SOLITON EQUATIONS AND THEIR HOLOMORPHIC SOLUTIONS

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PLAN OF THE COURSE

Lecture 1. Riemann problem and soliton equations.

Lecture 2. Local holomorphic inverse scattering.

Lecture 3. The Painlevé property.

Aims:

Describe all $(1+1)$-dim soliton equations of parabolic type and ALL of theit local holomorphic solutions using a local version of the inverse scattering method.

Discuss most interesting classes of solutions.

Give a CRITERION for solubility of the Cauchy problem in terms of the scattering data of the initial condition.

Prove the Painlevé property: ALL local holomorphic solutions are globally meromorphic and monodromy-free in the spatial variable.

LECTURE 1

RIEMANN PROBLEM AND SOLITON EQUATIONS

1. Examples and history

Korteweg–de Vries, nonlinear Schrödinger and Boussinesq eqs

$$(1) \qquad u_t = au_{xxx} + buu_x, \qquad a, b \in \mathbb{C} \setminus \{0\},$$

$$(2) \qquad iu_t = au_{xx} + bu|u|^2, \qquad a, b \in \mathbb{R} \setminus \{0\},$$

$$(3) \qquad u_{tt} = au_{xxxx} + buu_{xx} + bu_x^2, \qquad a, b \in \mathbb{C} \setminus \{0\}$$

with $|u(x,t)|^2$ in (2) understood as $u(x,t)\overline{u(\overline{x},\overline{t})}$.

$$(\mathrm{KdV}) \qquad u_t = au_{xxx} + buu_x$$

describes long waves on shallow water.

The inverse scattering method as discovered
by Gardner, Green, Kruskal, Miura 1967:
when the potential $u(x,\cdot)$ evolves according to (KdV),
the evolution of its scattering data
(spectral characteristsics of $L = \partial_x^2 + u(x,\cdot)$ on $L^2(\mathbb{R}^1_x)$)
is linear and can be integrated explicitly!

Explanation (Lax 1968): (KdV) with $a = 1/4$, $b = 3/2$ is the compatibility condition for the **auxiliary linear problem**

$$L\psi = \lambda\psi, \qquad \psi_t = P\psi \tag{4}$$

where $L := \partial_x^2 + u$ and $P := \partial_x^3 + (3/2)u\partial_x + (3/4)u_x$,
i.e. equation (KdV) can be written as $\boxed{L_t = [P, L]}$.
$\Longrightarrow$ the evolution of the operator L is its conjugaton by a t-dependent unitary operator on $L^2(\mathbb{R}^1_x)$
$\Longrightarrow$ the evolution of its spectral characteristics is simple.

$$\text{(NLS)} \qquad iu_t = au_{xx} + bu|u|^2$$

describes propagation of wave packages in nonlinear dispersive media.

Zakharov and Shabat (1971) replaced the scalar second-order ODE $L\psi = \lambda\psi$ in (4) by a first-order 2×2-matrix system:

$$(5) \qquad E_x = UE, \quad \text{and then} \quad E_t = VE$$

where $U(x,t,z)$ and $V(x,t,z)$ are polynomials of degree 1 and 2 in the spectral parameter $z \in \mathbb{C}$ related to λ in (4) by $\lambda = z^2$.

Hence (NLS) arises from the zero curvature equation

$$U_t - V_x + [U,V] = 0 \tag{6}$$

(the compatibility condition of the auxiliary linear problem (5); first explicitly written and used in soliton theory by Novikov 1974).

The Boussinesq equation (3) (describing water waves that can move left or right) was similarly studied by Zakharov (1973): replace 2×2-matrices in (5) or the second-order operator L in (4) by 3×3-matrices or a third-order operator.

2. Zero curvature equations

Lemma 1. *Let $\Omega \subset \mathbb{C}^2_{xt}$ be a simply connected domain and $U, V : \Omega \to \mathrm{gl}(n, \mathbb{C})$ holomorphic maps. Then the system*

$$E_x = UE, \quad E_t = VE \tag{ALP}$$

has a holomorphic solution $E : \Omega \to \mathrm{GL}(n, \mathbb{C})$ if and only if

$$U_t - V_x + [U, V] = 0 \quad \textit{everywhere in} \quad \Omega. \tag{ZCC}$$

Definition. A $(1 + 1)$-dimensional *soliton equation* is the ZCC $U_t - V_x + [U, V] = 0$, where $U, V : \Omega \times \mathbb{C}P^1_z \to \mathrm{gl}(n, \mathbb{C})$ are rational functions of $z \in \mathbb{C}P^1$ such that the expression $U_t - V_x + [U, V]$ is independent of z.

Unknown functions are the entries of the $\mathrm{gl}(n, \mathbb{C})$-valued coefficients of the partial fraction decompositions of U, V. Some of them can be expressed in terms of the others from the condition "$U_t - V_x + [U, V]$ is independent of z".

The type of a soliton equation is determined by the lists of poles of U and V counting multiplicities. For example,

(1) the poles of U and V are single, simple and distinct $\Longrightarrow$ **hyperbolic type** (PCM, Sine-Gordon);
(2) the poles of U and V are single and coincide $\Longrightarrow$ **parabolic type** (Kdv, NLS, Boussinesq).

Structure of soliton equations of parabolic type.

Consider the case when the common pole of U and V is ∞,
so U and V are polynomials in z, $\deg U = 1$ and $\deg V = m \geq 2$.
Then there is no loss of generality in assuming that

$$\begin{cases} U(x,t,z) = az + q(x,t), \\ V(x,t,z) = bz^m + r_1(x,t)z^{m-1} + \cdots + r_m(x,t), \end{cases}$$

where $a, b \in \mathrm{gl}(n,\mathbb{C})$ are **diagonal** matrices,
a has **simple spectrum** (all eigenvalues are distinct),
$q : \Omega \to \mathrm{gl}(n,\mathbb{C})$ is holomorphic and **off-diagonal**,
$r_1, \ldots, r_m : \Omega \to \mathrm{gl}(n,\mathbb{C})$ are holomorphic.

Claim: the condition "$U_t - V_x + [U, V]$ is independent of z" determines $r_1, \ldots, r_m$ as **differential polynomials** $r_j = F_j(q)$ (i.e. polynomials in q and its derviatives with respect to x) uniquely up to diagonal constants of integration at each step.

The functions $F_j(q)$ satisfy the **resolvent equation**

$$\boxed{F_x = [U, F]}, \quad \text{where} \quad U := az + q, \quad F := b + \sum_{j=1}^{\infty} \frac{F_j(q)}{z^j}$$

and we have $U_t - V_x + [U, V] = q_t - [a, F_{m+1}(q)]$. Hence

$$\text{ZCC takes the form} \quad \boxed{q_t = [a, F_{m+1}(q)]}.$$

In more detail,

Lemma 2. *Fix a, b as above and any point $x_0 \in \mathbb{C}$ and put*

$$\mathcal{O}(x_0) := \{\textit{holomorphic } \mathrm{gl}(n,\mathbb{C})\textit{-valued germs } q(x) \textit{ at } x_0\},$$
$$\mathcal{O}(x_0)^{od} := \{\textit{all off-diagonal } q \in \mathcal{O}(x_0)\}.$$

Then for every sequence $c_1, c_2, \dots \in \mathrm{gl}(n,\mathbb{C})$ of diagonal matrices there is a unique sequence of differential polynomials $F_j : \mathcal{O}(x_0) \to \mathcal{O}(x_0)$, $j = 0, 1, 2, \dots$ such that

1) *$F_0(q) \equiv b$ for all $q \in \mathcal{O}(x_0)$,*
2) *$F_j(0) \equiv c_j$ for all $j \geqslant 1$,*
3) *the formal power series $F(q, z) := \sum_{j=0}^{\infty} F_j(q) z^{-j}$ satisfies*

$$\partial_x F = [az + q, F] \quad \textit{for all } q \in \mathcal{O}(x_0)^{od}.$$

Proof is not competely trivial since the resolvent equation

$$\partial_x D_{j-1} = [q, N_{j-1}]_d,$$
$$[a, N_j] = \partial_x N_{j-1} - [q, F_{j-1}]_{od}$$

(where $j = 0, 1, 2, \ldots$, $F_{-1} := 0$, $D_j := (F_j)_d$, $N_j := (F_j)_{od}$) determines $F_j(q)$ only as integro-differential polynomials in q (since a has simple spectrum, the operator $X \mapsto [a, X]$ is invertible on the space of off-diagonal matrices in $\mathrm{gl}(n, \mathbb{C})$). Use the equalities $\partial_x F^k = [U, F^k] \Longrightarrow \partial_x \operatorname{tr} F^k = 0$ to get another formula for D_j, solving a linear system. $\square$

Lemma 3. *Fix a, b as above. Then*

$$U_t - V_x + [U,V] \text{ is independent of } z \iff r_j = F_j(q)(j = 1, \dots, m)$$

for some sequence $c_1(t), \dots, c_m(t)$ of diagonal matrices holomorphically depending on t in the projection of Ω to the t-axis.

In this case, the ZCC $\boxed{U_t - V_x + [U,V] = 0}$ with $U = az + q$ and $V = az^m + F_1(q)z^{m-1} + \cdots + F_m(q)$ takes the form

$$q_t = [a, F_{m+1}(q)], \tag{Eq_m}$$

where $q(x,t)$ is the unknown off-diagonal $\mathrm{gl}(n,\mathbb{C})$-valued germ at the point $(x_0, t_0) \in \mathbb{C}^2$.

Examples. (almost always $b = a$, $c_1 = c_2 = \cdots = 0$)

1) The hierarchy of the heat equation:

$$a = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \quad q(x,t) = \begin{pmatrix} 0 & u(x,t) \\ 0 & 0 \end{pmatrix} \quad \Longrightarrow$$

$$\Longrightarrow \quad (\mathrm{Eq}_m) \quad \text{takes the form} \quad \boxed{\partial_t u = \partial_x^m u}.$$

2) The Korteweg–de Vries equation:

$$a = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \quad q(x,t) = \begin{pmatrix} 0 & u(x,t) \\ 1 & 0 \end{pmatrix} \quad \Longrightarrow$$

$$\Longrightarrow \quad (\mathrm{Eq}_3) \quad \text{takes the form} \quad \boxed{u_t = u_{xxx} - 6uu_x}.$$

3) The nonlinear Schrödinger equation:

$$a = \begin{pmatrix} -i/2 & 0 \\ 0 & i/2 \end{pmatrix}, \quad q(x,t) = \begin{pmatrix} 0 & u(x,t) \\ -\overline{u(\overline{x},\overline{t})} & 0 \end{pmatrix} \quad \Longrightarrow$$

$$\Longrightarrow \quad (\mathrm{Eq}_2) \quad \text{takes the form} \quad \boxed{iu_t = -u_{xx} - 2u|u|^2}.$$

4) The modified Korteweg–de Vries equation ($C := (c_2)_{11} - (c_2)_{22}$):

$$a = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \quad q(x,t) = \begin{pmatrix} 0 & u(x,t) \\ u(x,t) & 0 \end{pmatrix} \quad \Longrightarrow$$

$$\Longrightarrow \quad (\mathrm{Eq}_3) \quad \text{takes the form} \quad \boxed{u_t = u_{xxx} + (6u^2 + C)u_x}.$$

3. Riemann problem and ZS dressing

$Riemann\ problem$: Let $D_+, D_- \subset \mathbb{C}P^1_z$ be disjoint open disks with $\overline{D}_+ \cup \overline{D}_- = \mathbb{C}P^1$. A continuous function $\gamma : \Gamma \to \mathrm{GL}(n, \mathbb{C})$ on the circle $\Gamma := \overline{D}_+ \cap \overline{D}_-$ is said to be **left-factorable** (resp. **right-factorable**) if there are continuous functions $\gamma_\pm : \overline{D}_\pm \to \mathrm{GL}(n, \mathbb{C})$, holomorphic on $D_\pm$ and satsifying $\boxed{\gamma = \gamma_+ \gamma_-^{-1}}$ on Γ (resp. $\boxed{\gamma = \gamma_-^{-1} \gamma_+}$ on Γ).

We regard γ as **data** of the Riemann problem, and the pair (γ_+, γ_-) as a **solution**. When a solution exists, it is unique up to right (resp. left) multiplication by $A \in \mathrm{GL}(n, \mathbb{C})$.

Zakharov–Shabat dressing method: Let (U_0, V_0) be a holomorphic solution of ZCC on $\Omega \subset \mathbb{C}^2_{xt}$, E_0 a holomorphic solution of ALP on Ω, $E_0(x_0, t_0) = I$, and $g : \Gamma \to \mathrm{GL}(n, \mathbb{C})$ right-factorable, where $\Gamma = \overline{D}_+ \cap \overline{D}_-$ does not pass through the poles of U_0, V_0.

For every $(x, t) \in \Omega$ we pose the Riemann problem of finding continuous $\theta_\pm : \overline{D}_\pm \to \mathrm{GL}(n, \mathbb{C})$ holomorphic on $D_\pm$ with

$$E_0(x, t, z) g(z) E_0^{-1}(x, t, z) = \theta_-^{-1}(x, t, z) \theta_+(x, t, z), \quad z \in \Gamma.$$

A solution $\theta_\pm$ exists in a neighbourhood of (x_0, t_0) and, actually, in Ω except maybe a pole along a complex curve.

For uniqueness require $\theta_-(x,t,z_0) = I$ for all (x,t) and some z_0. Then the pair (U_1, V_1) defined by the formulae

$$U_1(x,t,z) = \begin{cases} (\theta_+)_x \theta_+^{-1} + \theta_+ U_0 \theta_+^{-1}, & z \in \overline{D}_+, \\ (\theta_-)_x \theta_-^{-1} + \theta_- U_0 \theta_-^{-1}, & z \in \overline{D}_- \end{cases}$$

and similarly for $V_1(x,t,z)$ with $U_0 \longrightarrow V_0$ and $\partial_x \longrightarrow \partial_t$ is a meromorphic solution of ZCC on Ω with **the same divisors of poles in** z for U_1, V_1 and U_0, V_0.
Proofs will be given in a more concrete context below.

Definition. The solution (U_1, V_1) is said to be obtained by **dressing the solution** (U_0, V_0) **by means of** g.

A limiting case of dressing for SEPT.

For **soliton equations of parabolic type**, the divisors of poles of U and V are ∞ and $m\infty$ respectively. Hence it is natural to shrink the disk $D_- = \{|z| > R\} \cup \{\infty\}$ to a single point ∞ and expand $D_+ = \{|z| < R\}$ to the whole plane $\mathbb{C}$ as $R \to +\infty$. The dressing function $g(z)$ becomes an element of the set

$$\mathcal{D} := \{\text{holom } \mathrm{GL}(n, \mathbb{C})\text{-valued germs } f(z) \text{ at } \infty \text{ with } f(\infty) = I\}$$

We will dress the zero solution $(U_0, V_0) = (0, 0)$ of ZCC or, equivalently, the solution $q(x, t) \equiv 0$ of equation (Eq_m), by the invertible holomorphic germ $f^{-1}(z)$ for every $f \in \mathcal{D}$.

THEOREM 1. *Suppose that* $a, b, c_1, c_2, \dots \in \mathrm{gl}(n, \mathbb{C})$ *are diagonal matrices and* a *has simple spectrum. Fix an integer* $m \geq 2$ *and a point* $(x_0, t_0) \in \mathbb{C}^2$. *For every* $f \in \mathcal{D}$ *let* $\Omega(f)$ *be the set of all* $(x, t) \in \mathbb{C}^2$ *such that the function*

$$\gamma(x,t,z) := \exp\{az(x-x_0)+(bz^m+c_1 z^{m-1}+\cdots+c_m)(t-t_0)\}f^{-1}(z)$$

is right-factorable on some (and then any) circle $\{|z|=R\}$, $R_0 < R < +\infty$. *Then* $\Omega(f) \subset \mathbb{C}^2$ *contains a neighbourhood of* (x_0, t_0). *For every point* $(x, t) \in \Omega(f)$ *let* $(\gamma_+(x, t, z), \gamma_-(x, t, z))$ *be the solution of the Riemann problem*

$$\gamma(x,t,z) = \gamma_-^{-1}(x,t,z)\gamma_+(x,t,z), \quad R_0 < |z| < +\infty \tag{7}$$

with normalization $\gamma_-(x,t,\infty) = I$. *Put*

$$q(x,t) := \lim_{z\to\infty} z[\gamma_-(x,t,z) - I, a]. \tag{8}$$

Then $q : \Omega(f) \to \mathrm{gl}(n,\mathbb{C})$ *is off-diagonal, holomorphic at* (x_0,t_0) *and satisfies the soliton equation* (Eq_m)

$$q_t = [a, F_{m+1}(q)] \quad \text{in a neighbourhood of } (x_0,t_0), \tag{9}$$

where $F_j : \mathcal{O}(x_0) \to \mathcal{O}(x_0)$ $(j = 0,1,2,\dots)$ *is the sequence of differential polynomials determined by the sequence* $a, b, c_1, \dots$ *by Lemma* 2.

Remark. In what follows we denote this $q(x,t)$ by $q_f(x,t)$.

Proof. 1) The Riemann problem $\gamma = \gamma_-^{-1}\gamma_+$ is equivalent (for every $R \geq R_0$) to a linear equation $A\xi = \eta$ on the Banach space E of all $\mathrm{gl}(n, \mathbb{C})$-valued functions holomorphic for $|z| > R$, continuous for $|z| \geq R$ and vanishing at $z = \infty$.
(Here $\eta := -P\gamma$ and $A\xi := P(\xi\gamma)$, where P is the operator of taking the negative part of a Laurent series. The solutions $\gamma_\pm$ are recovered by the formulae $\gamma_- = I + \xi$, $\gamma_+ = (I + \xi)\gamma$.)
The operator A and the vector η depend holomorphically on $(x, t) \in \mathbb{C}^2$, and $A = \mathrm{Id}$ at $(x, t) = (x_0, t_0)$. Hence A is invertible in a neighbourhood of (x_0, t_0) and the solution ξ is holomorphic there. Thus $\Omega(f)$ contains a neighbourhood of (x_0, t_0) and $\gamma_\pm(x, t)$ are holomorphic there.

2) Claim: the function $\gamma_{+x}\gamma_+^{-1} - az$ is independent of z and equal to $q(x,t)$, so that for all (x,t) near (x_0,t_0) and all $z \in \mathbb{C}$ we have

$$\gamma_{+x} = U(x,t,z)\gamma_+, \quad \text{where} \quad U(x,t,z) := az + q(x,t). \tag{10}$$

Indeed, write $\gamma_+(x,t,z) = \gamma_-(x,t,z)E_0(x,t,z)f^{-1}(z)$ (where $E_0(x,t,z) := \exp\{az(x-x_0) + (bz^j + c_1 z^{j-1} + \cdots + c_j)(t-t_0)\}$) and differentiate with respect to x:

$$\begin{aligned}
&\gamma_{+x}\gamma_+^{-1} = (\gamma_- E_0 f^{-1})_x(\gamma_- E_0 f^{-1})^{-1} = \\
&\qquad \text{(since } f \text{ is independent of } x) = (\gamma_- E_0)_x(\gamma_- E_0)^{-1} = \\
&\qquad\qquad (\gamma_{-x} + \gamma_- az)E_0(\gamma_- E_0)^{-1} = \gamma_{-x}\gamma_-^{-1} + z \cdot \gamma_- a\gamma_-^{-1}.
\end{aligned}$$

As Laurent series in z, the left- and right-hand sides of

$$\gamma_{+x}\gamma_{+}^{-1} = \gamma_{-x}\gamma_{-}^{-1} + z \cdot \gamma_{-}a\gamma_{-}^{-1}$$

contain only powers $\geqslant 0$ and $\leqslant 1$ respectively. Hence both are polynomials of degree $\leqslant 1$ in z:

$$\gamma_{+x}\gamma_{+}^{-1} = \{z\gamma_{-}a\gamma_{-}^{-1}\}_{+},$$

where $\{\cdot\}_{+}$ is the "non-negative part" of a Laurent series: if $X = \sum_{n=-\infty}^{\infty} X_n z^n$, then $X_{+} = \sum_{n=0}^{\infty} X_n z^n$.
A calculation gives $\{z\gamma_{-}(x,t,z)a\gamma_{-}^{-1}(x,t,z)\}_{+} = az + q(x,t)$, where $q(x,t)$ is defined by (8). This proves (10).

3) Repeating part 2 with x replaced by t, we have

$$\gamma_{+t}\gamma_+^{-1} = \{z^m F(x,t,z)\}_+, \tag{11}$$

where $F(x,t,z) := \gamma_-(x,t,z)(b+c_1 z^{-1}+\cdots+c_m z^{-m})\gamma_-(x,t,z)^{-1}$. Claim: for every fixed t the function $F(x,t,z)$ satisfies the resolvent equation $\boxed{F_x = [U,F]}$ near x_0, where $U(x,t,z) = az + q(x,t)$. Indeed, write $F(x,t,z) = \gamma_-(x,t,z)\Phi(t,z)\gamma_-^{-1}(x,t,z)$, where $\Phi(t,z)$ is diagonal and independent of x. Differentiate $F\gamma_- = \gamma_-\Phi$ with respect to x and multiply by γ_-^{-1}. We get $F_x + F\gamma_{-x}\gamma_-^{-1} = \gamma_{-x}\Phi\gamma_-^{-1}$, i.e. $F_x = [\gamma_{-x}\gamma_-^{-1}, F]$. Here $\gamma_{-x}\gamma_-^{-1}$ may be replaced by U since $U - \gamma_{-x}\gamma_-^{-1} = \gamma_- az\gamma_-^{-1}$ commutes with $F = \gamma_-\Phi\gamma_-^{-1}$ (the matrices az and Φ are diagonal). This proves that $F_x = [U,F]$.

By (the proof of) the uniqueness in Lemma 2, it follows that

$$F(x,t,z) = \sum_{l=0}^{\infty} F_l^{(a,b,c_1'(t),\dots,c_l'(t))}(q)(x,t)z^{-l} \tag{12}$$

for some sequence $c_1'(t), c_2'(t), \dots$ of diagonal matrices depending only on t. For all $k = 1, 2, \dots$ and all z near ∞ we have

$$\operatorname{tr} F(x,t,z)^k = \begin{cases} \operatorname{tr}(b + c_1'(t)z^{-1} + c_2'(t)z^{-2} + \dots)^k, \\ \operatorname{tr}(b + c_1 z^{-1} + \dots + c_j z^{-j})^k, \end{cases}$$

(first by the proof of Lemma 2, second by the definition of F after (11)). Since all matrices on the right are diagonal, we conclude that $c_l'(t) \equiv c_l$ for $l = 1, \ldots, m$. Hence the differential polynomials in (12) coincide with the differential polynomials F_l determined by $a, b, c_1, \ldots$ by Lemma 2. Thus (11) and (12) yield

$$\gamma_{+t} = V(x,t,z)\gamma_+, \tag{13}$$

where $V(x,t,z) := \sum_{j=0}^{m} F_{m-j}(q)(x,t)z^j$.

4) Differentiate (13) with respect to x, (10) with respect to t, equate the results and divide by the invertible matrix γ_+. This gives the ZCC for U, V defined above, that is, by Lemma 3, the soliton equation (9) for $q(x,t)$. $\square$

Geometrical meaning of solutions of the RP.

$\boxed{\gamma_+(x,t)}$ is a parallel frame field with respect to the flat connection $\nabla(q) := d - U(q)\,dx - V(q)\,dt$,

$$E_x = UE, \quad E_t = VE, \quad \text{where } E = \gamma_+,$$

and $\boxed{\gamma_-(x,t)}$ is a gauge transformation of the trivial flat connection $\nabla(0) = d - az\,dx - bz^m\,dt$ to the connection $\nabla(q)$,

$$dm = \{U(q)\,dx + V(q)\,dt\}m - m\{U(0)\,dx + V(0)\,dy\},$$

where $m = \gamma_-$, $U(q) = az + q$,
$V(q) = bz^m + F_1(q)z^{m-1} + \cdots + F_m(q)$.

LEMMA 4 (NON-UNIQUENESS OF A GERM $f \in \mathcal{D}$ AS THE "SCATTERING DATA" OF THE "POTENTIAL" $q_f \in \mathcal{O}(x_0)^{od}$). *Functions $f, g \in \mathcal{D}$ determine the same solution $q_f(x,t) = q_g(x,t)$ of* Eq_m *near $(x_0, t_0) \in \mathbb{C}^2 \iff$ the function $f^{-1}g \in \mathcal{D}$ is diagonal $\iff q_f(x,t_0) = q_g(x,t_0)$ near $x_0 \in \mathbb{C}$.*

Why should the map $f \mapsto q_f$ be regarded as the inverse scattering transform?

Because it is so, under appropriate reduction of non-uniqueness, whenever the rapidly decaying or finite-gap versions apply.

Non-uniqueness is something useful (see "Addition of a soliton").

4. Examples of local inverse scattering transforms

A. Upper-triangular case and the Laplace transform.

Let $u : [0, +\infty) \to \mathbb{C}$ be a locally integrable function with $|u(x)| \le Ae^{Bx}$ for all $x \ge 0$ and some $A, B \in \mathbb{R}$. The *Laplace transform* of u is

$$\text{(LAP)} \qquad Lu(z) := \int_0^\infty e^{-zx} u(x) dx.$$

The function $Lu(z)$ is holomorphic for $\Re z > B$ and tends to 0 as $\Re z \to +\infty$. If $Lu_1 = Lu_2$, then $u_1 = u_2$. The function $Lu(z)$ is rational (and vanishes at $z = \infty$) $\iff$ $u(x)$ is an exponential polynomial, that is a finite linear combination of $x^k e^{cx}$ with integer $k \geq 0$ and complex c.

More generally, $Lu(z)$ is holomorphic in a neighbourhood of ∞ (and vanishes at $z = \infty$) $\Longleftrightarrow$ u is an entire function of exponential type, i.e. $u \in \mathcal{O}(\mathbb{C}^1)$ and there are $A, B > 0$ such that $|u(x)| \leq Ae^{B|x|}$ for all $x \in \mathbb{C}$. Then the inverse transform $u = B\varphi$ is called the *Borel transform* and is given by the formula

$$\text{(BOR)}\quad \varphi(z) = \sum_{j=1}^{\infty} \frac{\varphi_j}{z^j} \quad \Longrightarrow \quad B\varphi(x) = \sum_{k=0}^{\infty} \varphi_{k+1} \frac{x^k}{k!}, \quad x \in \mathbb{C}.$$

LEMMA 5. *Fix a diagonal matrix* $a = \mathrm{diag}(a_{11}, a_{22})$ *with* $\alpha := a_{11} - a_{22} \neq 0$ *and put* $x_0 = 0$, $E_0(x,z) := \exp(azx)$. *For every germ* $f \in \mathcal{D}$ *we regard the solution* $q_f : \Omega(f) \to \mathrm{gl}(2,\mathbb{C})$ *of* Eq_m *with* $b = c_1 = c_2 = \cdots = 0$ *as a function of* x *only. Then:*

(A) f *is upper-triangular for* $|z| > R_0 \iff q_f(x)$ *is upper-triabgular near* $x_0 = 0$.

(B) *If* $q \in \mathcal{O}(x_0)$ *is upper-triangular and* $q = q_f$ *for some* $f \in \mathcal{D}$, *then there is a unique upper-triangular* $f \in \mathcal{D}$ *such that* $q = q_f$ *and* $f_{11} = f_{22} \equiv 1$. *Define* $\mathbb{C}$*-valued functions* $\varphi(z)$ *and* $u(x)$ *by putting*

$$f(z) = \begin{pmatrix} 1 & \varphi(z) \\ 0 & 1 \end{pmatrix}, \quad q(x) = q_f(x) = \begin{pmatrix} 0 & u(x) \\ 0 & 0 \end{pmatrix}.$$

Then the bijection $f \leftrightarrow q$ corresponds to the following relation between $\varphi(z)$ and $u(x)$:

$$\text{if} \quad \varphi(z) = \sum_{j=1}^{\infty} \frac{\varphi_j}{z^j}, \quad \text{then} \quad u(x) = -\sum_{k=0}^{\infty} \frac{\alpha^{k+1} x^k \varphi_{k+1}}{k!}.$$

Thus $u(x) = -\alpha B\varphi(\alpha x)$, where $B\varphi$ is the Borel transform (BOR) *of φ. In particular, $u(x)$ is an entire function of exponential type. Conversely, any off-diagonal upper-triangular map $q : \mathbb{C}^1_x \to \mathrm{gl}(n, \mathbb{C})$ with $n = 2$ whose only non-zero entry $q_{12}(x)$ is an entrie function of exponential type, can be written in the form $q = q_f$ for some $f \in \mathcal{D}$.*

B. Constant potentials.

In the upper-triangular case, the maps $f \mapsto q_f$ and $q_f \mapsto f$ appear to be linear (after natural normalization). This is not the case in general, already for constant potentials (the simplest case of finite-gap potentials).

LEMMA 6. *Let $a \in \mathrm{gl}(n, \mathbb{C})$ be a diagonal matrix with simple spectrum and $C \in \mathrm{gl}(n, \mathbb{C})$ any off-diagonal matrix. Then there is a function $f \in \mathcal{D}$ such that $q_f(x, t_0) = C$ for all $x \in \mathbb{C}^1$. (The choice of $b, c_1, \ldots$ is irrelevant).*

Proof The matrix $a + Cz^{-1}$ is holomorphic in z and equals a at $z = \infty$. Hence, for all z in a neighbourhood of ∞, its eigenvalues are distinct and their eigenspaces depend holomorphically on z. Thus there is an $f \in \mathcal{D}$ such that the function

$$r(z) := f^{-1}(z)(a + Cz^{-1})f(z)$$

is *diagonal* and holomorphic in a neighbouhrood of ∞. Here the columns of $f(z)$ form a basis of eigenvectors of $a + Cz^{-1}$, which is uniquely determined by the condition that f is holomorphic and $f(\infty) = I$. Substituting the Laurent expansions

$$f(z) = I + f_1 z^{-1} + \dots \quad \text{and} \quad r(z) = r_0 + r_{-1} z^{-1} + \dots$$

into the definition ofe $r(z)$, we get $r_0 = a$ and $r_{-1} = C + [a, f_{-1}]$. But the left-hand side of the last equality is diagonal and the right-hand side is off-diagonal. Hence $r_{-1} = 0$. Thus the diagonal function $az - zr(z)$ vanishes at $z = \infty$. This means that the function $z \mapsto e^{(az - zr(z))x}$ belongs to $\mathcal{D}$ for every $x \in \mathbb{C}$. Since the matrices azx and $zr(z)x$ are diagonal, we have

$$e^{azx} f^{-1}(z) = e^{(az - zr(z))x} f^{-1}(z) \cdot f(z) e^{zr(z)x} f^{-1}(z)$$

for all $x \in \mathbb{C}$ and $|z| > R$. The first factor on the right belongs to $\mathcal{D}$, and the second is equal to $e^{(az+C)x}$. Thus the Riemann problem with $x_0 = 0$ and $t = t_0$ is solved for all $x \in \mathbb{C}$, and its solution involves $\gamma_+(x, t_0, z) = e^{(az+C)x}$. It follows that $q_f(x, t_0) \equiv C$. $\square$

C. Blaschkle factors and addition of a soliton.

For any $\alpha, \beta \in \mathbb{C}$ and any projection $P : \mathbb{C}^n \to \mathbb{C}^n$ (i.e. a linear operator with $P^2 = P$) we put

$$B_{\alpha\beta P}(z) := I + \frac{\beta - \alpha}{z - \beta} P = \mu P + (I - P), \qquad \mu := \mu(z) = \frac{z - \alpha}{z - \beta}.$$

For any vector subspaces A, B of $\mathbb{C}^n$ with $A \oplus B = \mathbb{C}^n$ there is a unique projection P such that $A = \operatorname{im} P$ and $B = \ker P$. It is called the *projection onto A along B*. When A, B are regarded as "coordinate axes" in $\mathbb{C}^n$, the operator $B_{\alpha\beta P}(z)$ multiplies all vectors of the first axis by $\mu(z)$ and leaves all vectors of the second axis fixed. The functions $B_{\alpha\beta P}(z)$ are called *Blaschke factors*.

LEMMA 7. *Suppose that* $E : \mathbb{C} \to \mathrm{GL}(n, \mathbb{C})$ *is an entire function,* $\alpha, \beta \in \mathbb{C}$ *and* $P : \mathbb{C}^n \to \mathbb{C}^n$ *is the projection onto* $A \subset \mathbb{C}^n$ *along* $B \subset \mathbb{C}^n$*. Then* $E(z)B^{-1}_{\alpha\beta P}(z)$ *is right-factorable* $\iff$ *the subspaces* $E(\alpha)A$ *and* $E(\beta)B$ *are transversal:*

$$E(\alpha)A \cap E(\beta)B = \{0\}.$$

If this condition holds, then the solution (E_1, f_1) *of the Riemann problem* $EB^{-1}_{\alpha\beta P} = f_1^{-1}E_1$ *is given by the formula*

$$f_1(z) = B_{\alpha\beta Q}(z),$$

where Q *is the projection to* $E(\alpha)A$ *along* $E(\beta)B$*.*

Let $q_f(x,t)$ be obtained from $f \in \mathcal{D}$ by Theorem 1. Given any $\alpha, \beta \in \mathbb{C}$ and any transversal subspaces $A, B \subset \mathbb{C}^n$, we put $h := B_{\alpha\beta P} f$, where P is the projector to A along B and say that the solution $q_h(x,t)$ of Eq_m is obtained from the solution $q_f(x,t)$ by *adding a soliton*. We claim that

$$\text{(AS)} \qquad q_h(x,t) = q_f(x,t) + (\beta - \alpha)[P_f(x,t), a],$$

where $P_f(x,t)$ is the projector to $\gamma_+^f(x,t,\alpha)A$ along $\gamma_+^f(x,t,\beta)B$. Indeed, suppose that the subspaces $\gamma_+^f(x,t,\alpha)A$ and $\gamma_+^f(x,t,\beta)B$ are transversal for some point $(x,t) \in \Omega(f)$. Put $E_0(x,t,z) := \exp\{az(x-x_0) + (bz^j + c_1 z^{j-1} + \cdots + c_j)(t-t_0)\}$. By Lemma 7, the expression

$$E_0(x,t,z)h^{-1}(z) = E_0(x,t,z)f^{-1}(z)B^{-1}_{\alpha\beta P}(z) = \\ = (\gamma^f_-)^{-1}(x,t,z)\gamma^f_+(x,t,z)B^{-1}_{\alpha\beta P}(z)$$

takes the form $(\gamma^f_-)^{-1}(x,t,z)B^{-1}_{\alpha\beta P_f(x,t)}(z)E_1(x,t,z)$ for some invertible entire function $E_1(x,t,\cdot)$. Hence $(x,t)\in\Omega(h)$ and

$$\gamma^h_-(x,t,z) = B_{\alpha\beta P_f(x,t)}(z)\gamma^f_-(x,t,z)$$

for all z near ∞. Comparing the Laurent coefficients for $|z|>R_0$, we obtain (AS).