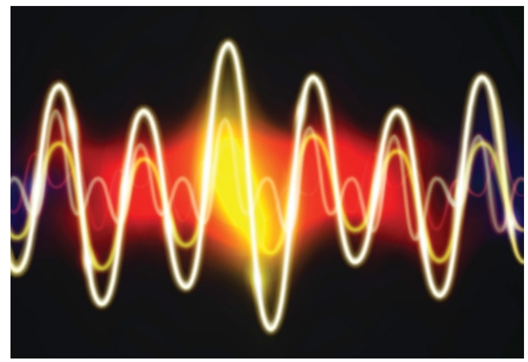
A fresh perspective on local photons and the Casimir effect

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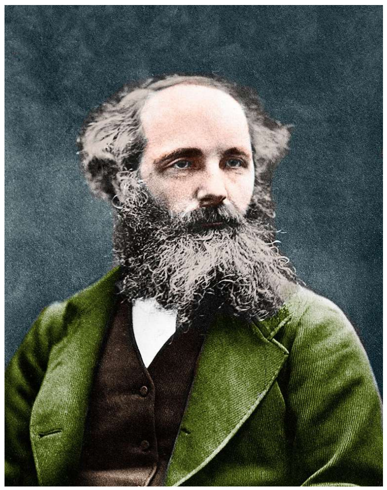
I

Classical descriptions of light

Light as waves

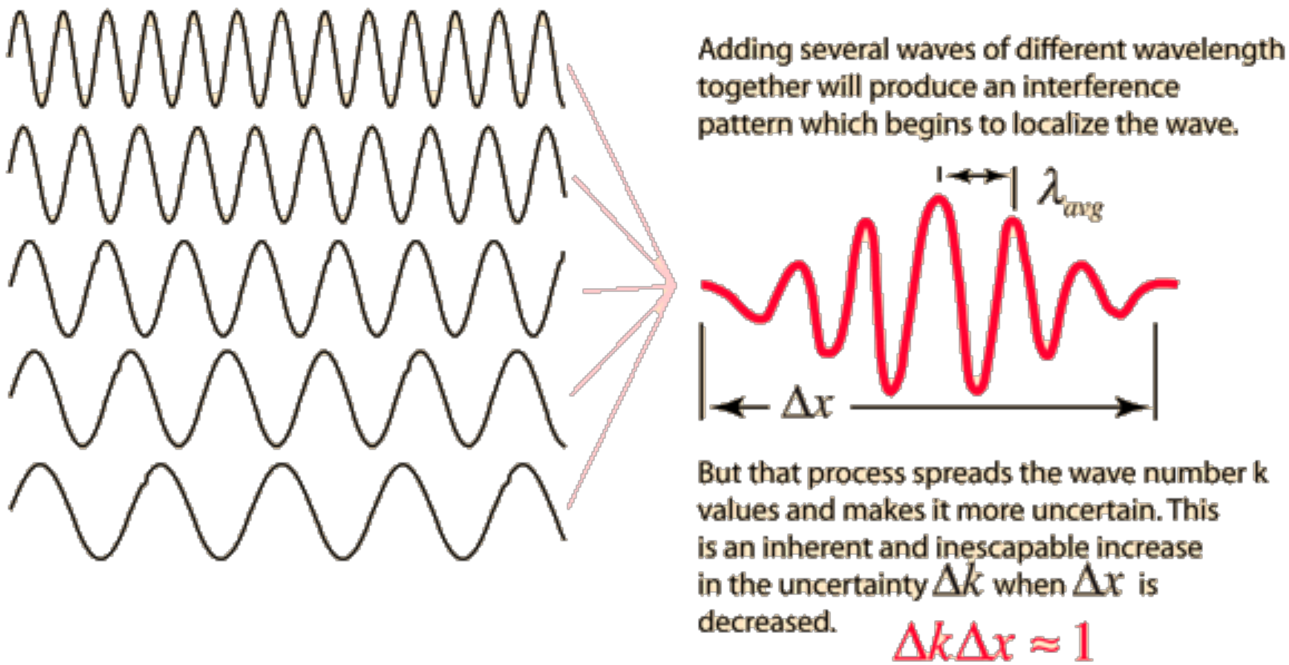


$$
\begin{aligned}
\vec{\nabla} \cdot \vec{E} &= 0 \\
\vec{\nabla} \cdot \vec{B} &= 0 \\
\vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\
\vec{\nabla} \times \vec{B} &= \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}
\end{aligned}
$$

When asked, most physicists probably say that the basic solutions MW's equations in free space are **monochromatic waves**:

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{O}(x,t) = 0 \text{ with } \mathbf{O} = \mathbf{E}, \mathbf{B}$$

Waves and wave packets



The basic solutions of MW's equations are plane travelling waves with wave numbers k and polarisations λ.

An alternative way of solving wave equations

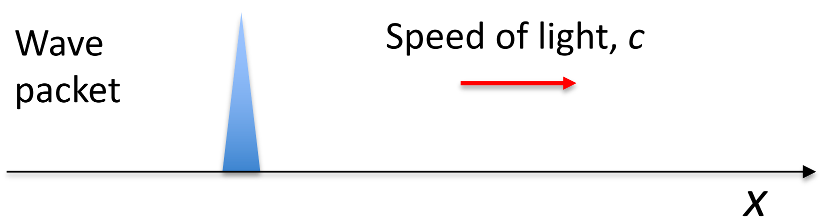
$$\begin{aligned}\vec{\nabla}\cdot\vec{E} &= 0\\ \vec{\nabla}\cdot\vec{B} &= 0\\ \vec{\nabla}\times\vec{E} &= -\frac{\partial\vec{B}}{\partial t}\\ \vec{\nabla}\times\vec{B} &= \epsilon_0\mu_0\frac{\partial\vec{E}}{\partial t}\end{aligned}$$

According to d'Alembert's principle, any **local wave packet** which moves at the speed of light solves MW's equations:

$$\left(\frac{\partial}{\partial x}+\frac{1}{c}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x}-\frac{1}{c}\frac{\partial}{\partial t}\right)\mathbf{O}(x,t)=0 \text{ with } \mathbf{O}=\mathbf{E},\mathbf{B}$$

The general solutions of MW's equations

Wave packets (WPs) of any shape are solutions of MW's equations, if they travel at the speed of light in one of two possible directions.



This includes highly-localised WPs which remain localised!

A comparison of the basic solutions

x-space solutions:

$x \in (-\infty, \infty)$
$\lambda = \mathsf{H}, \mathsf{V}$
$s = \pm 1$

k-space solutions:

$k \in (-\infty, \infty)$
$\lambda = \mathsf{H}, \mathsf{V}$
no s

There seems to be a degeneracy in the classical wave description of light.

Questions:

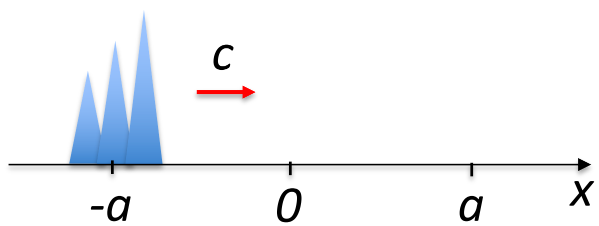
- What happens, if we quantise light directly in position space?
- Is there a local theory for photons?
- What are the implications of such a theory?

II

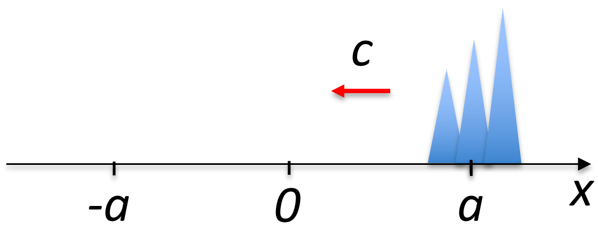
Suppose there is
a single-photon wave function.

Single-photon WPs

Situation 1:
a right-moving WP



Situation 2:
a left-moving WP

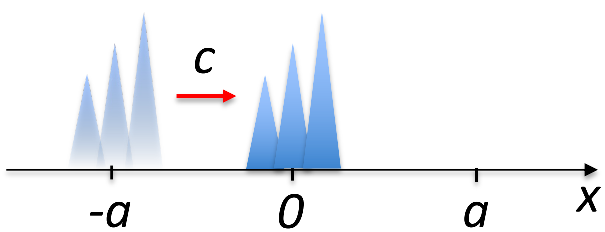


- If both WPs have the same shape, then

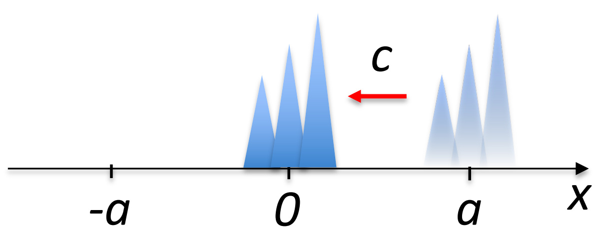
$$\psi_1(x,0) = \psi_2(x+2a,0)\,\mathrm{e}^{\mathrm{i}\varphi}\,.$$

Single-photon WPs

Situation 1:
a right-moving WP



Situation 2:
a left-moving WP



- If both WPs have the same shape, then

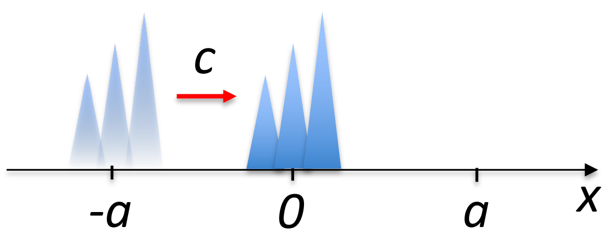
$$\psi_1(x,0) = \psi_2(x+2a,0)\,\mathrm{e}^{\mathrm{i}\varphi}\,.$$

- After a time $t = a/c$, the WPs become indistinguishable and

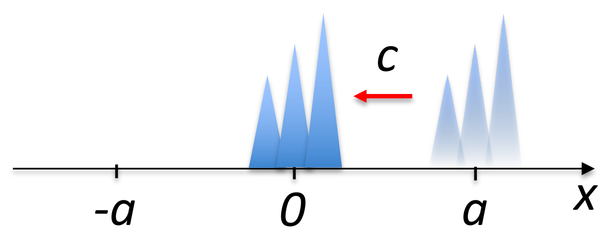
$$\psi_1(x,t) = \psi_2(x,t)\,\mathrm{e}^{\mathrm{i}\varphi}\,.$$

An additional degree of freedom is needed

Situation 1:



Situation 2:

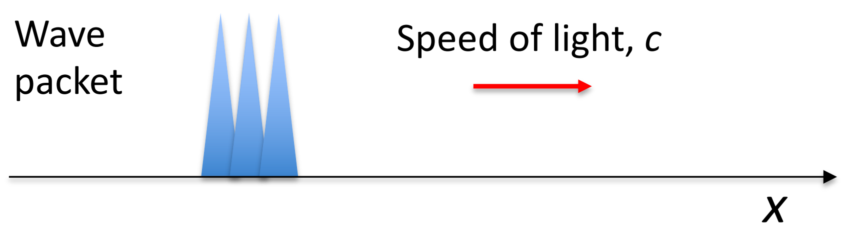


Problem: The dynamics is only unitary, if

$$\begin{aligned}\langle\psi_1(t)|\psi_2(t)\rangle &= \langle\psi_1(0)|U^\dagger(t,0)U(t,0)|\psi_2(0)\rangle \\ &= \langle\psi_1(0)|\psi_2(0)\rangle = 0\,.\end{aligned}$$

If want to model WPs of any shape which move without dispersion, the states $|\psi_1(t)\rangle$ and $|\psi_2(t)\rangle$ need to belong to different Hilbert space. In the following, we label them by $s = \pm 1$.

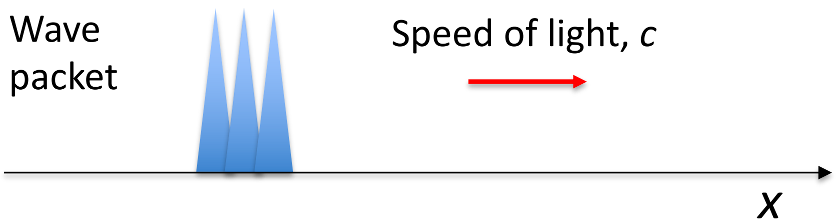
Dynamics in k-space



$s = -1$: left-moving WPs
$s = +1$: right-moving WPs

$$\psi_{s\lambda}(x,t) \quad = \quad \psi_{s\lambda}(x - sct, 0)$$

Dynamics in k-space



$s = -1$: left-moving WPs
$s = +1$: right-moving WPs

$$
\begin{aligned}
\psi_{s\lambda}(x,t) &= \psi_{s\lambda}(x - sct, 0) \\
\Rightarrow \quad \int_{-\infty}^{\infty} \mathrm{d}k\, \widetilde{\psi}_{s\lambda}(k,t)\, \mathrm{e}^{\pm \mathrm{i}kx} &= \int_{-\infty}^{\infty} \mathrm{d}k\, \widetilde{\psi}_{s\lambda}(k,0)\, \mathrm{e}^{\pm \mathrm{i}k(x - sct)} \\
\Rightarrow \quad \widetilde{\psi}_{s\lambda}(k,t) &= \widetilde{\psi}_{s\lambda}(k,0)\, \mathrm{e}^{\mp \mathrm{i}sckt}
\end{aligned}
$$

We need a field Hamiltonian with positive and negative eigenvalues.

III

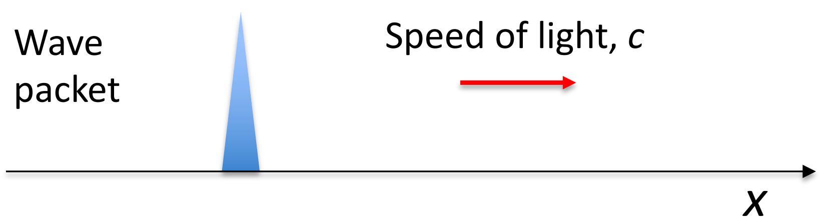
A local relativistic quantisation of the EM field

Southall, Hodgson, Purdy and Beige, *Locally acting mirror Hamiltonians*, J. Mod. Opt. **68**, 647 (2021).
Hodgson, Southall, Purdy and Beige, *Quantising one-dimensional electromagnetic fields in position space*, arXiv:2104.04499.

Local field excitations

Suppose the quantised EM field is made up of Bosons Localised in Position (**BLiPs**).

$$
\begin{aligned}
s = \pm 1 \quad &: \quad \text{direction of propagation} \\
\lambda = \mathsf{H}, \mathsf{V} \quad &: \quad \text{polarisation} \\
x \in (-\infty, \infty) \quad &: \quad \text{position}
\end{aligned}
$$



BLiPs travel at the speed of light.

Consistency with special relativity

Relativity tells us that we need to treat space and time equally.
We need to quantise the EM field in 1+1 dimension.

- **Relevant parameters:**

  s, λ, x and t $\quad\longrightarrow\quad$ $a_{s\lambda}(x,t)$

- **Relevant state vectors:**

  The possible state vectors $|\psi\rangle$ of the quantised EM field are obtained by applying $a^\dagger_{s\lambda}(x,t)$ operators to the vacuum state $|0\rangle$.

Bosonic commutator relations

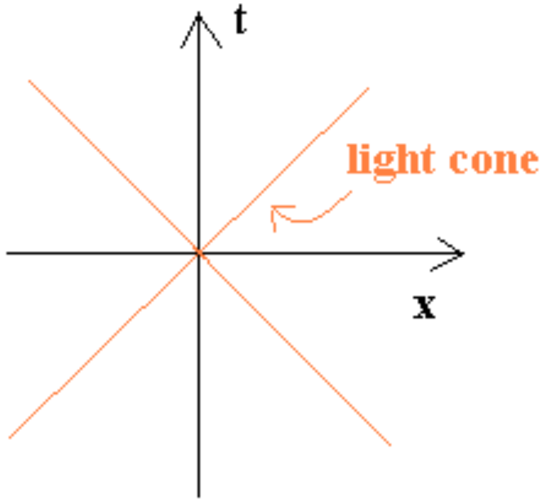
A single excitation (BLiP) state:
$$|1_{s\lambda}(x,t)\rangle = a^{\dagger}_{s\lambda}(x,t)|0\rangle$$

These states are only pairwise orthogonal if we assume that

$$\begin{aligned} \langle 1_{s\lambda}(x,t)|1_{s'\lambda'}(x',t)\rangle &= \langle 0|a_{s\lambda}(x,t)a^{\dagger}_{s'\lambda'}(x',t')|0\rangle \\ &= \left[a_{s\lambda}(x,t), a^{\dagger}_{s'\lambda'}(x',t)\right] \\ &= \delta_{s,s'}\,\delta_{\lambda,\lambda'}\,\delta(x-x')\,. \end{aligned}$$

A Hamiltonian constraint

The dynamics of light poses a constraint on the operators $a_{s\lambda}(x,t)$.



- All states belonging to the same worldline are the same.
- This equations resembles the Wheeler-DeWitt equation or a Dirac-like equation of light.

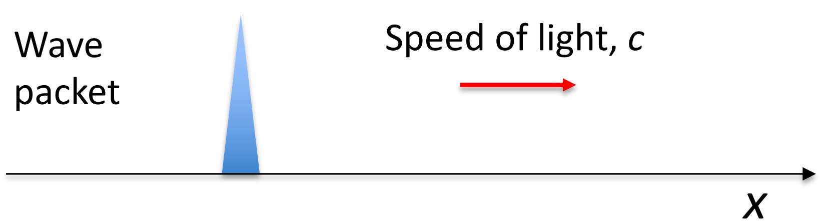
$$a_{s\lambda}(x,t) = a_{s\lambda}(x - sct, 0) \quad \Leftrightarrow \quad \left(\frac{\mathrm{d}}{\mathrm{d}x} + sc\,\frac{\mathrm{d}}{\mathrm{d}t}\right) a_{s\lambda}(x,t) = 0$$

The corresponding Hamiltonian

This dynamics is generated by a Hamiltonian:

$$\dot{a}_{s\lambda}(x,t) = -\frac{\mathrm{i}}{\hbar}\left[a_{s\lambda}(x,t), H_{\mathrm{dyn}}\right]$$

$$H_{\mathrm{dyn}} = \frac{1}{2\pi}\sum_{s=\pm1}\sum_{\lambda=1,2}\int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}x'\int_{-\infty}^{\infty}\mathrm{d}k\,\hbar ck\,\mathrm{e}^{\mathrm{i}sk(x'-x)} \times a^{\dagger}_{s\lambda}(x',t)a_{s\lambda}(x,t)\,.$$



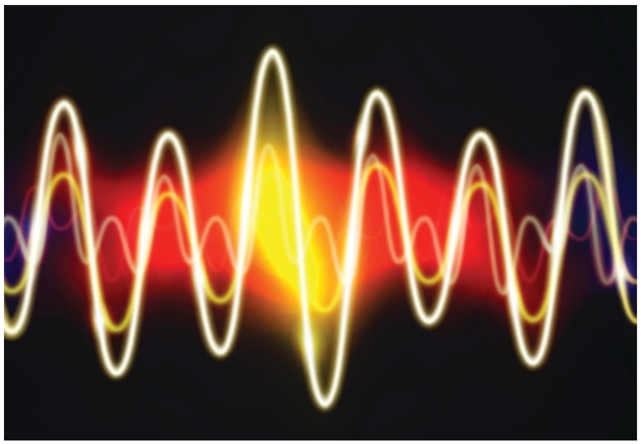
Beyond the usual Schrödinger equation

The energy observable H_{energy} has only positive eigenvalues,
while H_{dyn} has positive and negative eigenvalues.

$$
\begin{aligned}
H_{\mathrm{energy}} &= \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{A}{2} \left[\varepsilon \, \mathbf{E}(x,t)^2 + \frac{1}{\mu} \mathbf{B}(x,t)^2 \right] \\
&\neq H_{\mathrm{dyn}}
\end{aligned}
$$

H_{dyn} and H_{energy}
are no longer the same.

The basic building blocks of light



There are different ways of decomposing light into basic bosonic excitations:

1. **Monochromatic waves:**
   These correspond to standing or travelling waves.

2. **Localised wave packets:**
   BLiPs = Bosons Localised in Position

Waves and BLiPs can be superposed to obtain wave packets of finite energy, so-called photons.

IV

Electric and magnetic field observables

Southall, Hodgson, Purdy and Beige, *Locally acting mirror Hamiltonians*, J. Mod. Opt. **68**, 647 (2021).
Hodgson, Southall, Purdy and Beige, *Quantising one-dimensional EM fields in position space*, arXiv:2104.04499.

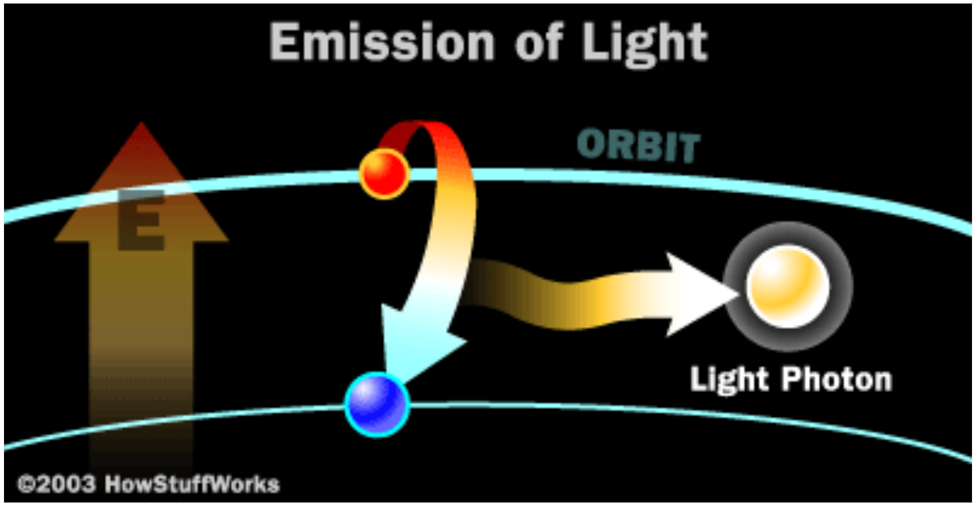
Electric and magnetic field observables

Consistency with MW's equations applies when

$$
\begin{aligned}
\mathbf{E}(x,t) &= \sum_{s=\pm 1} \mathcal{R}\big(a_{s\mathsf{H}}(x,t)\,\mathbf{y} + a_{s\mathsf{V}}(x,t)\,\mathbf{z}\big) + \mathrm{H.c.} \\
\mathbf{B}(x,t) &= -\frac{s}{c} \sum_{s=\pm 1} \mathcal{R}\big(a_{s\mathsf{V}}(x,t)\,\mathbf{y} - a_{s\mathsf{H}}(x,t)\,\mathbf{z}\big) + \mathrm{H.c.}
\end{aligned}
$$

$\mathcal{R}$: regularisation operator;
needed to impose properties,
cannot depend on x and t

The energy of a monochromatic photon



- Suppose **one** atom with energy $\hbar\omega_0$ emits exactly **one** photon.
- Due to resonance, the photon resembles a monochromatic ω_0-wave.
- Energy conservation implies that the photon has the energy $\hbar\omega_0$.

Photon annihilation operators

We link momentum and position space annihilation operators via Fourier transforms,

$$a_{s\lambda}(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}k \, \mathrm{e}^{\mathrm{i}skx} \, a_{s\lambda}(k,t)$$

$$a_{s\lambda}(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}x \, \mathrm{e}^{-\mathrm{i}skx} \, a_{s\lambda}(x,t) \, .$$

For $k \in (-\infty, \infty)$, these transformations are reversible and

$$\left[a_{s\lambda}(x,t), a_{s\lambda}^{\dagger}(x',t)\right] = \delta(x-x') \quad \Leftrightarrow \quad \left[a_{s\lambda}(k,t), a_{s\lambda}^{\dagger}(k',t)\right] = \delta(k-k') \, .$$

Field observables in momentum space

$$
\begin{aligned}
H_{\mathrm{energy}} &= \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{A}{2} \left[\varepsilon \, \mathbf{E}(x,t)^2 + \frac{1}{\mu} \mathbf{B}(x,t)^2 \right] \\
&= \int_{-\infty}^{\infty} \mathrm{d}k \, \hbar c |k| \, a_{s\lambda}^{\dagger}(k,t) a_{s\lambda}(k,t)
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \quad E_{s\lambda}(x,t) &\propto \int_{-\infty}^{\infty} \mathrm{d}k \, \sqrt{|k|} \, a_{s\lambda}(k,t) + \mathrm{H.c.}\,, \\
B_{s\lambda}(x,t) &\propto \frac{s}{c} \int_{-\infty}^{\infty} \mathrm{d}k \, \sqrt{|k|} \, a_{s\lambda}(k,t) + \mathrm{H.c.}
\end{aligned}
$$

. . . which are Lorentz covariant

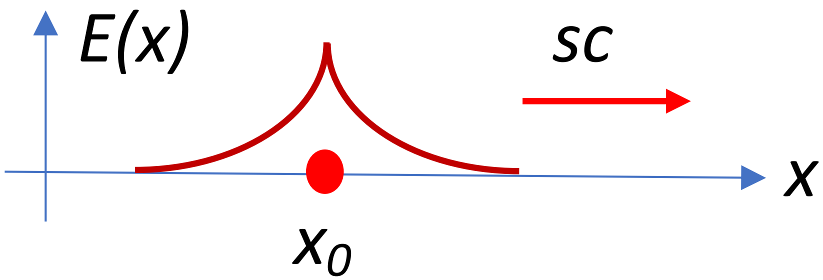
(In momentum space, $\mathcal{R}$ multiplies operators with a factor proportional to $\sqrt{|k|}$.)

Field observables in position space

$$
\Rightarrow \quad \begin{aligned}
E_{s\lambda}(x,t) &= \int_{-\infty}^{\infty} \mathrm{d}x'\, g(x,x')\, a_{s\lambda}(x',t) + \text{H.c.}\,, \\
B_{s\lambda}(x,t) &= \frac{s}{c}\int_{-\infty}^{\infty} \mathrm{d}x'\, g(x,x')\, a_{s\lambda}(x',t) + \text{H.c.}
\end{aligned}
$$

$$
\begin{aligned}
\text{with} \quad g(x,x') &\propto \int_{-\infty}^{\infty} \mathrm{d}k \left(\frac{2|k|}{\pi}\right)^{1/2} \mathrm{e}^{\mathrm{i}k(x-x')} \\
&= -|x-x'|^{-3/2}
\end{aligned}
$$

A physical picture of BLiPs



Comments:

- The fields of a local BLiP state at x_0 can be felt everywhere.
- Localised fields can only be created by a non-local source.

The dynamical Hamiltonian

In the Heisenberg picture, we find that

$$a_{s\lambda}(x,t) = a_{s\lambda}(x - sct, 0)$$
$$\Rightarrow \quad a_{s\lambda}(k,t) = \mathrm{e}^{-\mathrm{i}ckt}\, a_{s\lambda}(k,0)\,.$$

In the momentum representation, $H_{\rm dyn}$ simplifies
to the usual harmonic oscillator Hamiltonian

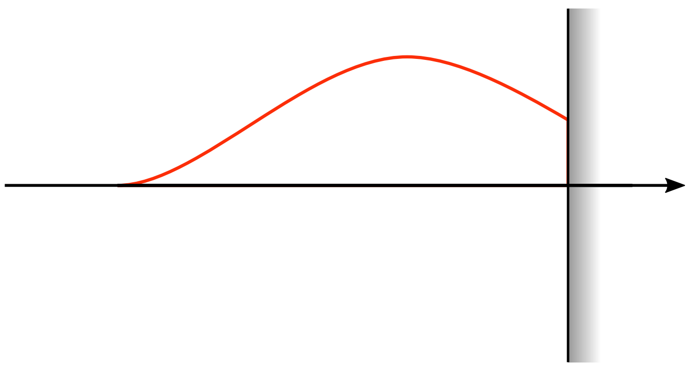
$$\begin{aligned} H_{\rm dyn} &= \sum_{s=\pm 1} \sum_{\lambda=1,2} \int_{-\infty}^{\infty} \mathrm{d}k\, \hbar ck\, a^\dagger_{s\lambda}(k,t) a_{s\lambda}(k,t)\,. \\ &= H_{\rm energy} \text{ if we set } a_{s\lambda}(k,t) \equiv 0 \text{ for } k < 0 \end{aligned}$$

V

Applications: Light reflection by mirrors

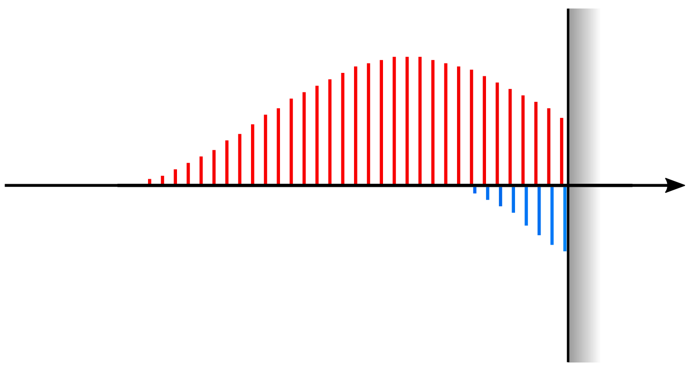
Southall, Hodgson, Purdy and Beige, *Locally acting mirror Hamiltonians*, J. Mod. Opt. **68**, 647 (2021).

Light reflection by mirrors



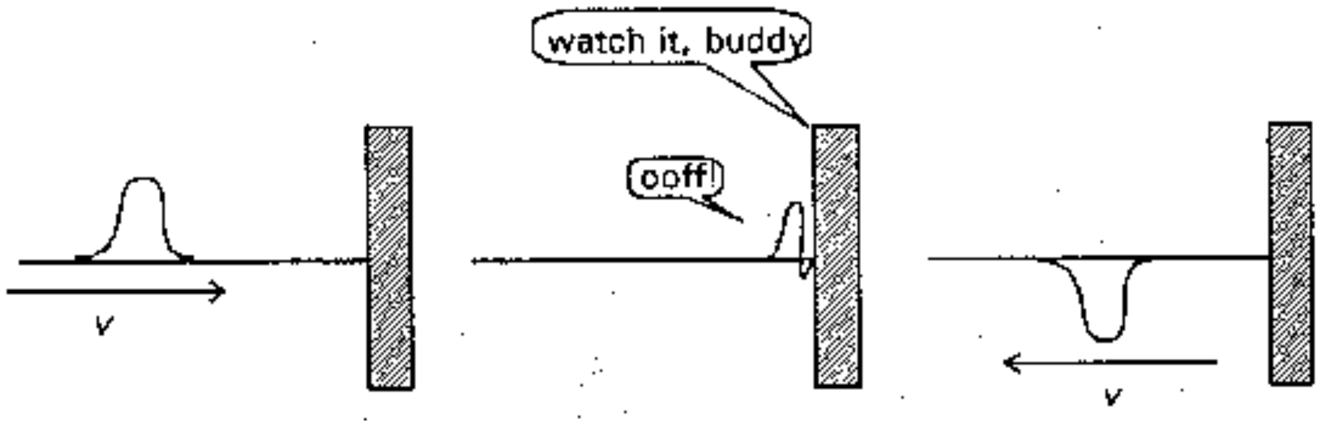
Modelling light reflection using waves is not very intuitive.
In classical ED, we describe light scattering in position space.

The mirror image method



A mirror changes the amplitude and direction of any incoming BLiP excitations.

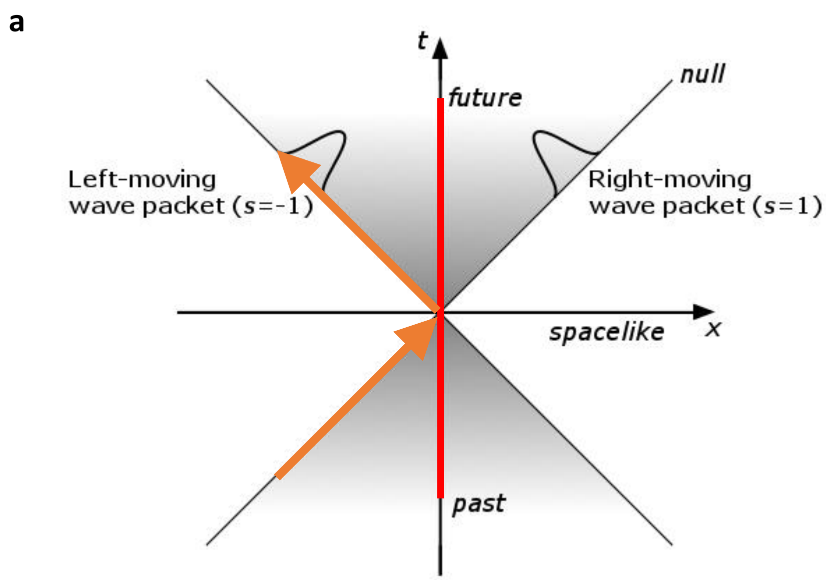
Light scattering by mirrors

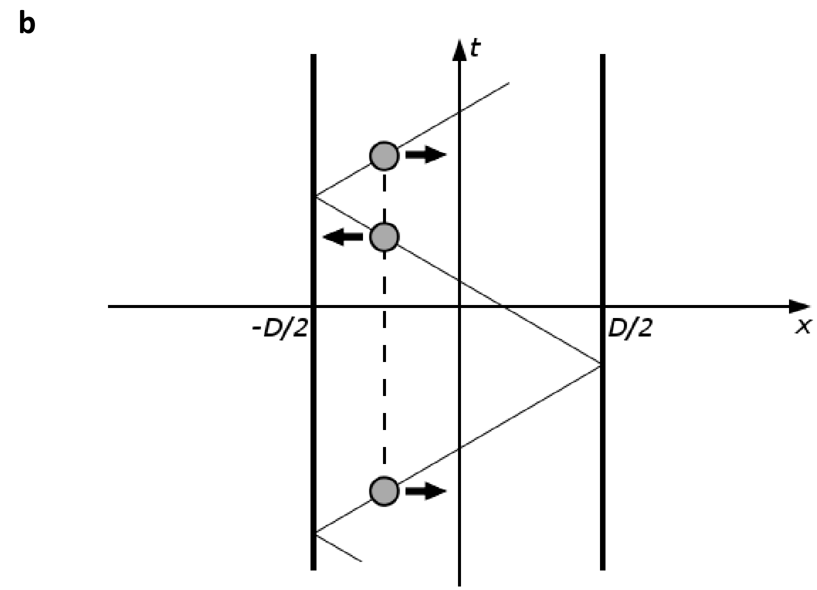


We can now describe two-sided semi-transparent mirrors with a locally acting mirror Hamiltonian.

Southall, Hodgson, Purdy and Beige, *Locally acting mirror Hamiltonians*, J. Mod. Opt. **68**, 647 (2021).

The dynamics of BLiPs inside an optical cavity





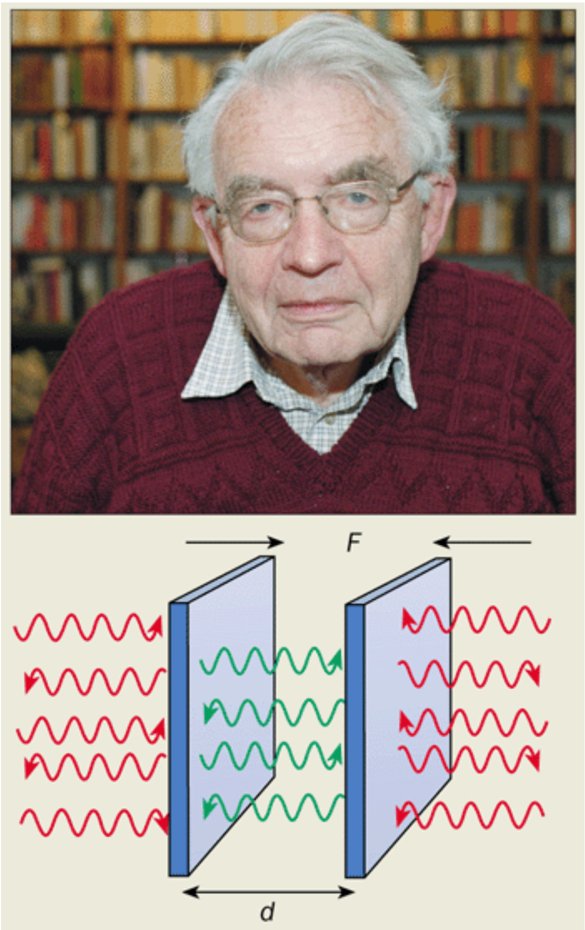
Inside an optical cavity, many different BLiP excitations travel back and force between both mirrors.

VI

Applications: The Casimir effect

Hodgson, Burgess, Altaie, Beige and Purdy, *An intuitive picture of the Casimir effect*, arXiv:2203.14385 (2022).

Experimental setup



- Already in 1948, Casimir predicted an attractive force between two metallic mirrors for small distances d.
- This effect can be understood as a consequence of boundary conditions imposed by the mirrors and is attributed to vacuum fluctuations.
- Obtaining a finite force requires regularisation procedures.

The electric field inside the cavity

Suppose $\mathcal{X}$ restricts the Hilbert space to BLiP excitations at positions $x \in (-D/2, D/2)$.

$$\boldsymbol{E}^{(\mathrm{in})}_{s\lambda}(x,t)$$

$$= \sum_{n=-\infty}^{\infty} \mathcal{X}\left(\boldsymbol{E}^{(\mathrm{free})}_{s\lambda}(x+2nD,t)\right.$$

$$\left. -\boldsymbol{E}^{(\mathrm{free})}_{-s\lambda}(-x+(2n-1)D,t)\right)$$

BLiPs inside the cavity cannot create fields outside and vice versa.

The electric field inside the cavity

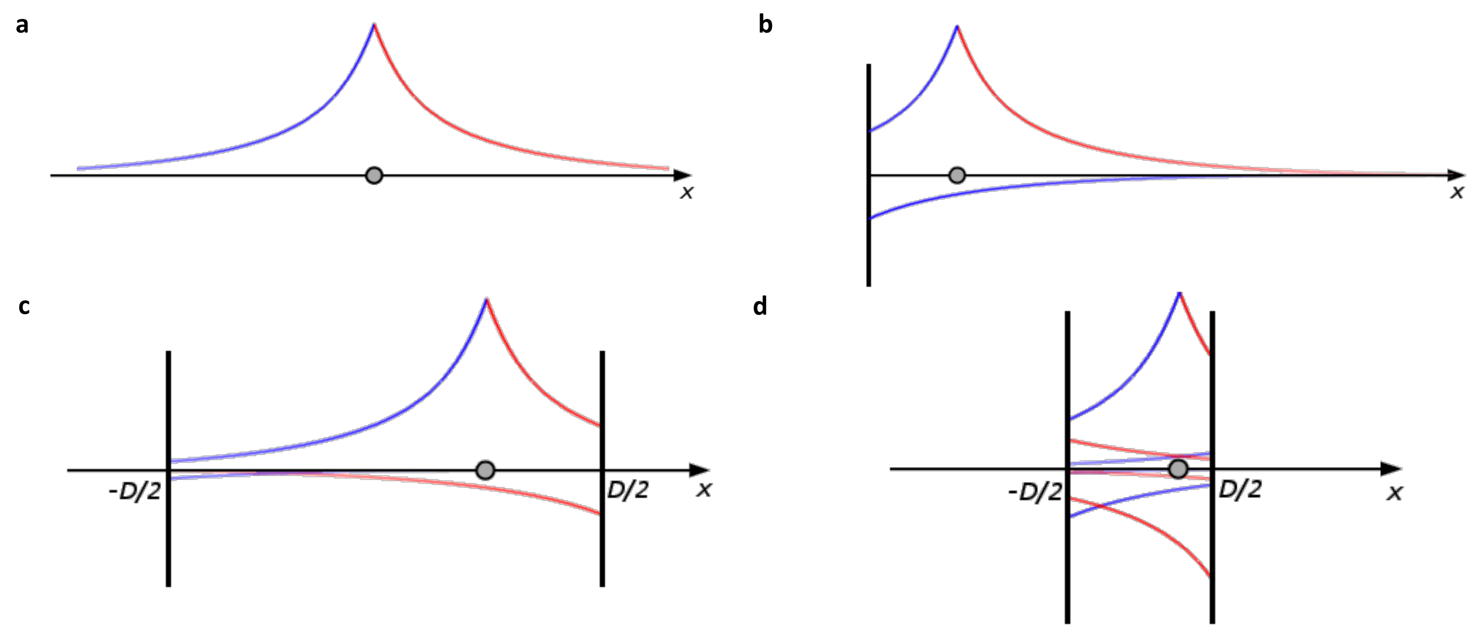


Figure 2: **a.** Because of the regularisation operator $\mathcal{R}$ in Eq. (6), local blip excitations contribute to local electric and magnetic field expectation values everywhere along the x axis (cf. Eq. (8)). **b.** Since a blip on one side of a highly reflecting mirror cannot contribute to the field expectation value on the other side, its field contribution must be folded back on itself. This effect alters the electric and magnetic field observables in the presence of a mirror. **c.** In the presence of two highly reflecting mirrors, blips outside the cavity cannot contribute to field expectation values on the inside. Moreover, the field contributions of blips on the inside need to be folded as in the case of one mirror. Now, however, the field contributions must be folded infinitely many times (cf. Eq. (18) in Methods). **d.** Comparing two cavities of different sizes, we see that the behaviour of the field contribution is now dependent on the cavity width.

The zero point energy of the EM field

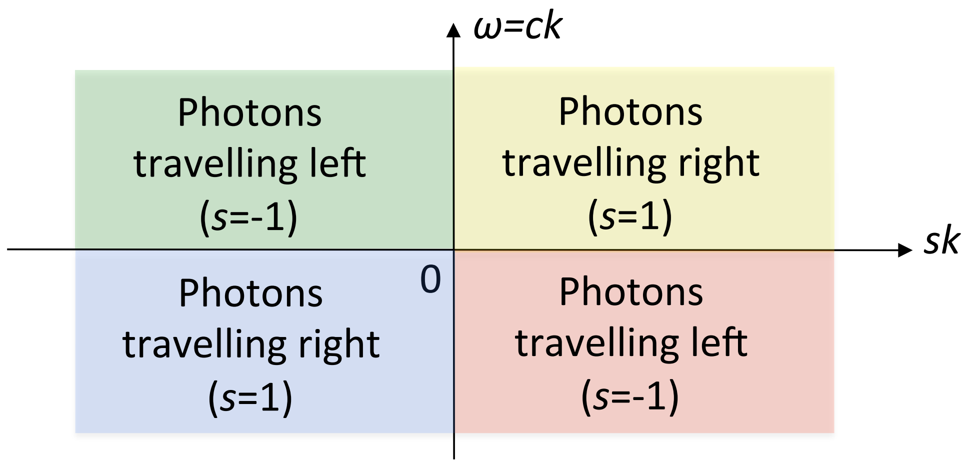
$$
\begin{aligned}
H_{\mathsf{ZPE}}^{(\mathsf{in})} &= \frac{\hbar c}{4\pi} \sum_{n,m=-\infty}^{\infty} \int_{-D/2}^{D/2} \mathsf{d}x \int_{-D/2}^{D/2} \mathsf{d}x' \\
&\qquad \left[\left| (x + x' + (2n-1)D)(x + x' + (2m-1)D) \right|^{-3/2} \right. \\
&\qquad \left. + \left| (x - x' + 2nD)(x - x' + 2mD) \right|^{-3/2} \right] \\
&= -\frac{\hbar c}{2\pi D} \sum_{m=-\infty}^{\infty} \frac{1}{m^2} \quad \Rightarrow \quad F_{\text{Casimir}} = -\frac{\mathrm{d}H_{\text{ZPE}}}{\mathrm{d}D} = -\frac{\pi \hbar c}{6D^2}
\end{aligned}
$$

The Casimir effect is due to interference effects of evanescent fields belonging to opposite sides of the cavity.

VII

Final remarks

The main message of this talk



The standard description of the quantised EM field in quantum optics is incomplete.

Comments

We quantised the EM field for light moving in 1D **in position space.** No-go theorems have been overcome by considering the positive and the negative-frequency solutions of MW's equations.

Our approach can be used to derive **Casimir forces without regularisation** (up to a factor 2).

Recommended references:

[1] **Locally acting mirror Hamiltonians**,
Southall, Hodgson, Purdy and Beige, J. Mod. Opt. **68**, 647 (2021).

[2] **Quantising one-dimensional electromagnetic fields in position space**,
Hodgson, Southall, Purdy and Beige, arXiv:2104.04499 (2022).

[3] **An intuitive picture of the Casimir effect**,
Hodgson, Burgess, Altaie, Beige and Purdy, arXiv:2203.14385 (2022).

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[4] **A physically-motivated quantisation of the electromagnetic field**,
Bennett, Barlow and Beige, Eur. J. Phys. **37**, 014001 (2016).

[5] **Time and Quantum Clocks: a review of recent developments**,
Altaie, Beige and Hodgson, Front. Phys. **10**, 897305 (2022).

See also related work by M. Hawton, R. J. Cook, J. Sipe, I. Bialynicky-Birula, Smith and Raymer and others.