Local inverse scattering

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Abstract. We develop a local version of the inverse scattering method for studying soliton equations of parabolic type (this includes, for example, Korteweg–de Vries, nonlinear Schrödinger, and Boussinesq equations, but not sine-Gordon). The potentials are germs of holomorphic matrix-valued functions, without any boundary conditions. The scattering data are matrix-valued formal power series in the spectral parameter. We give a precise description of all possible scattering data and exact criteria for solubility of the local holomorphic Cauchy problem for a soliton equation of parabolic type in terms of the scattering data of the initial conditions. As an application, we prove the strongest possible version of the Painlevé property for such equations: every local holomorphic solution admits a global meromorphic extension with respect to the space variable.

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1. Introduction

The first general result in the theory of partial differential equations was the Cauchy–Kowalevsky theorem [1,2]. We need only the following simple version of it. *Let P be a holomorphic function of x, t in a neighborhood of a given point $(x_0, t_0) \in \mathbb{C}^2$ and a polynomial in the other variables. Then the Cauchy problem*

$$\partial_t^m u = P(x, t, \{\partial_x^k \partial_t^l u\}_{k+l \le m, (k,l) \ne (0,m)});$$
$$\partial_t^j u(x, t_0) = \varphi_j(x), \quad 0 \le j \le m-1,$$

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has a unique local holomorphic solution $u(x,t)$ *in a neighborhood of* (x_0,t_0) *for all holomorphic germs* $\varphi_0,\varphi_1,\dots,\varphi_{m-1}\in\mathcal{O}(x_0)$. To explain the necessity of the conditions $k+l\le m$ and $(k,l)\ne(0,m)$, Kowalevsky [2] established the following theorem on forced analytic extension for solutions of the heat equation. *Each solution* $u\in\mathcal{O}(D)$ *of the equation* $u_t=u_{xx}$ *on an arbitrary bidisk* $D=\{(x,t)\in\mathbb{C}^2\,|\,|x-x_0|<\varepsilon_1,\ |t-t_0|<\varepsilon_2\}$ *admits an analytic continuation to a solution* $\widetilde{u}\in\mathcal{O}(S)$ *of the same equation on the strip* $S=\{(x,t)\in\mathbb{C}^2\,|\,|t-t_0|<\varepsilon_2\}$. In other words, every local holomorphic solution $u(x,t)$ extends to an entire function of x for each admissible value of t. Subsequent works of Salekhov [3], Kiselman [4] and Zerner [5] showed that the same assertion holds for all local holomorphic solutions $u(x,t)$ of all equations in the following larger classes:

$$\partial_t^p u=\partial_x^m u+\sum_{j=0}^{m-1}c_j\partial_x^j u, \tag{1}$$

$$\partial_x^m u=\sum_{k+l<m}c_{kl}\partial_x^k\partial_t^l u, \tag{2}$$

$$\partial_x^m u=\sum_{k+l<m}a_{kl}(x,t)\partial_x^k\partial_t^l u, \tag{3}$$

where $m\ge 2$ and p, $1\le p<m$ are integers, $c_j,c_{kl}\in\mathbb{C}$ are constant coefficients, and the functions $a_{kl}\in\mathcal{O}(\{(x,t)\in\mathbb{C}^2\,|\,|t-t_0|<\delta\})$ are assumed to be entire functions of x and holomorphic germs in t at the point $t_0\in\mathbb{C}$. Modern exposition of the results about analytic extension of holomorphic solutions of *linear* partial differential equations is given in Hörmander's well-known monograph [6, § 9.4], and papers of Henkin [7] and Rigat [8].

Up to now, all attempts to generalize these results and approaches to *nonlinear* equations and systems led only to partial and sporadic results (see, for example, [9, 10] and references therein). One can mention the extensive recent studies of the dissipative smoothing phenomenon (the regularizing effect of dispersive evolutionary equations of mathematical physics), which produce results looking very similar to the forced analytic extension (see various approaches in [11, 12] and references therein). However, the solution $u(x,t)$ in these results must always satisfy certain global restrictions as a function of x for $t=t_0\in\mathbb{R}$, and the conclusion about analytic extension to a neighborhood of the real axis $\mathbb{R}^1_x\subset\mathbb{C}^1_x$ is derived only for real $t>t_0$. There are several exceptions from this rule [13, 14], but neither of them gives any information about analytic extension of arbitrary local solutions that are holomorphic in x and t.

It was a long-standing challenge to obtain such information at least in the case of soliton equations[1], where it is referred to as "rigorous Painlevé analysis". As Kruskal *et al.* put it [15, p. 195], "To date there is no proof that the Korteweg–de Vries equation possesses the Painlevé property. The main problem lies in a lack of methods for obtaining the global analytic description

[1] Of parabolic type since the hyperbolic case is trivial by the Cauchy–Kowalevsky theorem.

of a locally defined solution in the space of several complex variables." In what follows we present such a method (which was suggested in [16, 17]) and use it to give a definitive answer to the question of analytic continuation of all local solutions (see Theorem 1 below).

Before doing this, we briefly explain the notion of a soliton equation starting with the most popular examples:

$$u_t = au_{xxx} + buu_x, \qquad a, b \in \mathbb{C} \setminus \{0\}, \tag{4}$$

$$u_{tt} = au_{xxxx} + buu_{xx} + bu_x^2, \qquad a, b \in \mathbb{C} \setminus \{0\}, \tag{5}$$

$$iu_t = au_{xx} + bu|u|^2, \qquad a, b \in \mathbb{R} \setminus \{0\}. \tag{6}$$

In (6), $|u|^2$ is understood as $u(x,t)\overline{u(\overline{x},\overline{t})}$. The inverse scattering method first appeared as a tool for solving the Korteweg–de Vries equation (4) (with real a, b), which describes long waves on shallow water. It was noted in the pioneering paper of Gardner, Green, Kruskal and Miura [18] that if the potential $u(x,t)$ evolves according to (4), then the evolution of its scattering data (certain spectral characteristics of the operator $L = \partial_x^2 + u(x,t)$ on the Hilbert space $L^2(\mathbb{R}_x^1)$) turns out to be linear and "explicitly integrable", which enables one to construct examples of solutions and study the properties of all solutions in certain classes. An explanation of the unexpected success of this approach was given by Lax [19], who showed that the equation (4) (to be definite, with $a = 1/4$, $b = 3/2$) is a necessary and sufficient condition for solubility of the *auxiliary linear problem*

$$L\psi = \lambda\psi, \qquad \psi_t = P\psi \tag{7}$$

for the operators $L := \partial_x^2 + u$ and $P := \partial_x^3 + (3/2)u\partial_x + (3/4)u_x$. In other words, (4) may be written in the form $L_t = [P, L]$. Since P is skew-Hermitian, it follows that the evolution of L consists in its conjugation by a t-dependent unitary operator on $L^2(\mathbb{R}_x^1)$. This conjugation clearly preserves the spectrum, and then it is no surprise that the more refined spectral charateristics (scattering data) also evolve in a simple and tractable way.

The nonlinear Schrödinger equation (6) describes the evolution of a slowly varying dispersive wave envelope in nonlinear media and arises in optics, hydrodynamics and plasma physics. It was first studied in terms of the inverse scattering method by Zakharov and Shabat [20], who modified (7) replacing the scalar second-order differential equation $L\psi = \lambda\psi$ by a matrix first-order 2×2-system of differential equations with subsequent reduction (that is, a choice of matrices of special algebraic structure: in the present case, skew-Hermitian). The auxiliary linear problem takes the form

$$E_x = UE, \qquad E_t = VE \tag{8}$$

for some matrix-valued polynomials $U(x,t,z)$ and $V(x,t,z)$ of degrees 1 and 2, respectively in the spectral parameter $z \in \mathbb{C}$ (which is related to the parameter λ in (7) by the formula $\lambda = z^n$ in case of $n \times n$-matrices). Hence the equation (6) turned out to be written, although implicitly, as a reduction of the zero curvature equation

$$U_t - V_x + [U, V] = 0, \tag{9}$$

which plays a fundamental role in our approach. The first explicit presentation of soliton equations as reductions of zero curvature equations and first corollaries of this presentation are due to Novikov [21].

Finally, the Boussinesq equation (5) describes water waves (like the Korteweg–de Vries equation) but admits wave motion in any direction (unlike the Korteweg–de Vries equation). It was studied in terms of the inverse scattering method by Zakharov (1973) and turned out to be the first physically relevant example where the 2×2-matrices in (8) or second-order operators L in (7) should be replaced by 3×3-matrices or third-order operators.

Thus all equations (4)–(6) are reductions of (9), where U and V are polynomials in z, the degree of U is equal to 1 and the degree of V is equal to $m \geq 2$. Taking this property for the definition[2] of a *soliton equation of parabolic type*, we shall give a complete answer to the question about analytic continuation of local holomorphic solutions of such equations. The situation appears to be almost the same as for the linear equations (1)–(3) with the only difference: the solutions now extend to globally meromorphic (not necessarily entire) functions of x.

Theorem 1. *For each of equations* (4)–(6)*, every local holomorphic solution* $u(x, t)$ *in a bidisk* $D = \{(x, t) \in \mathbb{C}^2 | |x - x_0| < \varepsilon_1, |t - t_0| < \varepsilon_2\}$ (*with real centre* $(x_0, t_0) \in \mathbb{R}^2$ *in case* (6)) *admits an analytic continuation to a meromorphic function* $\widetilde{u}(x, t)$ *in the strip* $S = \{(x, t) \in \mathbb{C}^2 \mid |t - t_0| < \varepsilon_2\}$.

It follows from the Cauchy–Kowalevsky theorem (with x as a time variable) that Theorem 1 is unimprovable: for each of equations (4)–(6) one can find a solution u whose extension $\widetilde{u}$ admits no further extension (holomorphic or meromorphic) beyond the strip S. In other words, the envelope of meromorphy of any local holomorphic (or meromorphic) solutions covers the whole complex line in the x-direction and may be arbitrary (any prescribed Riemann surface over the t-axis) in the t-direction.

To prove Theorem 1, we develop a local version of the inverse scattering method for soliton equations of parabolic type (this method was previously used only for equations of hyperbolic type, where the results and techniques are quite different; see [22, Ch. I] or [23, Part II, Ch. I, §§ 6–8]). The potentials are holomorphic germs without any boundary conditions. When one additionally imposes rapidly decaying or quasiperiodic (finite-gap) boundary conditions, the local version becomes naturally isomorphic to the corresponding standard version of the inverse scattering method. This may be regarded as a step towards solving another old puzzle: give a unified treatment of finite-gap solutions and rapidly decaying solutions (in the words of Bennequin [24, pp. 35–36], "…comment marier les solutions géometriques, attachées aux courbes algebriques […] avec les diffusons qui viennent du

[2] This definition will be sharpened in § 2.

scattering-inverse (solutions L^2 de KdV par exemple)?"). Many other applications of the local inverse scattering approach are yet to be developed.

Structure of the paper. In § 2 we recall the construction of some holomorphic solutions of zero curvature equations using a very simple version of the Riemann problem. It serves to motivate our approach and describe the algebraic structure of zero curvature equations. Then § 3 introduces the main definitions and results of the local inverse scattering method. We give only the barest sketches of proofs (along with references to their full versions) but preserve all motivations and accurate statements in the hope to present the logical structure of the theory clearly and comprehensively. In § 4 we formally deduce Theorem 1 from the results of § 3. Thus our exposition is organized as a proof of Theorem 1 and all other results should be regarded as lemmas. However, we call some of them theorems in view of their importance.

2. The Riemann problem and zero curvature equations

We start with the *zero curvature equations* (9), where $U(x,t,z)$, $V(x,t,z)$ are $\mathrm{gl}(n,\mathbb{C})$-valued[3] rational functions of an auxiliary parameter z with coefficients depending on the space and time variables x,t. Here the poles of U,V must be fixed in advance and independent of x,t, and the coefficients of a rational function are defined as the coefficients of its partial fraction expansion or, equivalently, as the coefficients of the principal parts of its Laurent expansions at all poles. A *holomorphic solution of* (9) *on a domain* $\Omega\subset\mathbb{C}^2_{xt}$ is a pair of rational $\mathrm{gl}(n,\mathbb{C})$-valued functions U,V of z with prescribed positions and multiplicities of poles such that all coefficients of U,V are defined and holomorphic on Ω and all coefficients of the rational function $U_t-V_x+[U,V]$ of z are identically equal to 0 on Ω. Equation (9) with a fixed z (different from the poles of U and V) holds on a simply connected domain $\Omega\subset\mathbb{C}^2_{xt}$ if and only if the auxiliary linear system (8) with the same value of z has a holomorphic solution $E:\Omega\to\mathrm{GL}(n,\mathbb{C})$. Note that this solution is unique up to right multiplication by an invertible matrix (possibly depending on z).

Our approach uses the Riemann problem (see, for example, in [25, Ch. III] or [23, Part I, Ch. II and Part II, Ch. I, §§ 6–8]) on factorization of matrix-valued functions on a circle, or rather a generalization of this problem to the case of divergent series of Gevrey type (see the next section). Let D_+,D_- be disjoint open disks whose closures cover the whole extended complex plane $\overline{D}_+\cup\overline{D}_-=\overline{\mathbb{C}}:=\mathbb{C}\cup\{\infty\}$. A continuous function $\gamma:\Gamma\to\mathrm{GL}(n,\mathbb{C})$ on the circle $\Gamma:=\overline{D}_+\cap\overline{D}_-$ is said to be *left-factorable* (resp. *right-factorable*) if there are continuous functions $\gamma_\pm:\overline{D}_\pm\to\mathrm{GL}(n,\mathbb{C})$ that are holomorphic on $D_\pm$ and satisfy $\gamma=\gamma_+\gamma_-^{-1}$ on Γ (resp. $\gamma=\gamma_-^{-1}\gamma_+$ on Γ). We regard the

[3]Throughout the paper $\mathrm{gl}(n,\mathbb{C})$ stands for the set of all $n\times n$-matrices with complex entries, $\mathrm{GL}(n,\mathbb{C})$ is the group of all invertible matrices in $\mathrm{gl}(n,\mathbb{C})$, and $[A,B]=AB-BA$ is the commutator of matrices $A,B\in\mathrm{gl}(n,\mathbb{C})$.

function γ as *data* of the Riemann problem and the pair (γ_+, γ_-) as a *solution*. If a solution exists, it is unique up to right (resp. left) multiplication of both elements of the pair by the same invertible constant matrix.

We now describe a holomorphic version of the Zakharov–Shabat *dressing method* [26]. Let (U_0, V_0) be a holomorphic solution (which may, for example, be identically equal to zero) of equation (9) on a domain $\Omega \subset \mathbb{C}^2$, and let E_0 be the corresponding solution of the auxiliary linear problem (8) normalized by the condition $E_0(x_0, t_0, z) = I$ (the identity matrix) for all z, where $(x_0, t_0) \in \Omega$ is a fixed point. Consider any covering of the extended complex plane $\overline{\mathbb{C}}$ by the disks D_+, D_- such that the circle $\Gamma = \overline{D}_+ \cap \overline{D}_-$ contains no poles of the rational functions U_0, V_0, and take any right-factorable continuous function $g : \Gamma \to \mathrm{GL}(n, \mathbb{C})$. For every $(x, t) \in \Omega$ pose the Riemann problem of finding invertible continuous functions $\theta_\pm : \overline{D}_\pm \to \mathrm{GL}(n, \mathbb{C})$ that are holomorphic on $D_\pm$ and satisfy

$$E_0(x, t, z) g(z) E_0^{-1}(x, t, z) = \theta_-^{-1}(x, t, z) \theta_+(x, t, z) \quad \text{for} \quad z \in \Gamma. \tag{10}$$

To make the solution $\theta_\pm$ unique, we fix a point $z_0 \in D_-$ and impose the additional condition $\theta_-(x, t, z_0) = I$ for all x, t. By a theorem of Malgrange [27, § 4], the set Ω_g of all points $(x, t) \in \Omega$ for which the Riemann problem (10) is soluble, is either the whole domain Ω, or the complement to a complex curve $C_g \subset \Omega$ that does pass through the point (x_0, t_0), and the matrix-valued functions $\theta_\pm(x, t, z)$ are meromorphic on $\Omega \times D_\pm$ with at most a pole in (x, t) along this curve for every fixed $z \in D_\pm$. We put

$$U_1(x, t, z) = \begin{cases} (\theta_+)_x \theta_+^{-1} + \theta_+ U_0 \theta_+^{-1} & \text{for} \quad z \in \overline{D}_+, \\ (\theta_-)_x \theta_-^{-1} + \theta_- U_0 \theta_-^{-1} & \text{for} \quad z \in \overline{D}_- \end{cases} \tag{11}$$

and define $V_1(x, t, z)$ by the same formula with U_0 replaced by V_0 and the derivatives in x replaced by the derivatives in t. Then the pair $(U_1(x, t, z), V_1(x, t, z))$ is a holomorphic solution of (9) on the domain $\Omega_g \subset \mathbb{C}^2_{xt}$ (or a meromorphic solution on Ω) *with the same positions and multiplicities of poles* of the rational functions U_1, V_1 as they were for the rational functions U_0, V_0. We say that this solution is obtained by *dressing the solution* U_0, V_0 *by means of the function* g.

In what follows we always assume that the divisors of poles of the rational functions U, V are equal to ∞ and $m\infty$ for some integer $m \geq 2$, that is, U is a polynomial of degree 1 in z, and V is a polynomial of degree $m \geq 2$ in z (see the definition of soliton equations of parabolic type in the Introduction). Then it is natural to consider a limiting case of the dressing method when the disk D_- contracts to the point ∞ and the disk D_+ expands to the whole plane $\mathbb{C}$. (An analogue of this construction was studied by Krichever [22, Ch. I] in the hyperbolic case when the sets of poles of U and V are disjoint[4].) For the dressing function $g(z)$ we take the germ at ∞

[4]This enabled him to present all local holomorphic solutions of (9) with disjoint sets of poles of U and V as a result of dressing of "trivial" solutions and write any local holomorphic solution as a non-linear superposition of two waves running along the characteristics

of an arbitrary holomorphic invertible matrix-valued map, and for the contour Γ we take any circle of large radius lying in the domain of that germ. To state the limiting version of the dressing method, we digress on the algebraic structure of zero-curvature equations (9) for polynomials U, V of the form specified above. It may be assumed from the very beginning that

$$\begin{cases} U(x,t,z) = az + q(x,t) \ , \\ V(x,t,z) = bz^m + r_1(x,t)z^{m-1} + \dots + r_m(x,t) \end{cases} \tag{12}$$

for some *diagonal* matrices $a, b \in \mathrm{gl}(n, \mathbb{C})$ and holomorphic matrix-valued functions $q, r_1, \dots, r_m : \Omega \to \mathrm{gl}(n, \mathbb{C})$ on a given domain $\Omega \subset \mathbb{C}^2$. Then (9) is a system of $m+1$ matrix equations for $m+1$ unknown matrix functions $q, r_1, \dots, r_m$. Assume for non-degeneracy that the matrix a has *simple spectrum* (that is, all of its eigenvalues are distinct) and the matrix-valued function $q(x,t)$ is *off-diagonal* (that is, $q_{ii}(x,t) \equiv 0$ for $i = 1, \dots, n$). Then the first m equations of the system and the diagonal part of the last $(m+1)$-th equation are soluble in a purely algebraic way. Hence the system can be reduced to one off-diagonal matrix equation for one off-diagonal unknown matrix-valued function $q(x,t)$.

To state this more precisely, we fix an arbitrary point $x_0 \in \mathbb{C}$ and introduce the set $\mathcal{R}(x_0)$ of all germs of holomorphic $\mathrm{gl}(n, \mathbb{C})$-valued maps at x_0 and the set $\mathcal{R}(x_0)^{\mathrm{od}}$ of all off-diagonal germs $q \in \mathcal{R}(x_0)$, that is, the germs with $q_{ii}(x) \equiv 0$ for $i = 1, \dots, n$. A map $F : \mathcal{R}(x_0) \to \mathcal{R}(x_0)$ is called a *differential polynomial* if each entry of the matrix-valued function $F(\kappa)$ is an ordinary polynomial (the same for all κ) in the entries of κ and their derivatives (of any order) with respect to x. We need the following assertion ([28, Lemma 1]) whose content and proof must be regarded as well known.

Lemma 1. *Let $a, b, c_1, c_2, \dots \in \mathrm{gl}(n, \mathbb{C})$ be diagonal matrices such that a has simple spectrum. Then there is a unique sequence of differential polynomials $F_j : \mathcal{R}(x_0) \to \mathcal{R}(x_0)$ $(j = 0, 1, 2, \dots)$ with the following properties:*

(a) $F_0(\kappa) \equiv b$,

(b) $F_j(0) \equiv c_j$ *for all* $j = 1, 2, \dots$,

(c) *the formal Laurent series* $F(\kappa, z) := \sum_{j=0}^{\infty} F_j(\kappa) z^{-j}$ *satisfies the differential equation* $\partial_x F(\kappa, z) = [az + \kappa, F(\kappa, z)]$ *identically with respect to x and z for all* $\kappa \in \mathcal{R}(x_0)^{\mathrm{od}}$.

Arguing as in the proof of [28, Theorem 1], we see that a pair of polynomials $U(x,t,z)$, $V(x,t,z)$ of the form (12) with diagonal matrices $a, b \in \mathrm{gl}(n, \mathbb{C})$ (where a has simple spectrum) and off-diagonal function $q(x,t)$ is a holomorphic solution of (9) in a domain $\Omega \subset \mathbb{C}^2$ if and only if the following two conditions hold. First, the coefficients $r_1, \dots, r_m : \Omega \to \mathrm{gl}(n, \mathbb{C})$ of the polynomial V must be expressed in terms of the off-diagonal function $q : \Omega \to \mathrm{gl}(n, \mathbb{C})$ by the formulae

$$r_1 = F_1(q), \quad \dots, \quad r_m = F_m(q)$$

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similarly to the d'Alembert formula for solutions of the wave equation. Clearly, none of these results holds in the case of parabolic equations, which we study here.

for some diagonal matrices $c_1(t),\dots,c_m(t)\in\mathrm{gl}(n,\mathbb{C})$ that depend holomorphically on t in the domain equal to the projection of Ω to the coordinate axis $\mathbb{C}^1_t$. Second, the holomorphic off-diagonal function $q:\Omega\to\mathrm{gl}(n,\mathbb{C})$ must satisfy the equation

$$q_t=[a,F_{m+1}(q)] \tag{13}$$

on Ω for the same choice of the diagonal matrices $c_1(t),\dots,c_m(t)$ as in the first condition and for an arbitrary diagonal matrix $c_{m+1}\in\mathrm{gl}(n,\mathbb{C})$ (the right-hand side of (13) actually does not depend on the choice of c_{m+1}).

Among all solutions U,V of the form (12) of the zero curvature equations (9), we are interested only in those that correspond to solutions of (13) for some *t-independent* diagonal matrices $c_1,\dots,c_j$. We also assume for nondegeneracy that both matrices a, b have simple spectrum. Then we call (13) the *soliton equation of parabolic type defined by the matrices* $a,b,c_1,\dots,c_m$. This equation is equivalent to the zero curvature equation $U_t-V_x+[U,V]=0$ for the polynomials

$$U(x,t,z)=az+q(x,t),\qquad V(x,t,z)=\sum_{j=0}^{m}F_{m-j}(q)(x,t)z^j, \tag{14}$$

where F_0, F_1, $\dots$, F_m are the differential polynomials determined by the sequence of matrices $a,b,c_1,\dots,c_m$ according to Lemma 1. Examples of reductions of soliton equations of parabolic type are the linear equations of the form $u_t=P(\partial_x)u$ for an arbitrary polynomial P of degree ≥ 2, the Korteweg–de Vries equation (4), the nonlinear Schrödinger equation (5) and others (see, for example, [28, end of § 2]).

We now state the limiting version of the dressing method for constructing holomorphic solutions of the equations studied. The identically zero solution U_0,V_0 will be dressed by means of any germ $g=f^{-1}\in\mathcal{D}$, where $\mathcal{D}$ is the set of all holomorphic $\mathrm{GL}(n,\mathbb{C})$-valued functions f on $\{z\in\mathbb{C}\mid |z|>R_0\}\cup\{\infty\}$ (R_0 depends on f) with $f(\infty)=I$. In other words, $\mathcal{D}$ consists of the germs of holomorphic $\mathrm{GL}(n,\mathbb{C})$-valued functions f at ∞ with $f(\infty)=I$. The following assertion ([28, Theorem 1]) must be regarded as well known, although it was not explicitly stated and completely proved anywhere in the literature. Versions of it are contained in [29, Proposition 2.7], [30, Theorem 3.2.6] and [31, Proposition 2.9].

Lemma 2. *Let $a,b,c_1,c_2,\dots\in\mathrm{gl}(n,\mathbb{C})$ be diagonal matrices such that a has simple spectrum. We fix an integer $m\ge 2$ and a point $(x_0,t_0)\in\mathbb{C}^2$. For every function $f\in\mathcal{D}$ let $\Omega(f)$ be the set of all $(x,t)\in\mathbb{C}^2$ such that the function*

$$\gamma(x,t,z):=\exp\{az(x-x_0)+(bz^m+c_1z^{m-1}+\dots+c_m)(t-t_0)\}f^{-1}(z)$$

is right-factorable on some (and then on any) circle $\{|z|=R\}$, $R_0<R<+\infty$. Then the set $\Omega(f)\subset\mathbb{C}^2$ is either the whole of $\mathbb{C}^2$ or the complement to an entire complex curve (the set of zeros of an entire function) not passing through (x_0,t_0). For every point $(x,t)\in\Omega(f)$ let $(\gamma_+(x,t,z),\gamma_-(x,t,z))$ be the solution of the Riemann problem

$$\gamma(x,t,z)=\gamma_-^{-1}(x,t,z)\gamma_+(x,t,z)\quad \textit{for}\quad R_0<|z|<+\infty, \tag{15}$$

normalized by the condition $\gamma_-(x,t,\infty)=I$. *We put*

$$q_f(x,t) := \lim_{z\to\infty} z[\gamma_-(x,t,z)-I,a]. \tag{16}$$

Then the function $q_f : \Omega(f) \to \mathrm{gl}(n,\mathbb{C})$ *is an off-diagonal holomorphic solution on* $\Omega(f)$ *of the soliton equation of parabolic type* (13) *determined by the matrices* $a, b, c_1, c_2, \dots$.

We note that the Riemann problem (15) coincides with (10) up to the notation $f = g^{-1}$, $\gamma_- = \theta_-$, $\gamma_+ = \theta_+ E_0$, and the definition (16) of the solution constructed in Lemma 2 is obtained by equating the coefficients of z^0 in second equation (11).

The class of solutions $q_f(x,t)$ constructed in Lemma 2 contains all *finite-gap* solutions (they correspond to those matrices $f \in \mathcal{D}$ whose columns are eigenvectors of some non-degenerate[5] rational $\mathrm{gl}(n,\mathbb{C})$-valued function $G(z)$; see [32, Theorem 5]) and many *rapidly decreasing* solutions (as described in [32, § 5]). In both cases the construction of Lemma 2 coincides with the corresponding version of the inverse scattering method if we understand the germ $f \in \mathcal{D}$ as the *scattering data* of a matrix-valued potential $q_f(x,t_0) \in \mathcal{R}(x_0)$. This is explained at length in [32, §§ 4, 5]. Note that our "potentials" determine their "scattering data" not uniquely, but only up to right multiplication by any diagonal germ in $\mathcal{D}$. This can be expressed in the following form (see [32, Theorem 4(A)] and its proof).

Lemma 3. *Two functions* $f, g \in \mathcal{D}$ *determine the same solution* $q_f(x,t) = q_g(x,t)$ *of equation* (13) *in a neighborhood of the point* $(x_0,t_0) \in \mathbb{C}^2$ *if and only if the function* $g^{-1}f \in \mathcal{D}$ *is diagonal. This condition is also necessary and sufficient for the equality* $q_f(x,t_0) = q_g(x,t_0)$ *in a neighbourhood of the point* $x_0 \in \mathbb{C}$.

3. The main definitions and results

The construction of solutions described in Lemma 2 is far from giving *all local holomorphic solutions* of (13) in a neighborhood of the given point $(x_0,t_0) \in \mathbb{C}^2$. (For example, all solutions constructed in Lemma 2 extend meromorphically to $\mathbb{C}^2$, while the Cauchy–Kowalevsky theorem stated in the introduction enables us to construct local solutions with any prescribed singularity in t.) We now present a natural modification of this construction which is free from this disadvantage as well as from the non-uniqueness (described in Lemma 3)

[5]Here non-degeneracy means that the complex curve $C_G := \{(z,w) \in \mathbb{C}^2 \mid \det(G(z) - wI) = 0\}$ splits into n distinct holomorphic branches over a punctured neighbourhood $\{|z| > R\}$ of the point $z = \infty$. This automatically holds if the matrix $G(\infty)$ has simple spectrum. The algebraic curve C_G is known as a *spectral curve* and plays an important role in the theory of finite-gap solutions. Replacing "rational" by "holomorphic at ∞" in the definition of G gives an equivalent description of the set of all solutions constructed in Lemma 2.

of the correspondence between potentials and their scattering data[6]. Recall that the set $\mathcal{D}$ of "scattering data" consists of all convergent (in a neighborhood of the point $z=\infty$) series of the form

$$f(z) = I + \frac{\varphi_0}{z} + \frac{\varphi_1}{z^2} + \dots,$$

where $\varphi_0, \varphi_1, \dots \in \mathrm{gl}(n,\mathbb{C})$. We now want to replace it by the set $\mathcal{D}_{1/m}$ of all formal power series of the same form with off-diagonal matrices $\varphi_k \in \mathrm{gl}(n,\mathbb{C})$ (this makes the correspondence between the germs of solutions and their scattering data one-to-one in contrast to Lemma 2) and with

$$\sum_{k=0}^{\infty} \frac{|\varphi_k|}{k!^{1/m}} A^k < \infty \quad \text{for some } A > 0,$$

where $m \geq 2$ is the number of the equation (13) in its hierarchy. The class $\mathcal{D}_{1/m}$ is natural because its elements are precisely those formal power series for which the left-hand side of (15) (that is, the data of the Riemann problem) is well defined as a formal Laurent series in z for all (x,t) in a neighborhood of the point $(x_0,t_0) \in \mathbb{C}^2$. (This follows from Lemma 4 below.) In the case when $m=1$ (or, equivalently, $b=c_1=c_2=\dots=0$), equation (13) takes a trivial form $q_t=0$, but its "solutions" (that is, all germs $q(x)$ of holomorphic off-diagonal $\mathrm{gl}(n,\mathbb{C})$-valued functions at the point $x_0 \in \mathbb{C}$) are also described by their scattering data. This is an important part of the whole method (see Theorem 3 below).

Let us describe appropriate Banach spaces of formal power series. For every $\alpha \geq 0$ we introduce the set Gev_α (referred to as *Gevrey class* α) of all formal power series of the form $\varphi(z) = \sum_{k=0}^{\infty} \varphi_k z^{-(k+1)}$ $\varphi_k \in \mathrm{gl}(n,\mathbb{C})$ such that the series $\sum_{k=0}^{\infty} (k!)^{-\alpha} |\varphi_k| x^k$ has a non-zero radius of convergence. Here $|\cdot|$ is any fixed norm on $\mathrm{gl}(n,\mathbb{C})$ with the property $|AB| \leq |A||B|$. The vector space Gev_α is the union of an increasing family of Banach spaces isometrically isomorphic to l_1. Namely, $\mathrm{Gev}_\alpha = \bigcup_{A>0} G_\alpha(A)$, where $G_\alpha(A)$ is the set of all formal power series $\varphi(z) = \sum_{k=0}^{\infty} \varphi_k z^{-(k+1)}$ with $\varphi_k \in \mathrm{gl}(n,\mathbb{C})$ such that $\|\varphi\|_{\alpha,A} := \sum_{k=0}^{\infty} (k!)^{-\alpha} |\varphi_k| A^k < \infty$.

In the same vein, for every $m \geq 1$ we write the vector space Ent_m of all $\mathrm{gl}(n,\mathbb{C})$-valued entire functions of order $\leq m$ and finite type (for order exactly m) in the form $\mathrm{Ent}_m = \bigcup_{B>0} E_m(B)$, where $E_m(B)$ is the set of all formal power series $\varepsilon(z) = \sum_{l=0}^{\infty} \varepsilon_l z^l$ with $\varepsilon_l \in \mathrm{gl}(n,\mathbb{C})$ such that $\|\varepsilon\|_{m,B} := \sup_{l \geq 0} |\varepsilon_l| (l!)^{1/m} B^{-l} < \infty$ (this condition guarantees that the series converges for all $z \in \mathbb{C}$). Clearly, each $E_m(B)$ is a Banach space isometrically isomorphic to l_∞.

[6]Note, however, that this non-uniqueness is sometimes an asset: it provides a flexible and natural language in some important constructions. For example, adding a soliton to a given solution q_f is very conveniently described in the notation of Lemma 2 as multiplication of f by a Blaschke factor (see, for example, [25, Ch. III, § 2] or [31, Proposition 4.2]), but this description becomes cumbersome if we insist on using the normalized scattering data, which are introduced below.

An important property of the Banach spaces $G_\alpha(A)$ and $E_m(B)$ is the possibility to multiply their elements for $\alpha m \le 1$ and $B < A$ (and, generally speaking, these inequalities are unimprovable). This fact is expressed by the following assertion [17, Lemma 1], where $\{\cdot\}_+$ and $\{\cdot\}_-$ stand respectively for the positive and negative parts of a Laurent series: $\{\sum_{k\in\mathbb{Z}} a_k z^k\}_+ = \sum_{k\ge 0} a_k z^k$, $\{\sum_{k\in\mathbb{Z}} a_k z^k\}_- = \sum_{k\le -1} a_k z^k$.

Lemma 4. *Suppose that $A > B > 0$, $m \ge 1$ and $0 \le \alpha \le 1/m$. Then the product of elements of $G_\alpha(A)$ and $E_m(B)$ in any order is a well-defined formal Laurent series belonging to the direct sum $G_\alpha(A-B) + E_m(B)$. The maps $(\varphi,\varepsilon) \mapsto \{\varphi\varepsilon\}_\pm$ and $(\varphi,\varepsilon) \mapsto \{\varepsilon\varphi\}_\pm$ are continuous bilinear forms on $G_\alpha(A) \times E_m(B)$ with values in $G_\alpha(A-B)$ and $E_m(B)$.*

We can now state the main result (a slightly extended version of [17, Theorem 3] with basically the same proof) on the solubility of the Riemann problem (15) in the context of divergent power series and on the analytic properties of its solutions as functions of parameters. We actually need only two very special cases: first, when Ω is $\mathbb{C}^2_{xt}$ and the polynomial $P(x,t,z)$ is of the form $a(x-x_0)z + (bz^m + c_1 z^{m-1} + \cdots + c_m)(t-t_0)$ for some integer $m \ge 2$ with the same diagonal matrices $a, b, c_1, c_2, \dots \in \mathrm{gl}(n,\mathbb{C})$ as in Lemma 2 and, second, when Ω is $\mathbb{C}^1_x$, the polynomial $P(x,z)$ is equal to $a(x-x_0)z$, and $m = 1$. In part (B) we use the notation $\mathrm{Gev}_{\alpha-0} := \bigcup_{0\le s<\alpha} \mathrm{Gev}_s$.

Theorem 2.

(A) *Let Ω be a complex manifold, $m \ge 1$ an integer, $p_0, p_1, \dots, p_m : \Omega \to \mathrm{gl}(n,\mathbb{C})$ holomorphic maps, and $\xi_0 \in \Omega$ a point with $p_k(\xi_0) = 0$ for $k = 0, 1, \dots, m$. Put $P(\xi, z) := \sum_{k=0}^m p_k(\xi) z^k$ for all $\xi \in \Omega$, $z \in \mathbb{C}$. Then for every series $f \in I + \mathrm{Gev}_{1/m}$ one can find a neighborhood $\Omega(f)$ of the point ξ_0 in Ω, numbers $A, B > 0$ and holomorphic maps $\gamma_- : \Omega(f) \to I + G_{1/m}(A)$ and $\gamma_+ : \Omega(f) \to E_m(B)$ such that the following equality of formal Laurent series holds for all $\xi \in \Omega(f)$:*

$$e^{P(\xi,z)} f^{-1}(z) = \gamma_-^{-1}(\xi, z)\gamma_+(\xi, z) \tag{17}$$

and all values of the entire function $z \mapsto \gamma_+(\xi, z)$ belong to the group $\mathrm{GL}(n,\mathbb{C})$ of invertible complex $n \times n$-matrices and satisfy the equality

$$\det \gamma_+(\xi, z) = e^{\operatorname{tr} P(\xi,z)} \quad \textit{for all} \quad z \in \mathbb{C}. \tag{18}$$

(B) *Under the hypotheses of part (A), if we additionally know that $f \in I + \mathrm{Gev}_{(1/m)-0}$ and Ω is a Stein manifold[7] with $H^2(\Omega,\mathbb{Z}) = 0$, then there is a holomorphic non-vanishing at ξ_0 function $\tau_f \in \mathcal{O}(\Omega)$ with the following properties.*

  (a) *The germs of the holomorphic maps $\xi \mapsto \tau_f(\xi)(\gamma_-(\xi,z) - I)$ and $\xi \mapsto \tau_f(\xi)(\gamma_-^{-1}(\xi,z) - I)$ at the point ξ_0 admit an analytic continuation to holomorphic maps $\Omega \to G_{1/m}(A)$ for every $A > 0$.*

[7] A Stein manifold may be defined as a closed complex submanifold of $\mathbb{C}^N$. The additional requirements on Ω in part (B) guarantee the solubility of the second Cousin problem on Ω (see, for example, [33, subsections 41 and 49]). All the hypotheses of part (B) automatically hold in our cases when Ω is either $\mathbb{C}^2$ or $\mathbb{C}^1$.

(b) *For every exhaustion* $\{\xi_0\} = K_0 \subset K_1 \subset \dots$ *of the manifold* Ω *by holomorphically convex compact sets* $K_j \subset \operatorname{int} K_{j+1}$ *with* $H^2(\operatorname{int} K_j, \mathbb{Z}) = 0$ *there is a sequence of numbers* $B_j > 0$ *such that the germs* $\xi \mapsto \tau_f(\xi)\gamma_+(\xi, z)$ *and* $\xi \mapsto \tau_f(\xi)\gamma_+^{-1}(\xi, z)$ *admit an analytic continuation to holomorphic maps* $\operatorname{int} K_j \to E_m(B_j)$ *for every* $j = 1, 2, \dots$.

(c) *The equalities* (17) *and* (18) *hold for all* $\xi \in \Omega$ *with* $\tau_f(\xi) \neq 0$.

To say this in simpler words, if a formal $\mathrm{gl}(n, \mathbb{C})$-valued power series $f(z) = I + \varphi_0 z^{-1} + \varphi_1 z^{-2} + \dots$ belongs to a Gevrey class such that the left-hand side of (17) is well defined in a neighborhood of ξ_0 by Lemma 4, then the Riemann problem (17) is soluble in some neighborhood of ξ_0, and its solution $\gamma_\pm(\xi, z)$ is holomorphic with respect to ξ in this neighborhood. This fact further supports the idea of natural appearance of the Gevrey classes in our approach. But if we additionally assume (as in part (B)) that the series $f(z)$ belongs to a strictly smaller Gevrey class than in part (A), then the Riemann problem (17) becomes soluble everywhere on Ω except possibly for a complex hypersurface $\{\xi \in \Omega \mid \tau_f(\xi) = 0\}$ that does not pass through ξ_0, and the solution $\gamma_\pm(\xi, z)$ is globally meromorphic with respect to ξ in Ω with at most poles along this hypersurface. The hypotheses of part (B) certainly hold (for any m) when $f(z)$ is an ordinary convergent series in a neighborhood of $z = \infty$ (this situation was described in Lemma 2), and we thus recover the (needed part of the) result of Malgrange [27, mentioned in § 1].

To prove part (A) of Theorem 2, we reduce the Riemann problem (17) to a linear inhomogeneous equation of the form $X(\xi)\varphi = u(\xi)$ on the Banach space $E = G_\alpha(A)$ for appropriate values of $\alpha \leq 1/m$ and $A > 0$, where $\varphi = \varphi(\xi) \in E$ is the unknown vector, $X(\xi) : E \to E$ is a known linear operator (a slightly modified version of the Toeplitz operator on the Hardy space) and $u(\xi) \in E$ is a known vector. Here $X(\xi)$ and $u(\xi)$ depend holomorphically on ξ in a neighborhood of ξ_0 and $X(\xi_0) = I$ is the identity operator. Once this is done, it is clear that the solution $\varphi(\xi) = X(\xi)^{-1}u(\xi)$ exists, is unique and depends holomorphically on ξ in a neighborhood of ξ_0. The details are given in [17, § 5].

To prove part (B) of Theorem 2, we note that under the hypotheses of part (B) the operator $X(\xi)$ and the vector $u(\xi)$ are defined and holomorphic with respect to ξ on the whole parameter space Ω and, moreover, the operator $Y(\xi) := X(\xi) - I$ is compact for every $\xi \in \Omega$. Therefore the desired conclusion follows from the "meromorphic Fredholm alternative" contained in the following lemma[8], which can be found along with a proof in [17, Lemma 8].

Lemma 5. *Let* Ω *be a Stein manifold with* $H^2(\Omega, \mathbb{Z}) = 0$ *and let* $Y : \Omega \to \mathcal{B}(E)$ *be a holomorphic map from* Ω *to the Banach space* $\mathcal{B}(E)$ *of all linear*

[8]This result seems to be first stated at the needed level of generality (that is, for operators on general Banach spaces and not only on the Hilbert space) by Gokhberg (1953). Then it was rediscovered many times by various authors. Amazingly, references to the possible authorship of this result in the well-known monographs of Kato (1966), Lang (1975), Reed and Simon (1978) and Yafaev (1993) give us four *mutually disjoint* sets of authors.

continuous operators on a Banach space E. Suppose that the operators $Y(\xi)$ are compact for all $\xi \in \Omega$ and the operator $I + Y(\xi_0)$ is invertible for some point $\xi_0 \in \Omega$. Then there is a holomorphic function $\tau \in \mathcal{O}(\Omega)$ with $\tau(\xi_0) = 1$ such that the following assertions hold.

(i) *The operator $I + Y(\xi)$ is invertible for those and only those $\xi \in \Omega$ that satisfy $\tau(\xi) \neq 0$.*

(ii) *$\xi \mapsto \tau(\xi)(I + Y(\xi))^{-1}$ is a holomorphic map $\Omega \to \mathcal{B}(E)$.*

Theorem 2(A) enables us to define the inverse scattering transform for all $f \in I + \mathrm{Gev}_1$ (recall that Lemma 2 did so only for $f \in I + \mathrm{Gev}_0$) by the formula (16) with $t = t_0$ (or, equivalently, with $b = c_1 = c_2 = \cdots = 0$). We now describe this definition in more detail. Fix a diagonal matrix $a \in \mathrm{gl}(n, \mathbb{C})$ with simple spectrum and an arbitrary point $x_0 \in \mathbb{C}$. For every formal power series $\varphi \in \mathrm{Gev}_1$ consider the solution $\gamma_\pm(x, z)$ of the Riemann problem (17) with $P(x, z) = a(x - x_0)z$ and $f(z) = I + \varphi(z)$. Let $\mathcal{R}(x_0)$ be the set of all germs of holomorphic $\mathrm{gl}(n, \mathbb{C})$-valued functions at x_0, and let $\mathcal{R}(x_0)^{\mathrm{od}}$ be the set of all off-diagonal germs $q \in \mathcal{R}(x_0)$ (that is, those with $q_{ll}(x) \equiv 0$ for $l = 1, \dots, n$). Then all coefficients $g_k(x)$ of the expansion $\gamma_-(x, z) = I + \sum_{k=0}^{\infty} g_k(x) z^{-(k+1)}$ belong to $\mathcal{R}(x_0)$, and the formula (basically (16) with $t = t_0$)

$$B\varphi(x) := [g_0(x), a] \quad \text{for} \quad x - x_0 \in \Omega(f) \tag{19}$$

determines a map $B : \mathrm{Gev}_1 \to \mathcal{R}(x_0)^{\mathrm{od}}$. We call this map the *inverse scattering transform*. The notation $B\varphi$ is chosen in honor of the classical Borel transform, to which (19) reduces for upper-triangular $\mathrm{gl}(2, \mathbb{C})$-valued convergent series $\varphi \in \mathrm{Gev}_0$ (as explained in [28, § 6], or [17, § 2]).

The *direct scattering transform* $L : \mathcal{R}(x_0)^{\mathrm{od}} \to \mathrm{Gev}_1$ is defined by the formula

$$Lq(z) := \mu(x_0, z) - I, \tag{20}$$

where $\mu(x, z) = I + \sum_{k=0}^{\infty} m_k(x) z^{-(k+1)}$ is a unique solution of the differential equation $\mu_x = (az + q(x))\mu - \mu az$ in the class of formal power series of the form indicated with $m_k \in \mathcal{R}(x_0)$, $k = 0, 1, 2, \dots$, such that all coefficients of the series $\mu(x_0, z) - I$ are off-diagonal (the existence and uniqueness of this solution are proved in [17], § 6, the paragraph before Lemma 10). The notation Lq is chosen in honor of the classical Laplace transform

$$Lu(z) = \int_0^\infty u(x) e^{-xz}\, dx,$$

to which (20) reduces in case of upper-triangular $\mathrm{gl}(2, \mathbb{C})$-valued potentials $q(x)$ that are entire functions of exponential type (see [28, § 6], [17, § 2]). The definition of Lq may seem strange (where does the differential equation $\mu_x = (az + q(x))\mu - \mu az$ come from?), but it is natural in view of the following observation (which is rather standard in the Riemann-problem approach to integrable systems). Consider the Riemann problem (17) with $\xi = x$ and $P(\xi, z) = a(x - x_0)z$, differentiate it with respect to x (the Leibniz rule for the derivative of a product still holds because of the last assertion of Lemma 4) and separate the positive and negative powers of z in the resulting

Laurent series. This yields that the components of the solution of the Riemann problem satisfy the differential equations

$$\partial_x\gamma_+ = (az + q(x))\gamma_+, \qquad \partial_x\gamma_- = (az + q(x))\gamma_- - \gamma_- az \tag{21}$$

with initial conditions $\gamma_+(x_0, z) = I$, $\gamma_-(x_0, z) = f(z)$, where $q(x) := Bf(x)$ is defined in (19). Thus we see that the differential equation for $\mu(x, z)$ just selects the candidates for the role of $\gamma_-(x, z)$, and the initial condition restores $f(z)$ by the formula (20). This observation also motivates the first part of the following theorem (an extended version of [17, Theorem 1]) which says that the maps L and B are indeed inverse to each other if we restrict ourselves by only off-diagonal series in Gev_1 (as already mentioned in the definition of $\mathcal{D}_{1/m}$ above, this restriction removes the non-uniqueness described in Lemma 3). The second part of Theorem 3 follows from the first part and Theorem 2(B). It says that all potentials whose scattering data belong to strictly smaller Gevrey classes than necessary for the Riemann problem (17) to be well defined, are globally meromorphic in x.

Theorem 3.

(A) *The map $q \mapsto Lq$ is a bijection of the set $\mathcal{R}(x_0)^{\mathrm{od}}$ onto the set $\mathrm{Gev}_1^{\mathrm{od}}$ of all off-diagonal series in Gev_1. The inverse map is the restriction to $\mathrm{Gev}_1^{\mathrm{od}}$ of the map $B : \mathrm{Gev}_1 \to \mathcal{R}(x_0)^{\mathrm{od}}$ defined in* (19).

(B) *If $q \in \mathcal{R}(x_0)^{\mathrm{od}}$ and $Lq \in \mathrm{Gev}_{1-0}$, then the germ $q(x)$ admits an analytic continuation to a globally meromorphic off-diagonal $\mathrm{gl}(n, \mathbb{C})$-valued function on $\mathbb{C}^1_x$ (denoted again by $q(x)$) such that for every $z \in \mathbb{C}$ the auxiliary linear system $E_x = (az + q(x))E$ has a globally meromorphic fundamental system of solutions.*

A key role in the proof of part (A) of Theorem 3 is played by the following particular case of a theorem of Sibuya on formal solutions of singularly perturbed ordinary differential equations (see [34, Theorem A.5.4.1 on pp. 254–256] or [35, Theorem XII-5-2]). *Let $m, \nu \geq 1$ be integers, $A : \mathbb{C}^\nu \to \mathbb{C}^\nu$ an invertible linear operator, and $y(x, z) = \sum_{k=0}^{\infty} a_k(x) z^{-k}$ a formal power series whose coefficients $a_k(x)$ are $\mathbb{C}^\nu$-valued holomorphic germs at $x_0 \in \mathbb{C}$. Suppose that*

$$\frac{dy}{dx} = z^m Ay + \sum_{j=0}^{m-1} z^j B_j(x, y) \tag{22}$$

for some $\mathbb{C}^\nu$-valued polynomials $B_j(x, y)$ in the components of the vector y with coefficients in $\mathcal{O}(x_0)$. Then the series $y(x_0, z)$ belongs to $\mathrm{Gev}_{1/m}$.

In our applications of this result, $\mathbb{C}^\nu$ is the vector space of all off-diagonal matrices $X \in \mathrm{gl}(n, \mathbb{C})$ and the operator $AX := [C, X]$ sends every such matrix to its commutator with a given diagonal matrix $C \in \mathrm{gl}(n, \mathbb{C})$. The role of C is played by a in the proof of Theorem 3 and b in the proof of Theorem 4 below. Since the operator A is invertible if and only if the matrix C has simple spectrum, this explains our non-degeneracy assumptions (made in the definition of soliton equations of parabolic type in § 2) that the matrices a and b have simple spectrum.

The detailed proof of part (A) of Theorem 3 is given in [17, §§6, 7] and we shall not repeat it here. Once part (A) (or rather the equality $q = BLq$ for all $q \in \mathcal{R}(x_0)^{\mathrm{od}}$) is proved, part (B) follows easily. Indeed, if $Lq \in \mathrm{Gev}_{1-0}$, then the germ $BLq(x)$ admits a global meromorphic extension from a neighborhood of x_0 to the whole of $\mathbb{C}^1_x$ by Theorem 2(B). Since $BLq = q$, this proves the first assertion of Theorem 3(B). To prove the second assertion, note that the component $\gamma_+(x,z)$ of the solution of the Riemann problem (17) with $P(x,z) = a(x-x_0)z$ satisfies the auxiliary linear system $E_x = (az + q(x))E$ for all $z \in \mathbb{C}$ (this follows from the first equality in (21)) and its columns are linearly independent by (18). Hence its columns form a fundamental system of solutions. On the other hand, it follows from Theorem 2(B), assertion (b), that $\gamma_+(x,z)$ is a globally meromorphic function on $\mathbb{C}^1_x$ with denominator $\tau_f(x)$ for every fixed z. This proves the second assertion of Theorem 3(B), which is also known as the *trivial-monodromy property.*

We can now state a criterion for solubility of the local holomorphic Cauchy problem for soliton equations of parabolic type. Consider any system of evolution equations of the form (13), where $q(x,t)$ is an unknown off-diagonal $\mathrm{gl}(n,\mathbb{C})$-valued function, $m \geq 2$ is a given integer and $F_0, F_1, F_2, \dots$ is the sequence of differential polynomials in x corresponding to a given sequence of diagonal matrices $a, b, c_1, c_2, \dots \in \mathrm{gl}(n,\mathbb{C})$ according to Lemma 1. We always assume that the non-degeneracy condition holds: *the matrices a, b have simple spectrum.* Let $\mathcal{R}(x_0,t_0)$ be the set of all germs of holomorphic $\mathrm{gl}(n,\mathbb{C})$-valued maps at the point $(x_0,t_0) \in \mathbb{C}^2$, and let $\mathcal{R}(x_0,t_0)^{\mathrm{od}}$ be the set of all off-diagonal germs in $\mathcal{R}(x_0,t_0)$. The *local holomorphic Cauchy problem* for (22) is posed as follows. Given an off-diagonal holomorphic germ $q_0 \in \mathcal{R}(x_0)^{\mathrm{od}}$, it is required to find a germ $q \in \mathcal{R}(x_0,t_0)^{\mathrm{od}}$ that satisfies equation (13) and the initial condition $q(x,t_0) = q_0(x)$. The following theorem is an extended version of [17, Theorem 2].

Theorem 4.

(A) *The Cauchy problem $q(x,t_0) = q_0(x)$ for equation* (13) *admits a local holomorphic solution at the point $(x_0,t_0) \in \mathbb{C}^2$ if and only if $Lq_0 \in \mathrm{Gev}_{1/m}$. If such a solution $q(x,t)$ exists, it is unique.*

(B) *Every local holomorphic solution $q(x,t)$ of equation* (13) *in an arbitrary bidisk $D := \{(x,t) \in \mathbb{C}^2 \mid |x-x_0| < \delta_1,\ |t-t_0| < \delta_2\}$ admits an analytic continuation to a meromorphic function in the strip $S := \{(x,t) \in \mathbb{C}^2 \mid |t-t_0| < \delta_2\}$ possessing the trivial-monodromy property with respect to x (in the sense of Theorem* 3(B)) *for every fixed t. On the other hand, one can find a holomorphic solution $q_0(x,t)$ of* (13) *in D that admits no further analytic extension beyond the strip S.*

(C) *The envelope of meromorphy of any local holomorphic solution $q \in \mathcal{R}(x_0,t_0)^{\mathrm{od}}$ of equation* (22) *can be written in the form $\mathbb{C}^1_x \times X$, where X is a Riemannian domain over $\mathbb{C}^1_t$. Conversely, for every Riemannian domain $\pi : X \to \mathbb{C}^1_t$ over $\mathbb{C}^1_t$ and every point $(x_0,t_0) \in \mathbb{C} \times \pi(X)$ one can find a local holomorphic solution $q \in \mathcal{R}(x_0,t_0)^{\mathrm{od}}$ of equation* (22) *whose envelope of meromorphy is equal to $\mathbb{C}^1_x \times X$.*

(D) *In the notation of part* (A), *if the germ* $q_0(x) := q(x, t_0)$ *satisfies* $Lq_0 \in \mathrm{Gev}_{(1/m)-0}$, *then the solution* $q(x,t)$ *of the corresponding Cauchy problem admits an analytic continuation to a meromorphic off-diagonal* $\mathrm{gl}(n,\mathbb{C})$*-valued function on* $\mathbb{C}^2$ *possessing the following trivial-monodromy property with respect to* x *and* t. *For every* $z \in \mathbb{C}$ *the auxiliary linear system* $E_x = (az + q(x,t))E$, $E_t = (bz^m + \sum_{j=1}^m F_j(q)(x,t)z^{m-j})E$ (*which is defined by the formulae* (8) *on account of* (14)) *has a globally meromorphic fundamental system of solutions on* $\mathbb{C}^2_{xt}$.

In connection with the terminology in part (C) of the theorem we recall that a *Riemannian domain over* $\mathbb{C}^N$ is a complex manifold X together with a holomorphic locally invertible map $\pi : X \to \mathbb{C}^N$ (see [33, subsection 22]), and the *envelope of meromorphy* of an arbitrary family of germs of holomorphic functions at a point $\zeta_0 \in \mathbb{C}^N$ is defined as the largest holomorphically separable Riemannian domain over $\mathbb{C}^N$ such that all the germs in this family can be analytically continued to meromorphic functions on this Riemannian domain (see [33, subsection 41]). This domain over $\mathbb{C}^N$ admits a more constructive description as the union of the results of all possible analytic extensions along chains of polydisks, similarly to the definition of a complete analytic function in the sense of Weierstrass ([33, Russian page 276]). By the envelope of meromorphy of a $\mathrm{gl}(n,\mathbb{C})$-valued germ (or a family of such germs) we understand the envelope of meromorphy of all entries of these germs.

To prove the necessity of the condition $Lq_0 \in \mathrm{Gev}_{1/m}$ for the existence of a local holomorphic solution $q(x,t)$ of the Cauchy problem, one should reduce the ordinary differential equation for $\mu(x_0,t,z)$ (where $\mu(x,t,z)$ is the formal series from the definition (20) of the scattering data $Lq(t,z)$) to the form (22) with x replaced by t and then apply Sibuya's theorem mentioned above (using the assumption that the matrix b has simple spectrum). This part of the argument is done in [17] by a reference to [28], but the exposition of this proof in [28, § 5] contains an inaccuracy that will be corrected now. Contrary to the last paragraph of [28, § 5], one cannot in general remove all terms with negative powers of z from the formula (5.1) of [28] by making the transformation indicated there. However, there is no actual need to remove them. Just replace the last paragraph of [28, § 5] by the following paragraph (which uses our current notation $\mu(x,t,z)$ for what was denoted by $m(x,t,z)$ in [28]; the other notation is from [28]).

By the definition of the series $\widetilde{V}$ in [28], the off-diagonal series $N(t,z) := \mu(x_0,t,z) - I$ satisfies the differential equation $N_t = V(I+N) - (I+N)\widetilde{V}$, where V is defined by (14). Taking the diagonal parts of both sides of this equation, we have $0 = V_d + (V_{od}N)_d - \widetilde{V}$. Now, substituting $\widetilde{V} = V_d + (V_{od}N)_d$ into the equality of the off-diagonal parts, we obtain the following equation of the form (22) for $N(t,z)$:

$$N_t = VN - NV_d + V_{od} - (I+N)(V_{od}N)_d,$$

where the subscripts d and od denote the diagonal and off-diagonal part respectively. To verify that this equation is indeed of the form (22) (with the variable t instead of x and after rearranging all entries of the matrix N into one vector $y \in \mathbb{C}^\nu$, $\nu = n(n-1)$), we note the following. First, V_{od} and the difference between $VN - NV_d$ and $[bz^m, N]$ are polynomials of degree at most $m-1$ in z whose coefficients depend holomorphically on t and polynomially on N. Second, the linear operator $N \mapsto [b, N]$ is invertible[9] on the space of all off-diagonal matrices. Therefore all the hypotheses of Sibuya's theorem hold, and we arrive at the desired conclusion: the formal series $Lq_0(z) = N(t_0, z)$ belongs to $\mathrm{Gev}_{1/m}$. This proves the necessity in part (A).

To prove the sufficiency of the condition $Lq_0 \in \mathrm{Gev}_{1/m}$ for the existence of the local holomorphic solution of the Cauchy problem, we consider the Riemann problem (17) with parameter $\xi = (x,t) \in \mathbb{C}^2$, the polynomial $P(\xi, z) = az(x - x_0) + (bz^m + c_1 z^{m-1} + \cdots + c_m)(t - t_0)$, and the formal series $f(z) = I + Lq_0(z)$. By Theorem 2(A), the solution $\gamma_\pm(x,t,z)$ of this problem exists in a neighborhood of the point $(x_0, t_0) \in \mathbb{C}^2$ and depends holomorphically on x, t. Putting $q(x,t) := [g_0(x,t), a]$, where $g_0(x,t)$ is the coefficient at z^{-1} in the expansion $\gamma_-(x,t,z) = I + \sum_{k=0}^\infty g_k(x,t) z^{-(k+1)}$, we claim that the holomorphic off-diagonal $\mathrm{gl}(n,\mathbb{C})$-valued function $q(x,t)$ satisfies equation (13) in a neighborhood of (x_0,t_0) along with the initial condition $q(x,t_0) = q_0(x)$. Indeed, the initial condition $q(x,t_0) = q_0(x)$ follows from the equality $BLq_0 = q_0$, which holds by Theorem 2(A). Furthermore, the first equality (21) shows that in a neighborhood of (x_0,t_0) we have $E_x = UE$, where $E(x,t,z) := \gamma_+(x,t,z)$ and $U(x,t,z) := az + q(x,t)$. Repeating verbatim the proof of Lemma 2 (which is legitimate in our case because of Lemma 4), we obtain that $E_t = VE$, where $V(x,t,z)$ is given by the formula (14) with the same differential polynomials $F_j : \mathcal{R}(x_0) \to \mathcal{R}(x_0)$ as in (13). The resulting equations $E_x = UE$ and $E_t = VE$ form the auxiliary linear system (8) whose solubility (with invertible E) implies that we have the zero curvature condition (9): $U_t - V_x + [U,V] = 0$, which is equivalent to the equation (13). This completes the proof of part (A) of Theorem 4.

Once part (A) is proved, part (B) follows easily from it and Theorem 3(B) since we always have $1/m < 1$ for all $m \geq 2$. Examples mentioned in the last assertion of part (B) can be constructed in abundance using the Cauchy–Kowalevsky theorem (this was done in [36, § 4], for all equations appearing in Theorem 1). The rest of Theorem 4 can also be easily obtained from part (A) and Theorems 2, 3, but we omit the details since these results have no direct use in the proof of Theorem 1.

4. Proof of Theorem 1

We start by showing that every local holomorphic solution $u(x,t)$ of any of equations (4)–(6) induces a local holomorphic solution $q(x,t)$ of an appropriate system (13). Indeed, if u satisfies (4), then a rescaling of x and t yields that $u_t = u_{xxx} - 6uu_x$, which is equivalent to (13) for $m = 3$, $a =$

[9] This is the only place where we use the assumption that b has a simple spectrum.

$b = \mathrm{diag}(1/2, -1/2)$, $c_1 = c_2 = c_3 = 0$ and $q(x,t) = \left(\begin{smallmatrix} 0 & u(x,t) \\ 1 & 0 \end{smallmatrix}\right)$. If u satisfies (6), then a rescaling of x and t by real factors yields that $iu_t + u_{xx} = \pm u|u|^2$, which is equivalent to (13) for $m = 2$, $a = b = \mathrm{diag}(-i/2, i/2)$, $c_1 = c_2 = 0$ and $q(x,t) = \begin{pmatrix} 0 & u(x,t) \\ \pm\overline{u(\overline{x},\overline{t})} & 0 \end{pmatrix}$. If u satisfies (5), then the reduction is more complicated. It is described, for example, in [36] and follows Drinfeld and Sokolov [37]. First, a rescaling of x and t yields that $u_{tt} = -1/3u_{xxxx} - 4/3(uu_x)_x$, which is the condition for solubility of the system

$$\varphi_x = u_t, \quad \varphi_t = -1/3u_{xxx} - 4/3uu_x$$

in the bidisk D. This enables us to write the rescaled equation (6) in the form $L_t = [P, L]$ (see (7)), where $L := \partial_x^3 + u\partial_x + 1/2(\varphi + u_x)$ and $P := \partial_x^2 + 2/3u$. Second, writing $L = (\partial_x - v_3)(\partial_x - v_2)(\partial_x - v_1)$ for some $v_1, v_2, v_3 \in \mathcal{O}(D)$, we define an off-diagonal (because $v_1 + v_2 + v_3 = 0$) matrix-valued function $q(x,t) := K^{-1}\,\mathrm{diag}(v_1(x,t), v_2(x,t), v_3(x,t))K$ with $K \in \mathrm{GL}(3,\mathbb{C})$ being the matrix with entries $K_{ij} = (\alpha_j)^{i-1}$, where $\alpha_1, \alpha_2, \alpha_3$ are the cubic roots of 1 written in an arbitrary order. Then the rescaled equation (6) is equivalent to (13) for $m = 2$, $a = \mathrm{diag}(\alpha_1, \alpha_2, \alpha_3)$, $b = a^2$ and $c_1 = c_2 = 0$.

Now, to prove Theorem 1, we apply Theorem 4(B) to $q(x,t)$ and conclude that $q(x,t)$ extends to a global meromorphic function of x for every fixed t. Recovering $u(x,t)$ from $q(x,t)$ by the formulae above, we see that the same conclusion holds for $u(x,t)$, as required.

References

[1] A.-L. Cauchy, *Mémoire sur les systèmes d'equations aux dérivées partielles d'ordre quelconque et sur leur réduction à systèmes d'equations linéaires du premier ordre.* C. R. Acad. Sci. Paris, **40**(1842), 131–138.

[2] S. von Kowalevsky, *Zur Theorie der partiellen Differentialgleichungen.* J. reine angew. Math., **80**(1875), 1–32.

[3] G. S. Salekhov, *On the Cauchy–Kowalevsky problem for a class of linear equations in the domain of arbitrarily smooth functions.* Izv. Akad. Nauk SSSR Ser. Mat., **14**(1950), 355–366. (Russian)

[4] C. Kiselman, *Prolongement des solutions d'une équation aux dérivées partielles à coefficients constants.* Bull. Soc. Math. France, **97**(1969), 329–356.

[5] M. Zerner, *Domaines d'holomorphie des fonctions vérifiànt une équation aux dérivées partielles.* C. R. Acad. Sci. Paris **272**(1971), 1646–1648.

[6] L. Hörmander, *The analysis of linear partial differential operators.* Vol. I, Berlin–Heidelberg–New York–Tokyo, Springer-Verlag, 1983.

[7] G. M. Henkin, *Jean Leray and several complex variables.* // in: J. Leray, *Oeuvres Scientifiques*, vol. III, Berlin and Paris, Springer and SMF, 1998, 1–31.

[8] S. Rigat, *Application of functional calculus to complex Cauchy problems.* Comput. Meth. Funct. Theory, **7**(2007), 509–526.

[9] H. Tahara, *On the singularities of solutions of nonlinear partial differential equations in the complex domain* // Introduction to path integrals and microlocal analysis (N. Kumano-go, S. Yamazaki, Y. Okado, eds.), RIMS Kôkyûroku **1723**(2011), 101–109.

[10] H. Hannach, A. A. Himonas and G. Petronilho, *Gevrey regularity in time for generalized KdV type equations.* Contemp. Math. **400**(2006), 117–127.

[11] F. Linares and G. Ponce, *Introduction to nonlinear dispersive equations*, New York, Springer, 2009.

[12] A. Rybkin, *The Hirota τ-function and well-posedness of the Korteweg–de Vries equation with an arbitrary step-like initial profile decaying on the right half-line.* Nonlinearity **24**(2011), 2953–2990.

[13] J. B. McLeod and P. J. Olver, *The connection between partial differential equations soluble by inverse scattering and ordinary differential equations of Painlevé type.* SIAM J. Math. Anal. **14**(1983), 488–506.

[14] N. Hayashi and K. Kato, *Regularity in time of solutions to nonlinear Schrödinger equations.* J. Funct. Anal. **128**(1995), 253–277.

[15] M. D. Kruskal, N. Joshi and R. Halburd, *Analytic and asymptotic methods for nonlinear singularity analysis: a review and extension of tests for the Painlevé property*, Lect. Notes Phys. **638**(2004), 175–208.

[16] A. V. Domrin, *Local holomorphic Cauchy problem for soliton equations of parabolic type.* Dokl. Ross. Akad. Nauk, **420**(2008), 14–17. English transl.: Doklady Math. **77**(2008), 332–335.

[17] A. V. Domrin, *Meromorphic extension of solutions of soliton equations.* Izv. Ross. Akad. Nauk Ser. Mat., **74**(2010), no. 3, 19–40. English transl.: Izvestiya Math. **74**(2010), 461–480.

[18] C. S. Gardner, J. M. Greene, M. D. Kruskal and R. M. Miura, *Method for solving the Korteweg–de Vries equation.* Phys. Rev. Lett. **19**(1967), 1095–1097.

[19] P. D. Lax, *Integrals of nonlinear equations of evolution and solitary waves.* Comm. Pure Appl. Math. **21**(1968), 467–490.

[20] V. E. Zakharov and A. B. Shabat, *Exact theory of two-dimensional self-focusing and one-dimensional automodulation of waves in nonlinear media.* Zh. Exper. Teor. Fiz. **61**(1971), 118–134. English transl.: Sov. Phys. JETP **34**(1972), 62–69

[21] S. P. Novikov, *The periodic problem for the Korteweg–de Vries equation.* Funkts. Anal. Prilozhen. **8**(1974), no. 3, 54–66. English transl.: Funct. Anal. Appl. **8**(1974), 236–246.

[22] I. M. Krichever, *Nonlinear equations and elliptic curves.* // Itogi Nauki Tekhn. Sovrem. Probl. Mat., **23**(1983), pp. 79–136. English transl.: J. Sov. Math. **28**(1985), 51–90.

[23] L. D. Faddeev and L. A. Takhtajan, *Hamiltonian methods in the theory of solitons.* Moscow, Nauka, 1986. English transl.: Berlin–Heidelberg–New York, Springer-Verlag, 1987.

[24] D. Bennequin, *Hommage à Jean–Louis Verdier: Au jardin des systèmes intégrables.* // Integrable systems. The Verdier memorial conference (O. Babelon et al. eds), Boston, Birkhäuser, 1993, 1–36.

[25] S. P. Novikov, S. V. Manakov, L. P. Pitaevskii and V. E. Zakharov, *Theory of solitons. The inverse scattering method.* Moscow, Nauka, 1980. English transl.: New York, Plenum, 1984.

[26] V. E. Zakharov and A. B. Shabat, *Integration of nonlinear equations of mathematical physics by the inverse scattering method.* II. Funkts Anal. Prilozhen. **13**(1979), no. 3, 13–22. English transl.: Funct. Anal. Appl. **13**(1979), 166–174.

[27] B. Malgrange, *Sur les deformations isomonodromiques.* I. *Singularites regulieres.* // Mathematique et Physique. Sem. Ecole Norm. Sup. 1979–1982 (L. Boutet de Monvel et al. eds), Basel, Birkhäuser, 1983, 401–426.

[28] A. V. Domrin, *Remarks on the local version of the inverse scattering method.* Trudy Mat. Inst. Steklova, **253**(2006), 46–60. English trans.: Proc. Steklov Math. Inst., **253**(2006), 37–50.

[29] G. Wilson, *The τ-functions of the* gAKNS *equations.* // Integrable systems. The Verdier memorial conference (O. Babelon et al. eds), Boston, Birkhäuser, 1993, 131–145.

[30] D. H. Sattinger and J. Szmigielski, *Factorization and the dressing method for the Gel'fand–Dikii hierarchy.* Physica D **64**(1993), 1–34.

[31] C.-L. Terng and K. Uhlenbeck, *Bäcklund transformations and loop group actions.* Comm. Pure Appl. Math. **53**(2000), 1–75.

[32] A. V. Domrin, *Riemann problem and matrix-valued potentials with a convergent Baker–Akhiezer function.* Teor. Mat. Fiz., **144**(2005), 453–471. English transl.: Theor. Math. Phys. **144**(2005), 1264–1273.

[33] B. V. Shabat, *Introduction to complex analysis Part II. Functions of several variables.* Moscow, Nauka, 1985. English transl.: Providence, AMS, 1992.

[34] Y. Sibuya, *Linear differential equations in the complex domain: problems of analytic continuation.* Providence, AMS, 1990.

[35] P.-F. Hsieh and Y. Sibuya, *Basic theory of ordinary differential equations.* Berlin, Springer-Verlag, 1999.

[36] A. V. Domrin, *On holomorphic solutions of equations of Korteweg–de Vries type*, Trudy Mosk. Mat. Obsch. **73**(2012), 241–257. English transl.: Trans. Moscow Math. Soc. 2012, 193–206.

[37] V. G. Drinfeld and V. V. Sokolov, *Lie algebras and equations of Korteweg–de Vries type.* // Itogi Nauki Tekhn. Sovrem. Probl. Mat. **24**(1984), 81–180. English transl.: J. Sov. Math. **30**(1985), 1975–2036.

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