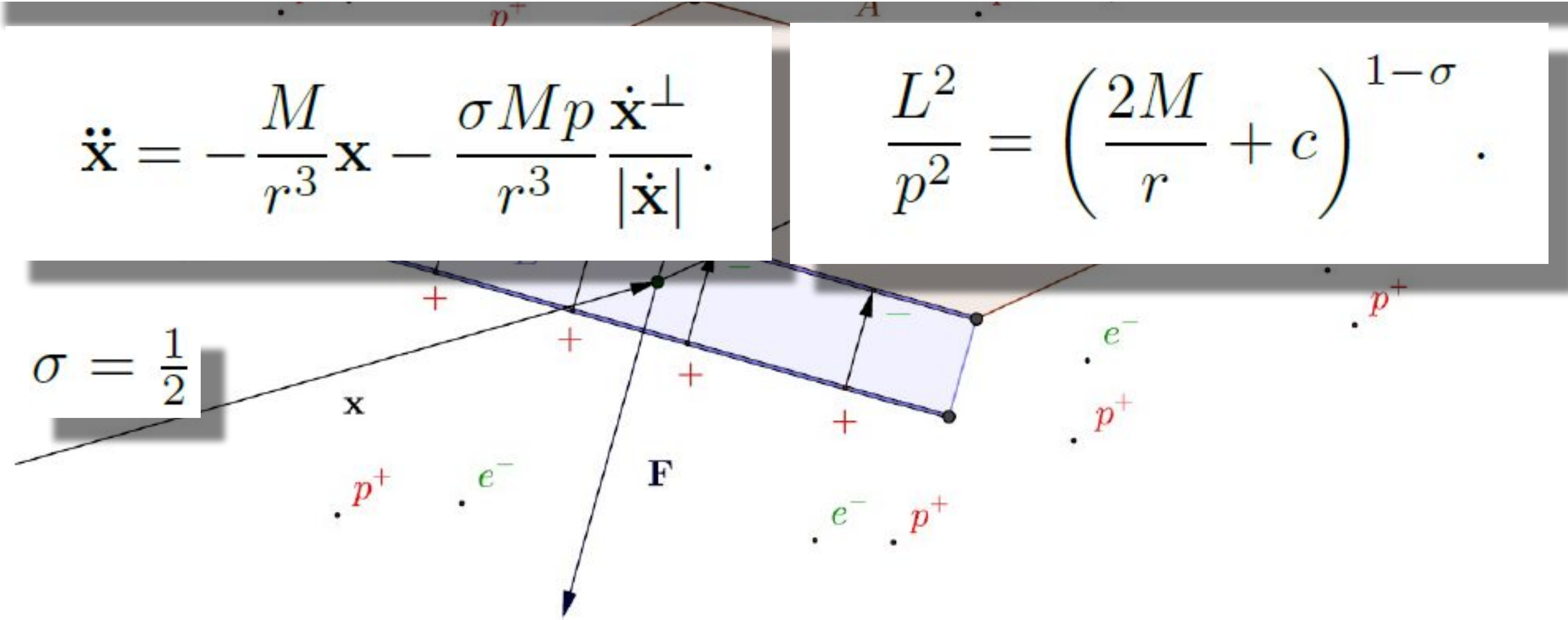
$\sigma = \frac{1}{2}$



Orbits of Dipole drive pointing in direction perpendicular to motion.

$$\ddot{\mathbf{x}} = -\frac{M}{r^3}\mathbf{x} - \frac{\sigma M p}{r^3}\frac{\dot{\mathbf{x}}^{\perp}}{|\dot{\mathbf{x}}|}.$$

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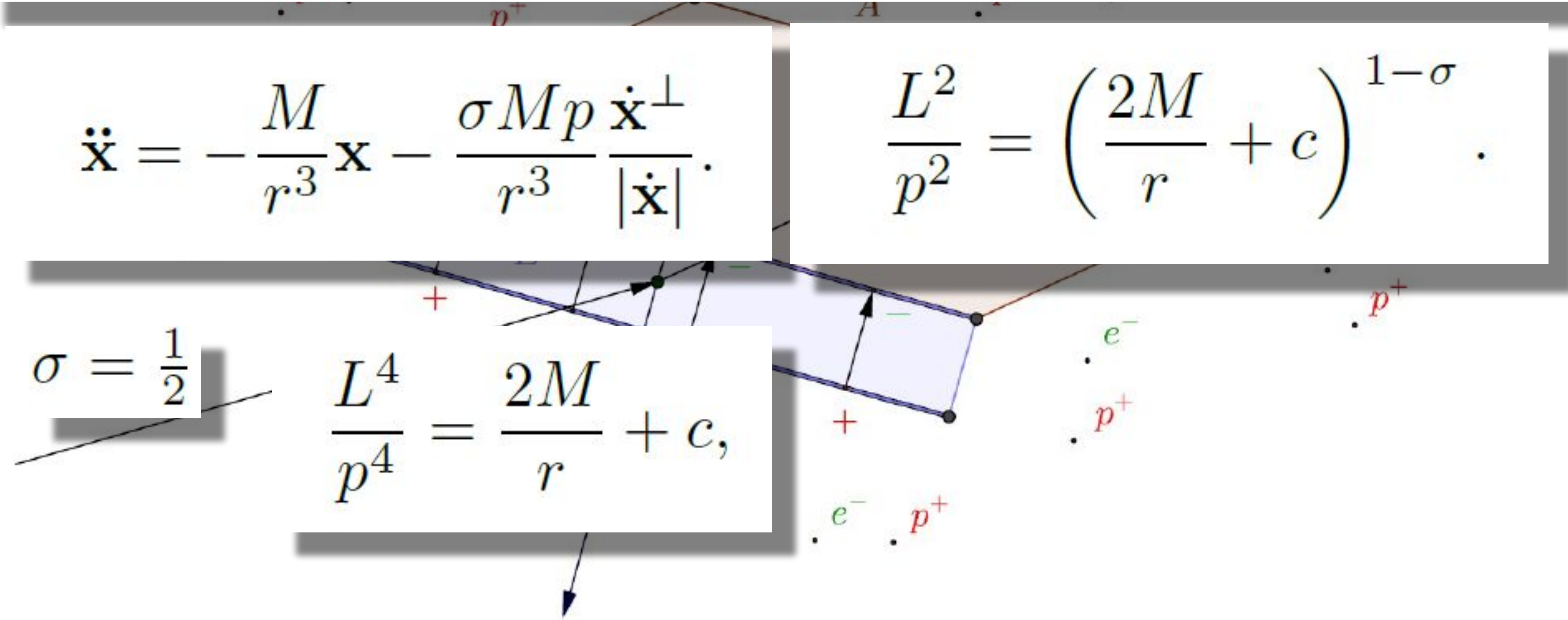
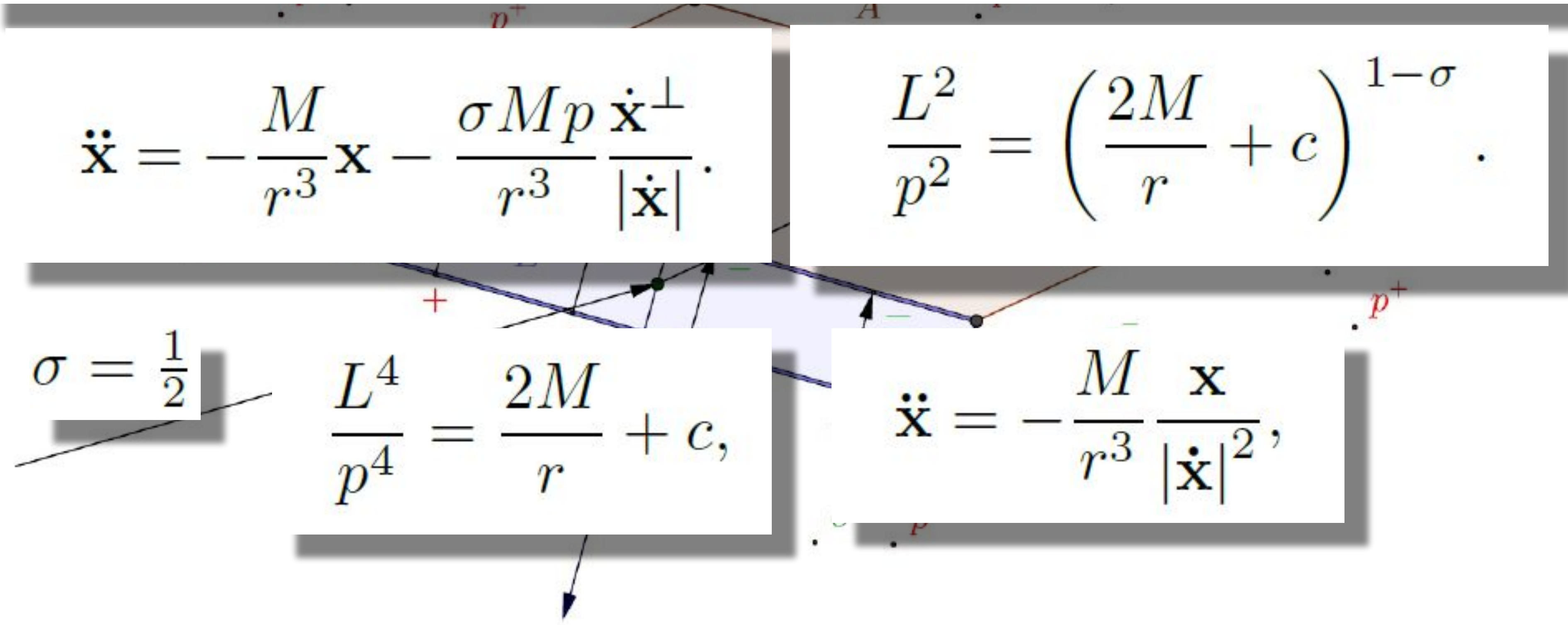
$$\frac{L^4}{p^4} = \frac{2M}{r} + c,$$



Orbits of Dipole drive pointing in direction perpendicular to motion.



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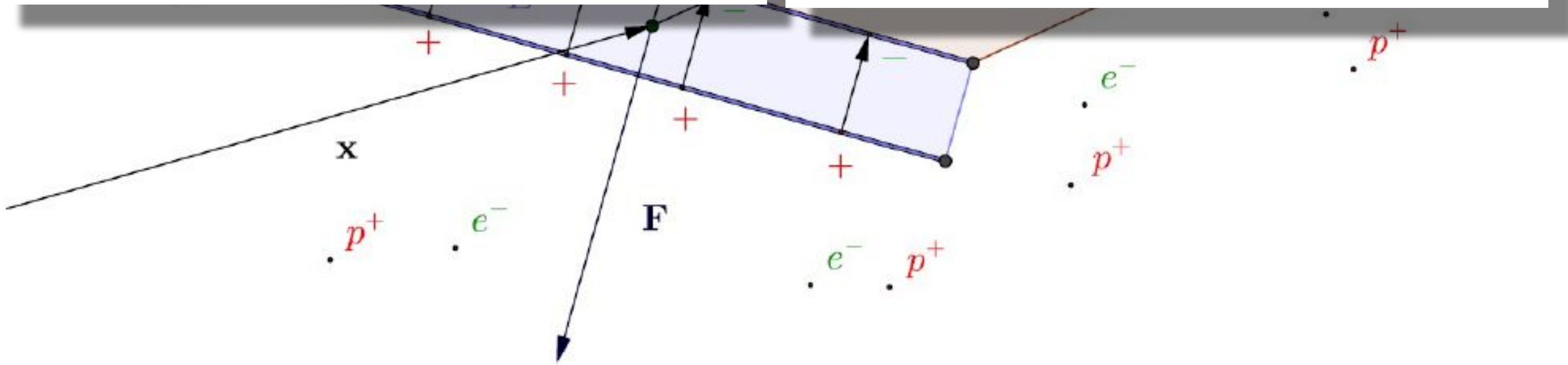
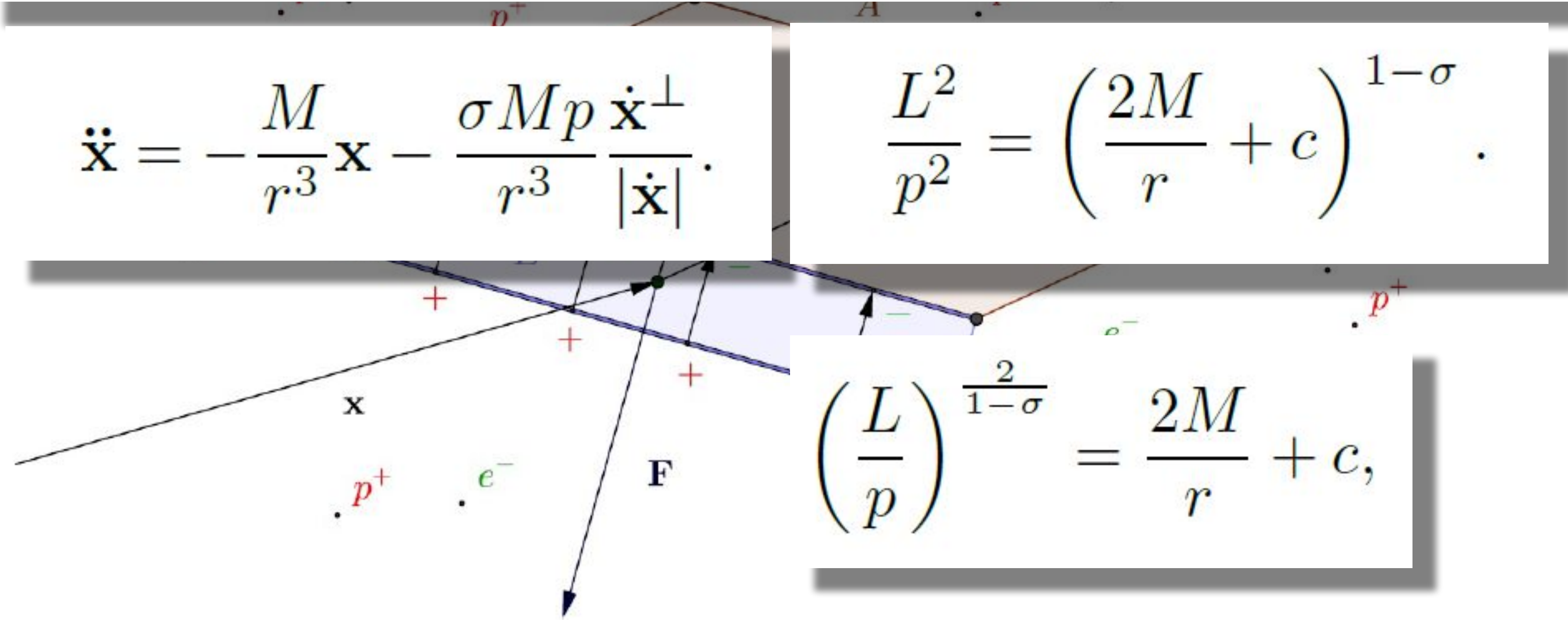


Orbits of Dipole drive pointing in direction perpendicular to motion.



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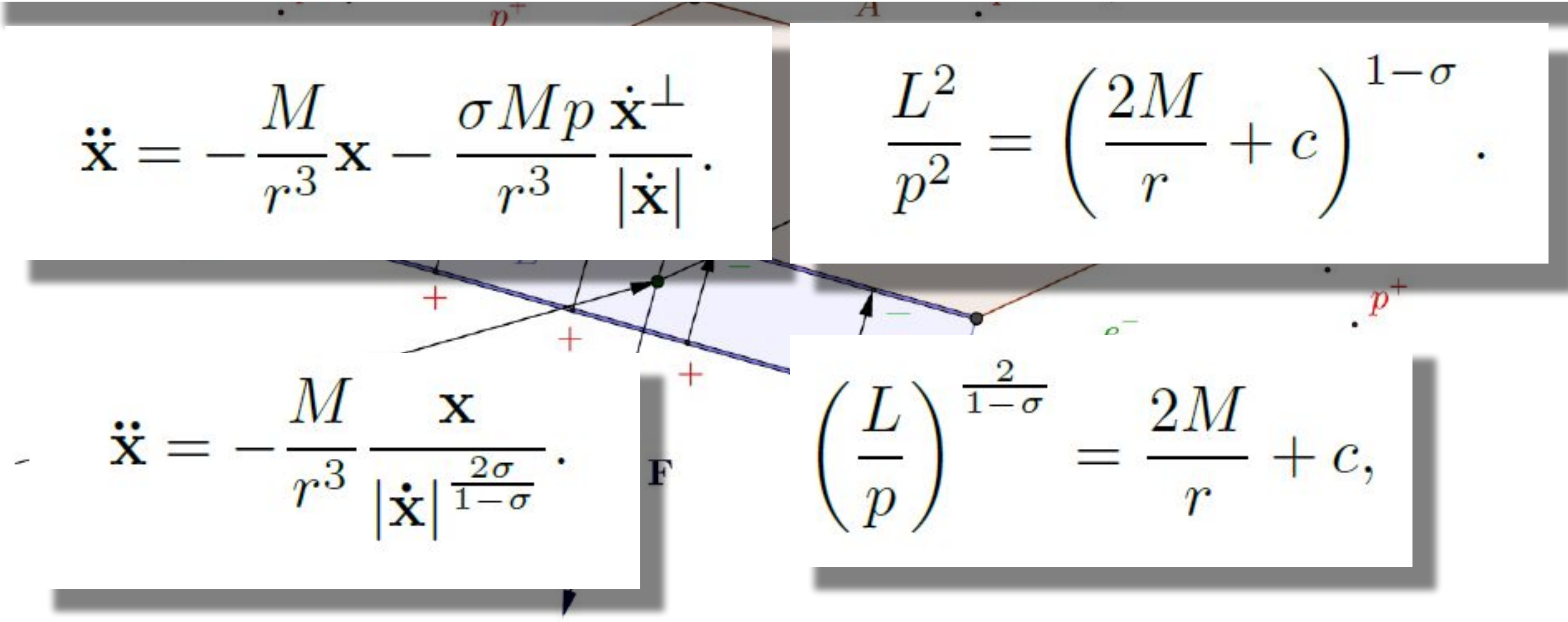
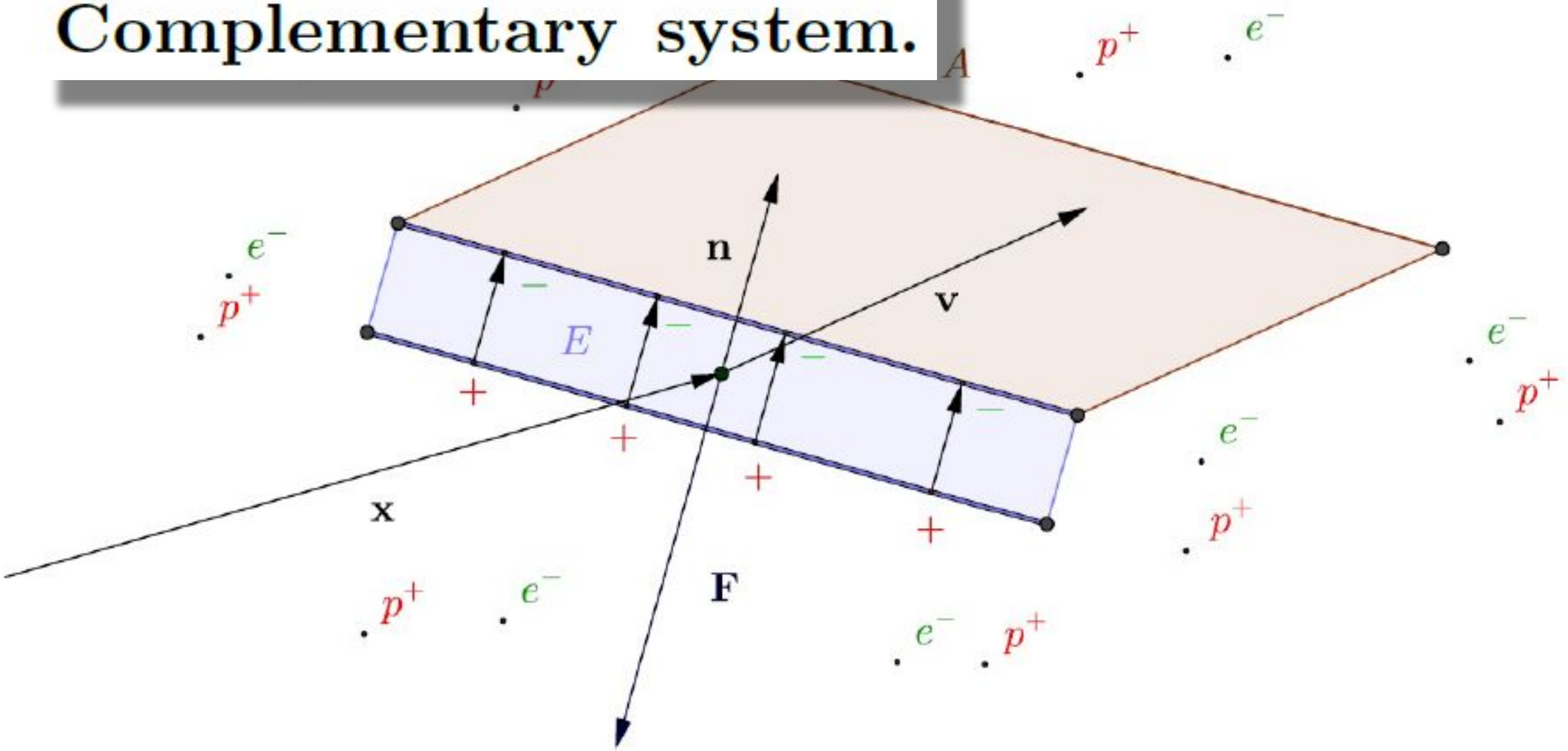
$$\left(\frac{L}{p}\right)^{\frac{2}{1-\sigma}} = \frac{2M}{r} + c,$$

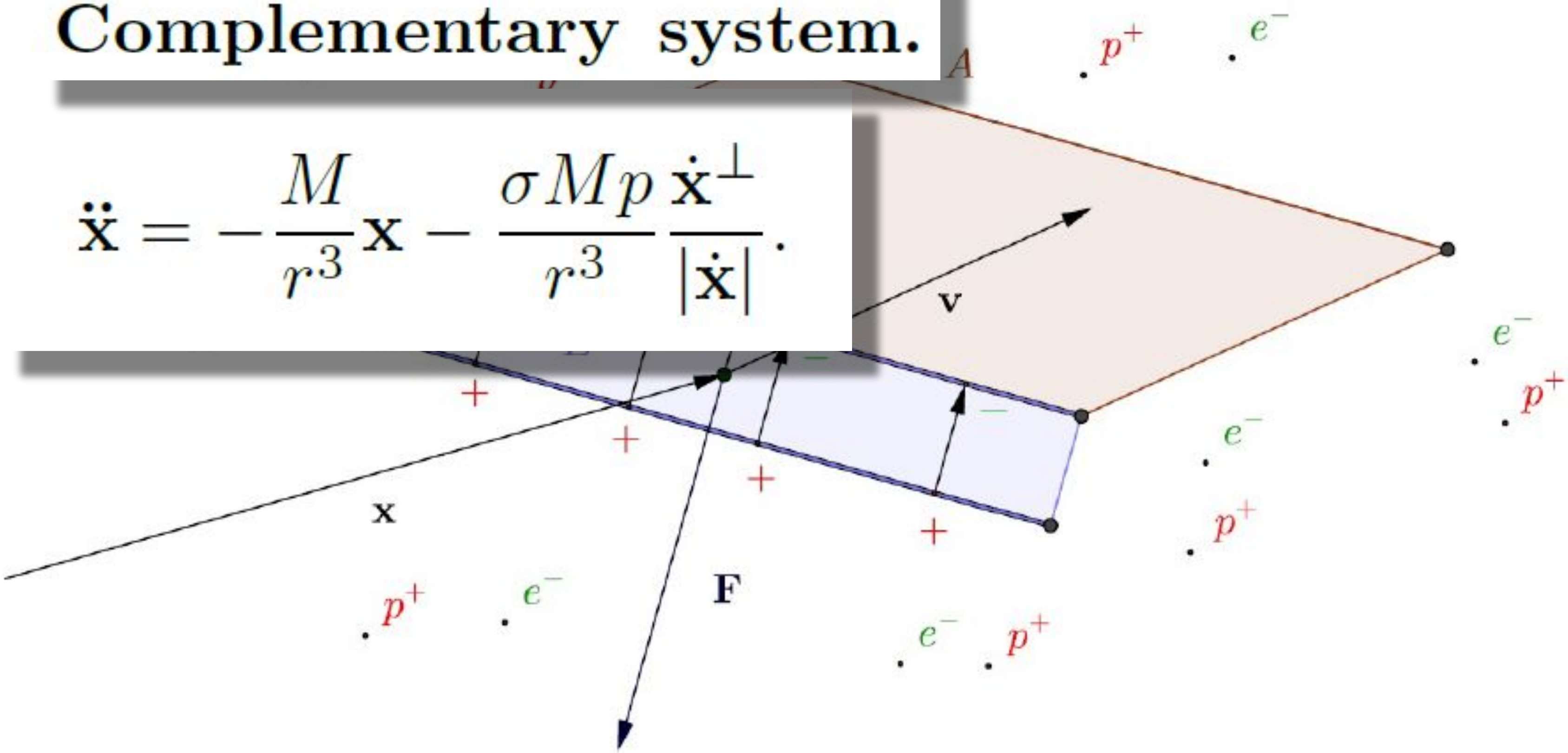


Complementary system.



Complementary system.

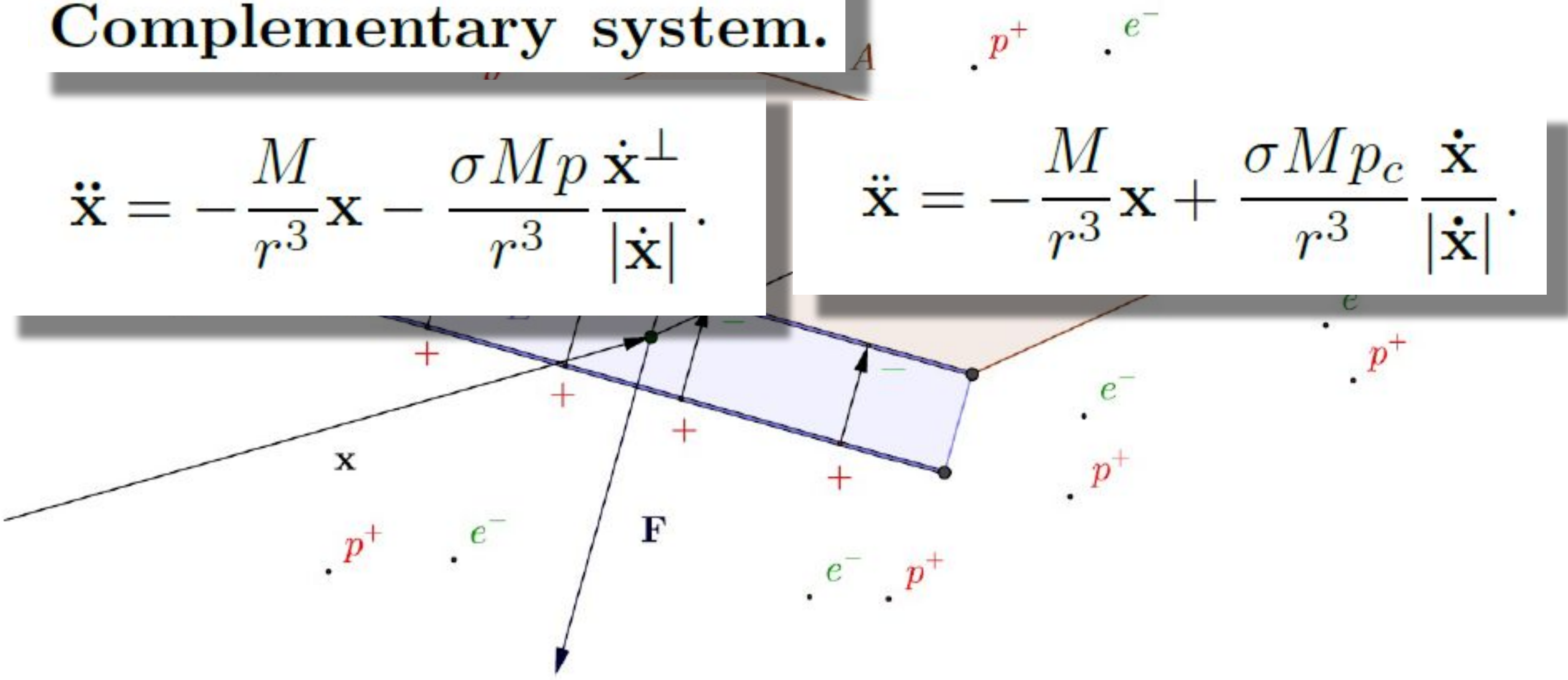
$$\ddot{\mathbf{x}} = -\frac{M}{r^3}\mathbf{x} - \frac{\sigma M p}{r^3}\frac{\dot{\mathbf{x}}^{\perp}}{|\dot{\mathbf{x}}|}.$$



Complementary system.

$$\ddot{\mathbf{x}} = -\frac{M}{r^3}\mathbf{x} - \frac{\sigma M p}{r^3}\frac{\dot{\mathbf{x}}^{\perp}}{|\dot{\mathbf{x}}|}.$$

$$\ddot{\mathbf{x}} = -\frac{M}{r^3}\mathbf{x} + \frac{\sigma M p_c}{r^3}\frac{\dot{\mathbf{x}}}{|\dot{\mathbf{x}}|}.$$



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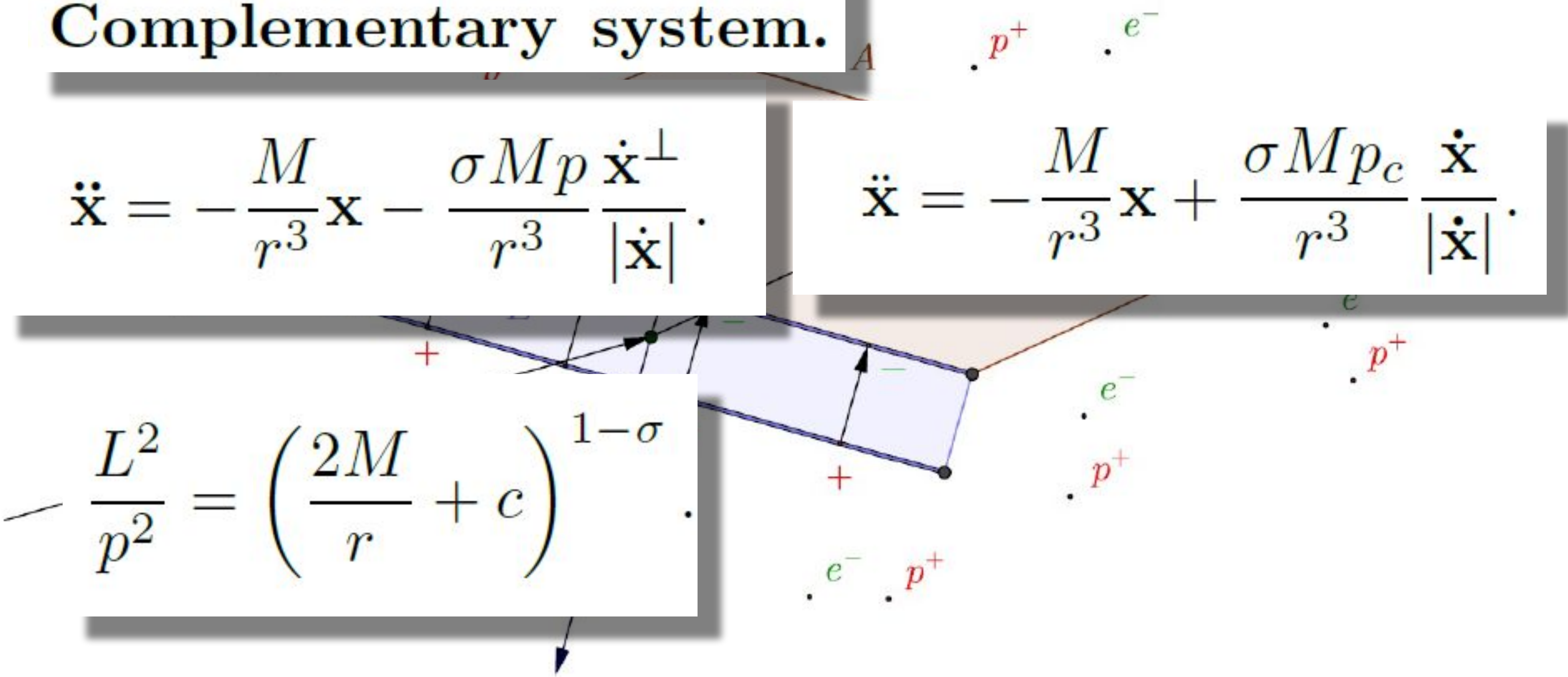

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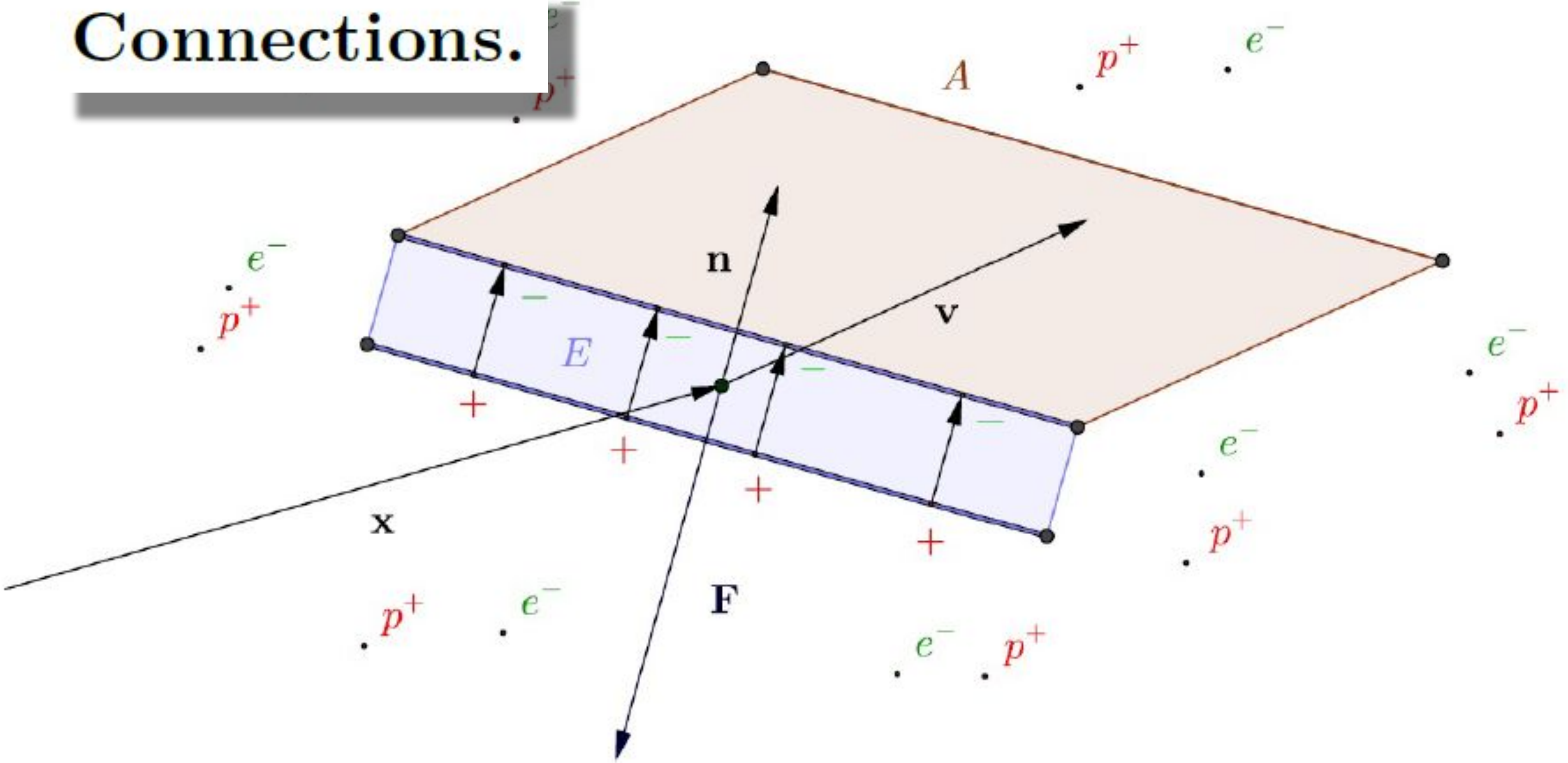
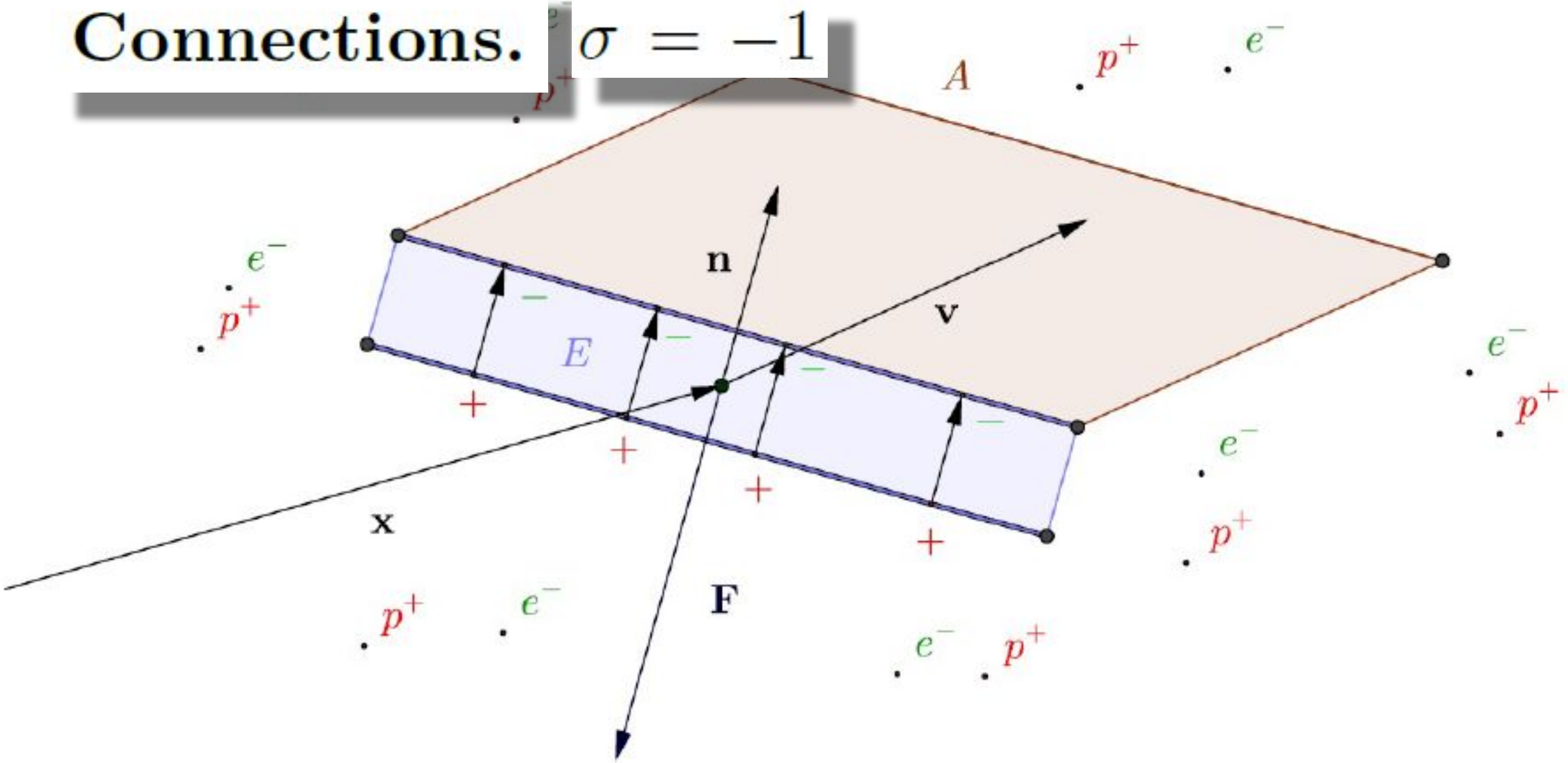
$$\frac{L^2}{p^2} = \left(\frac{2M}{r} + c\right)^{1-\sigma}.$$

$$\frac{L^2}{p^2} = \left(\frac{2M(1-\sigma)}{r} + c\right)^{\frac{1}{1-\sigma}}.$$

Connections.



Connections. $\sigma = -1$



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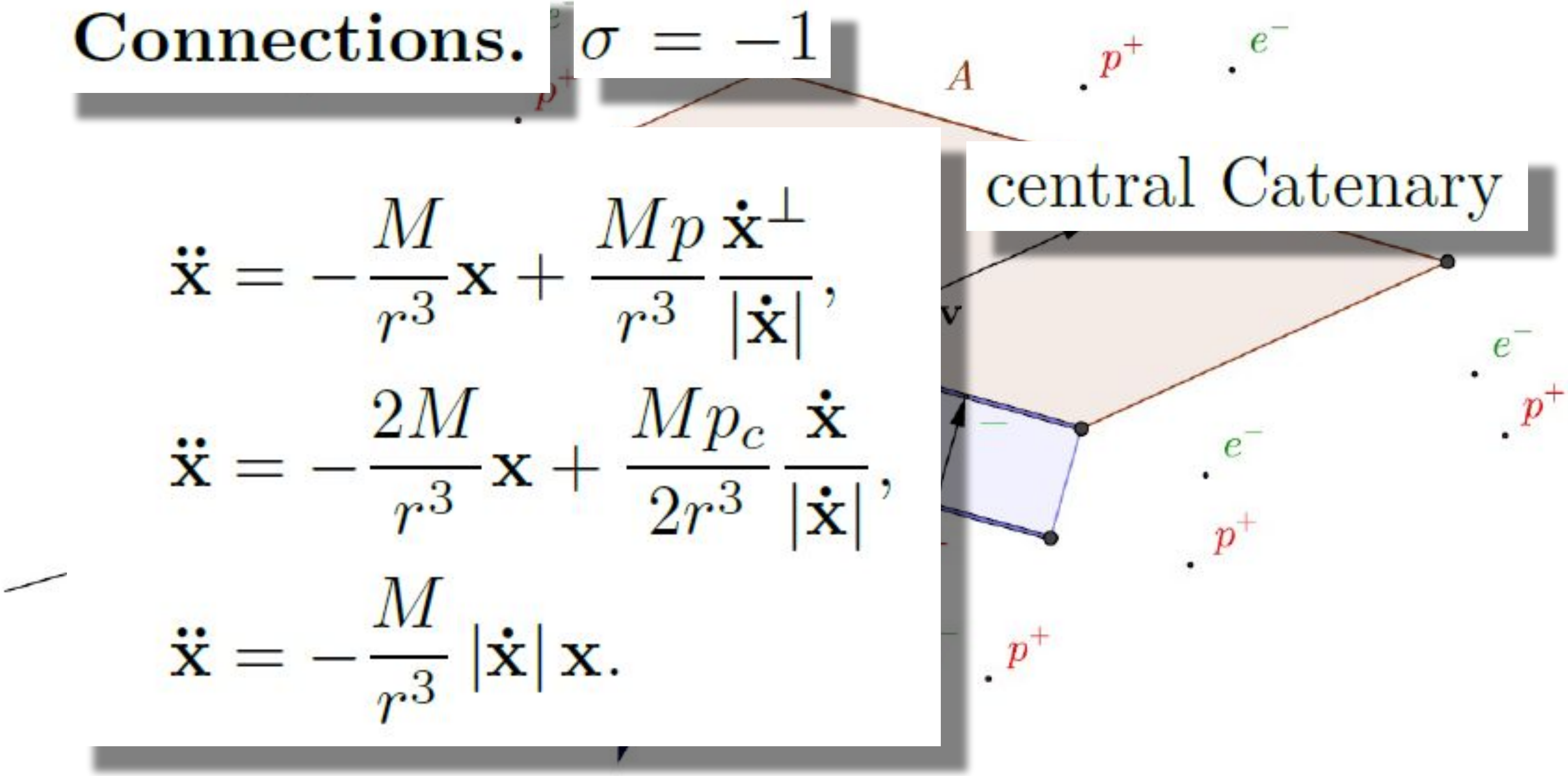

$$\ddot{\mathbf{x}} = -\frac{M}{r^3}\mathbf{x} + \frac{Mp}{r^3}\frac{\dot{\mathbf{x}}^\perp}{|\dot{\mathbf{x}}|},$$

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central Catenary

Connections. $\sigma = -1$

central Catenary

$$\ddot{\mathbf{x}} = -\frac{M}{r^3}\mathbf{x} + \frac{Mp}{r^3}\frac{\dot{\mathbf{x}}^{\perp}}{|\dot{\mathbf{x}}|},$$

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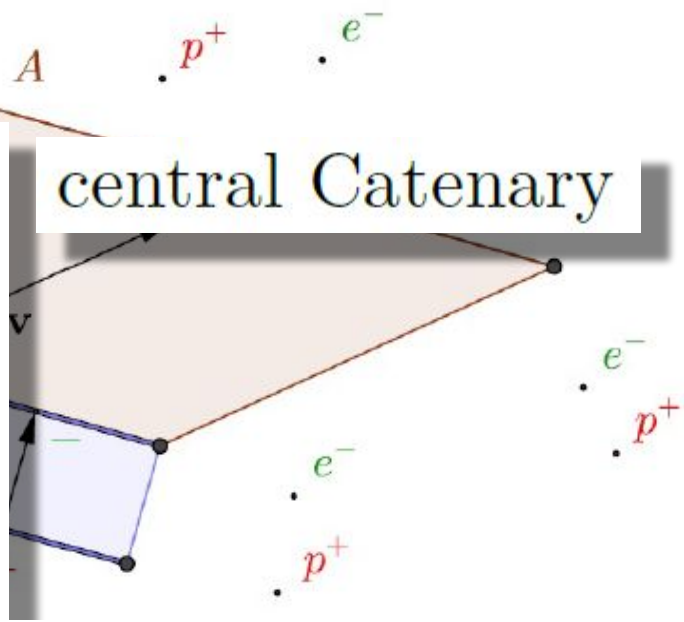

Same orbits in Pedal coordinates!

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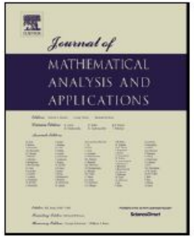

Pedal coordinates, solar sail orbits, Dipole drive and other force problems

Petr Blaschke

P. Blaschke: *Pedal coordinates, solar sail orbits, Dipole drive and other force problems*, Journal of Mathematical Analysis and Applications, Volume 506, Issue 1, 2022, 125537, ISSN 0022-247X, https://doi.org/10.1016/j.jmaa.2021.125537.

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Pedal coordinates, solar sail orbits, Dipole drive and other force problems

Petr Blaschke

JOURNAL OF MATHEMATICAL PHYSICS **58**, 063505 (2017)

Pedal coordinates, dark Kepler, and other force problems

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Thank you