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LECTURES ON DIFFEOMORPHISM GROUPS IN QUANTUM PHYSICS

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Infinite-dimensional groups and algebras play an increasingly important role in physics. This presentation describes from an elementary starting point how diffeomorphism groups and their unitary representations enter nonrelativistic quantum theory, making connections with local current algebras and various topics of current interest. Along the way some fundamental ideas from group theory and quantum field theory are introduced and discussed, with illustrative examples of unitary group representations and the physical systems they describe.

1. Introduction: Groups, Representations, and Symmetries

My goal in these lectures is to give a broad, mostly self-contained introduction to local current algebras and diffeomorphism groups. I hope to highlight how these infinite-dimensional algebras and groups help to unify certain ideas in quantum theory, and connect with other topics in physics. Thus we shall focus on elementary ideas, overarching themes, and important physical intuitions, rather than on rigorous proofs of theorems.

As we proceed, fundamental concepts will be introduced from group theory, quantum field theory, and topology. We shall construct illustrative examples of unitary group representations and the physical systems they describe, explaining and making use of some necessary techniques from infinite-dimensional functional analysis and differential geometry. Hopefully the result will be a fully accessible presentation that develops the connections among a number of areas often considered separately.

For those familiar with other work on infinite-dimensional algebras and groups in physics, the content of these notes can be considered complementary to the excellent books by Pressley and Segal,[1] by Mickelsson,[2] and by Kac,[3] with which there are important points of contact but only a little direct overlap. The book by Ottesen[4] contains a much more extensive discussion of canonical commutation and anticommutation relations in Fock space than that presented here, but has only a minor discussion of diffeomorphism groups despite their central role in quantum physics. The most closely related comprehensive reference I know is the monograph of Ismagilov,[5] which provides full mathematical details on certain of the topics I shall discuss, but which perhaps is not so accessible to students of physics. The monograph by Omori[6] develops many related topics, including deformation quantization and quantum groups, from a highly mathematical standpoint. Readers interested in the diffeomorphism group in hydrodynamics are also referred to the superb book by Arnold and Khesin.[7]

Let us begin by stepping back to view a wider landscape, and look at the study of diffeomorphism groups in the context of what has been one of the most important themes of physics during the last century — the growth in our understanding of the role of symmetry.

To say that a physical system has a symmetry is to assert that in some important sense the system — or, at least, some aspect of the system — remains invariant under a transformation of some kind. Fundamental unifications in physics have followed from the discovery of symmetries that underlie known laws of nature. If the dynamical equations governing a physical system are invariant under a set of symmetry transformations, the symmetries allow whole families of solutions to be obtained from particular solutions to the equations. Furthermore, dynamical symmetries are closely associated with conservation laws. Symmetries in a system may be exact, or they may be approximate. And symmetry may be spontaneously broken, as when the ground state of a system is not itself invariant under the symmetry transformations.

To describe symmetry mathematically, we make use of groups and group representations. We shall consider, in succession, symmetry groups that are finite or infinite discrete groups, and compact or noncompact finite-dimensional Lie groups. Then we shall turn our attention to certain infinite-dimensional groups and algebras, including groups of diffeomorphisms of compact or noncompact manifolds and their semidirect products. The introductory discussion also provides a context for us to introduce notation and definitions.

1.1. *Discrete Groups and Symmetries*

A *group* G is a set closed under an associative binary operation, possessing an identity element, and for which every element has an inverse. Essentially, the group structure is the appropriate one for describing symmetry algebraically, because two symmetry transformations can be performed successively so as to result in a third transformation, while any symmetry transformation can normally be inverted. A group for which the binary operation is commutative is called Abelian. An element of G that commutes with every group element is called *central*.

For example, a molecule such as ammonia (chemical formula NH_3) can be visualized as consisting of three hydrogen atoms positioned at the vertices of an equilateral triangle, and a nitrogen atom that is somewhat above the plane of the triangle, located equidistant from the hydrogen atoms (but not so as to form a regular tetrahedron). The axis through the nitrogen atom perpendicular to the plane of the hydrogen atoms is called a 3-fold axis of rotational symmetry. There are here six distinct transformations that leave the molecular configuration invariant — the identity, rotation by 120° or 240° about the 3-fold axis, and three different reflections through mirror planes. These transformations form a group, which is named C_{3v} in the physical chemistry context. It is called a *point group* because each transformation leaves at least one point invariant. Note that C_{3v} is not Abelian: a rotation followed by a reflection does not give the same transformation when the reflection is performed first.

For NH_3 there exists a symmetry transformation that implements *any* permutation of the three hydrogen atoms. Thus the symmetry group happens to be isomorphic to the *symmetric group* S_3, where S_N denotes the group of all permutations of N objects (and thus contains $N!$ elements).

As another example, the symmetries of the cube include three mutually perpendicular 4-fold axes, four 3-fold axes (the long diagonals of the cube), and nine mirror planes. The point group here contains 48 distinct elements.

A *group representation* is a concrete realization of the elements of G by linear operators acting on a vector space $\mathcal{V}$ (usually taken to be an inner product space) that respects the group operation. That is, letting $\mathcal{L}(\mathcal{V})$ be the algebra of linear operators on $\mathcal{V}$, we write $\pi : G \to \mathcal{L}(\mathcal{V})$, and we require that $\pi(g_1 g_2) = \pi(g_1)\pi(g_2)$ $(\forall g_1, g_2 \in G)$. Thus we represent symmetry transformations g by matrices when the vector space $\mathcal{V}$ is finite-dimensional, or by (bounded) linear operators when we are in an infinite-dimensional Hilbert space.

A linear operator $Q : \mathcal{V}_1 \to \mathcal{V}_2$ is *unitary* if it is invertible and preserves the respective inner products in $\mathcal{V}_1$ and $\mathcal{V}_2$. In particular, $U : \mathcal{V} \to \mathcal{V}$ is unitary if and only if $U^* = U^{-1}$, where "$*$" denotes the adjoint operator. Two representations π_1, π_2 of G, in $\mathcal{V}_1$ and $\mathcal{V}_2$ respectively, are called *unitarily equivalent* if there exists a unitary operator $Q : \mathcal{V}_1 \to \mathcal{V}_2$ such that $(\forall g \in G)\ Q\pi_1(g)Q^{-1} = \pi_2(g)$. Especially important in physics are unitary representations of groups, which we denote by $U(g)$, $g \in G$. Thus $U(g)$ preserves the inner product of all pairs of vectors in $\mathcal{V}$. When $\mathcal{V}$ is the one-dimensional complex vector space $\mathbb{C}^1$, so that $U(g)$ assigns to each $g \in G$ a complex number of modulus 1, U is called a *character* of G.

A representation π of a group G in $\mathcal{V}$ is called *irreducible* when $\mathcal{V}$ has no closed subspaces, other than $\{0\}$ and $\mathcal{V}$ itself, that are invariant under all the operators $\pi(g)$, $g \in G$. Equivalently, any linear operator in $\mathcal{V}$ that commutes with all the $\pi(g)$ is a multiple of the identity. Under appropriate conditions, an arbitrary representation of G may be decomposed into a direct sum of irreducible representations, so that the unitary equivalence classes of irreducible representations of the group become fundamental objects in the study of the symmetry the group describes. It is evident that 1-dimensional group representations are always irreducible.

For the case of NH_3, we have two inequivalent characters of S_3 (or equivalently C_{3v}) — the trivial one, where all group elements are represented by 1, and the alternating representation, where group elements obtained through an odd number of elementary (pairwise) exchanges are represented by -1. A class of two-dimensional irreducible unitary representations of S_3 also exists. Once obtained, these representations of the symmetry group pertain immediately to the electronic structure of the ammonia molecule (via symmetry-adapted molecular orbitals), and to the description of molecular vibrations (via symmetry-adapted displacement coordinates).[8]

The trivial and the alternating 1-dimensional unitary representations of S_N exist for all $N \geq 2$, and are pertinent in quantum mechanics to the description of bosons and fermions. Higher-dimensional representations of S_N are classified by means of Young tableaux,[9] and pertain to the description of particles obeying parastatistics (see Sec. 5 below).[10]

Now a crystal in three-dimensional space consists of multiple copies of a fundamental (bounded) region, the unit cell, arranged in a periodic lattice that we idealize as being of infinite extent. The symmetry of the configuration within the unit cell is described by means of a finite group, the point group H. The lattice (called a Bravais lattice) is described by an Abelian group L of discrete translations in physical space that leave the lattice structure invariant. The symmetry of the whole crystal is described by a larger group called the *space group*, obtained by combining point group elements and translations, together with screw axes and glide planes. The crystallographic point groups and space groups have been completely classified — there are 32 point groups, and 230 space groups. Their study is important to the theory of molecular orbitals in quantum chemistry, to the relationship of X-ray diffraction patterns to crystal structures, to the theory of correlated electron systems, and to many other topics in the fundamental physics of condensed matter.[11]

Let us write the elements of the lattice group L as vectors $\mathbf{a} \in \mathbb{R}^3$. The group operation in L is then vector addition, denoted by the $+$ sign. Any element $h \in H$ also acts naturally on L; for $\mathbf{a} \in L$, we write this action as $h\mathbf{a}$. By applying elements of L and H successively and keeping track of what happens, we obtain a *semidirect product* of L with H. The semidirect product group is the set $L \times H$, with the group law given by $(\mathbf{a}_1, h_1)(\mathbf{a}_2, h_2) = (\mathbf{a}_1 + h_1\mathbf{a}_2, h_1h_2)$.

Symmetry groups may be finite or infinite. The point groups associated with many geometric shapes (*e.g.* the tetrahedron, the cube, the octahedron) are finite, as is the symmetric group S_N. The groups of translations describing periodic lattices, on the other hand, are infinite discrete groups, while the symmetry groups of the circle, the cylinder, or the sphere are infinite continuous groups.

A *normal subgroup* of a group G is a subgroup $N \subseteq G$ having the property that $(\forall n \in N)(\forall g \in G)\ g^{-1}ng \in N$. That is, N is invariant as a set under conjugation by elements of G. For any subgroup H of G, and for $g \in G$, the *right coset* Hg is $\{\, hg \mid h \in H\}$. It is easy to show that any two right cosets Hg_1 and Hg_2 are either equal or disjoint. Similarly the *left coset* gH is $\{gh \mid h \in H\}$. A normal subgroup of G is thus a subgroup for which $gN = Ng$ $(\forall g \in G)$. Let us denote the space of all left cosets by G/H, and the space of right cosets by $H\backslash G$. When G has a normal subgroup N, one can form the *quotient group* G/N, whose elements are the distinct (right or left) cosets by N, endowed with the group law for cosets, $(Ng_1)(Ng_2) = N(g_1g_2)$.

A *simple* group is a group G, containing more than one element, whose only normal subgroups are itself and $\{e\}$, where e is the identity element in G. Thus the simple groups are the analogues in group theory of the prime numbers in number theory — they are the fundamental building blocks from which other groups may be constructed. The complete classification of the finite simple groups, long a dream of mathematicians, required about 500 journal articles by about 100 authors, mainly published from 1955 to 1983. One of the most interesting of the finite simple groups is the largest of the sporadic groups, dubbed "the Monster" due to its extraordinary size — it contains $2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$ elements. The Monster is the automorphism group of a vertex operator algebra, and has deep connections with Kac–Moody algebras and groups and with quantum field theory.[12]

But groups do not describe everything that is crystalline in nature. A new horizon opened in the mathematical and physical study of crystal symmetry with the 1984 discovery in nature of *quasicrystals* — materials with *quasiperiodic* structures.[13] These are configurations in which (ideally) every *local* structure repeats infinitely often, though there are no global symmetry transformations. The X-ray diffraction patterns of quasicrystals show, for example, pentagonal symmetry, which cannot occur in any of the 230 permitted space groups. Indeed, it is likely that the well-known existence of a complete mathematical classification of the possible crystal structures in $\mathbb{R}^3$ actually *inhibited* the discovery of quasicrystals. Thus history teaches us not to be too rigidly constrained by known structures, but to be alert to ways of relaxing constraints or generalizing existing categories.

Ordinary group theory no longer suffices to characterize quasiperiodic patterns, and one must make use of other mathematical techniques.[14]

1.2. *Lie Groups and Lie Algebras*

To this point our examples have been discrete groups, but equally important to physics are the continuous groups. Cylindrical symmetry is described by the group $SO(2)$, of rigid rotations of $\mathbb{R}^2$ about the origin. Spherical symmetry is likewise described by $SO(3)$. Choosing an orthonormal basis for $\mathbb{R}^3$, we can realize $SO(3)$ as the group of real 3×3 orthogonal matrices, under the operation of matrix multiplication.

A *Lie group* is simultaneously a group and an analytic manifold (real or complex), where the group operations of multiplication and inversion are analytic mappings. The group manifold for $SO(2)$ is the circle S^1, while the manifold for $SO(3)$ is three-dimensional. Another example of a Lie group is the group $SU(2)$ consisting of the complex 2×2 unitary matrices, well-known as the two-sheeted universal covering group of $SO(3)$. The irreducible unitary representations of $SO(3)$ and $SU(2)$ give us, respectively, the orbital angular momentum states of the hydrogen atom (or any other quantum system having rotational symmetry), and the spin states of fundamental particles (or composites). The group $SU(3)$ describes an approximate symmetry of the strong interactions; its irreducible unitary representations give us quarks and antiquarks, as well as multiplets of hadrons describing families of baryons and mesons.

These are examples of Lie groups that are *compact*. Intuitively, compactness means that their group manifolds neither "extend to infinity" nor are "open" in any direction. Closed intervals in $\mathbb{R}$ are compact, while open or half-open intervals are not; spheres are compact, while infinitely-long cylinders are not. Mathematically, a *compact topological space* is a space for which every covering by open sets has a finite subcovering; or equivalently, one for which every continuous real-valued function assumes its maximum value. The symmetries of space-time, on the other hand, are described by *noncompact* Lie groups — the Lorentz group, or its semidirect product with space-time translations, the Poincaré group. The Poincaré group is actually the group of all transformations of Minkowskian space-time leaving invariant the indefinite form $[(x_\mu - y_\mu)(x^\mu - y^\mu)]$, where we use the common notation $\mu = 0, 1, 2, 3$, $x^\mu = (ct, \mathbf{x})$, and sum over repeated indices: $x_\mu x^\mu = c^2t^2 - \mathbf{x} \cdot \mathbf{x}$. Sometimes we restrict ourselves to Poincaré transformations that preserve the directionality of the time coordinate x^0, or preserve spatial parity, or both. Irreducible unitary representations of the Poincaré group are labeled by particle masses and spins.[15]

While Lie groups describe physical symmetries globally, the local (or infinitesimal) description of symmetry is achieved through *Lie algebras*. A Lie algebra $\mathcal{G}$ is a vector space (here taken to be real or complex), equipped with an additional binary operation that is written as the bracket of two elements: for all $X, Y \in \mathcal{G}$, $[X, Y] \in \mathcal{G}$. The bracket $[X, Y]$ is bilinear with respect to scalar multiplication. It is antisymmetric, *i.e.i* $[X, Y] = -[Y, X]$; and it satisfies the famous Jacobi identity,

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0, \quad (\forall X, Y, Z \in \mathcal{G}). \qquad (1)$$

Then, associated with a Lie group G, we have the corresponding Lie algebra $\mathcal{G}$, whose elements are *tangent vectors* to the Lie group manifold at the identity. The elements of $\mathcal{G}$ are the infinitesimal generators of 1-parameter subgroups of G; for $X \in \mathcal{G}$, the exponential map $a \to \exp(aX)$, $a \in \mathbb{R}$, defines the corresponding 1-parameter subgroup.

Alternatively, one can think of the elements of the Lie algebra $\mathcal{G}$ as left-invariant vector fields on the group manifold of G. If X and Y are two elements of $\mathcal{G}$, their *Lie bracket* $[X, Y]$ as vector fields is defined as the vector field that corresponds to the infinitesimal outcome of flowing infinitesimally by each of the two vector fields, in succession, and then flowing backward infinitesimally by each of the vector fields. That is, taken to order a^2, the equation

$$\exp(-aX)\exp(-aY)\exp(aX)\exp(aY) \approx \exp(a^2[X,Y]), \qquad (2)$$

serves to define the bracket in the Lie algebra $\mathcal{G}$ of the Lie group.[16]

When a Lie group is represented by unitary operators in $\mathcal{V}$, the corresponding Lie algebra may be represented by *self-adjoint* operators in $\mathcal{V}$, with the bracket in the Lie algebra corresponding to the *commutator* of linear operators in $\mathcal{V}$. We shall use the same bracket notation for the commutator of linear operators, namely $[A, B] = AB - BA$. It should always be clear from the context whether we refer to the Lie bracket of a pair of vector fields, the bracket operation applied to elements of a Lie algebra, or the commutator of linear operators. If $[X, Y] = Z$ in the Lie algebra $\mathcal{G}$, we shall require for a self-adjoint representation of $\mathcal{G}$ that the commutator $[\sigma(X), \sigma(Y)] = i\sigma(Z)$, where $\sigma(Z)$ is the self-adjoint operator representing Z. The extra factor of i on the right is needed to allow for a self-adjoint, rather than a skew-adjoint representation. The concepts of irreducible representations (for which there are no nontrivial invariant subspaces), and of unitarily equivalent representations, apply equally well at the level of the Lie algebra as at the level of the group.

The Lie algebra $so(3)$ of $SO(3)$, for example, is three-dimensional, and isomorphic to the Lie algebra $su(2)$ of $SU(2)$. The three generators can be represented in $\mathcal{V} = \mathbb{C}^2$ by the well-known Pauli spin matrices $\sigma_1, \sigma_2, \sigma_3$, whose commutators satisfy

$$[\sigma_j, \sigma_k] = 2i\,\epsilon_{jk\ell}\,\sigma_\ell\,, \quad j,k,\ell = 1,2,3\,. \qquad (3)$$

The components of $\hat{\mathrm{S}} = (\hbar/2)(\sigma_1, \sigma_2, \sigma_3)$ correspond respectively to the x, y, and z components of angular momentum for a spin-$\frac{1}{2}$ particle. For each value of the spin $s = \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$, there is an irreducible, self-adjoint representation σ_j $(j = 1,2,3)$ of the Lie algebra $su(2)$, obeying Eq. (3), acting in the complex vector space $\mathbb{C}^{2s+1}$; and there is a corresponding unitary representation of $SU(2)$. For integer s, this representation is also a unitary representation of $SO(3)$; for half-integer s, it is a projective representation of $SO(3)$. Unitary representations of $SO(3)$ also describe the *orbital* angular momentum $\hat{\mathrm{L}}$, in which case we label the generators $(\hat{L}_1, \hat{L}_2, \hat{L}_3)$ corresponding to the x, y, and z components of $\hat{\mathrm{L}}$.

Another important example for quantum mechanics is the Heisenberg algebra. Consider the Lie algebra with three generators Q, P and C,

together with the bracket operation

$$[Q, P] = C\,, \quad [Q, C] = [P, C] = 0\,. \qquad (4)$$

Since C is a central element, it must be represented by a multiple of the identity operator in an irreducible representation acting in $\mathcal{V}$; that is, $C \to i\hbar I$, where $\hbar$ is a real constant. For any two such representations acting in spaces $\mathcal{V}_1$ and $\mathcal{V}_2$ respectively, a unitary transformation $Q: \mathcal{V}_1 \to \mathcal{V}_2$ self-evidently obeys $Q(i\hbar I_1)Q^{-1} = i\hbar I_2$ (where I_1 and I_2 are the respective identity operators). Thus distinct values of $\hbar$ must correspond to unitarily inequivalent representations of (4).

Fixing $\hbar$ and letting $Q \to \hat{q}$, $P \to \hat{p}$, we have the famous Heisenberg algebra obtained by quantizing the particle position q and momentum p coordinates,

$$[\hat{q}, \hat{p}] = i\hbar I, \qquad (5)$$

where $\hbar = h/2\pi$ (h being Planck's constant). But Eq. (5), unlike Eq. (3), does not have any nontrivial finite-dimensional self-adjoint representations. Rather we have an irreducible representation, well-known from quantum mechanics, that acts in the infinite-dimensional Hilbert space of complex-valued, Lebesgue square-integrable functions on the real line, $\mathcal{H} = L^2_{dq}(\mathbb{R})$. Writing $\Psi(q) \in \mathcal{H}$, we have

$$\hat{q}\,\Psi(q) = q\,\Psi(q)\,,$$
$$\hat{p}\,\Psi(q) = -i\hbar\frac{d\,\Psi(q)}{dq}\,. \qquad (6)$$

The *uniqueness* of this representation up to unitary equivalence (demonstrated by von Neumann), together with the symmetry between position and momentum variables that is evident under Fourier transformation of Eqs. (6), are beautiful properties that have been regarded for many years as advantages of the usual, simple prescription for quantization of kinematics based on position and momentum operators. We shall see shortly an exquisite contrast with representations of Lie algebras of local currents and diffeomorphism groups in quantum mechanics, where we have a different kind of beauty — a rich multiplicity of unitarily inequivalent representations that describe physically distinct quantum-mechanical systems.

Notice too that in Eqs. (6), $\hat{q}$ and $\hat{p}$ are *unbounded* self-adjoint operators. This means that the ratios $||\hat{q}\,\Psi||/||\Psi||$ and $||\hat{p}\,\Psi||/||\Psi||$ have no upper bound, where $||\Psi|| = [\int_{\mathbb{R}} |\Psi(q)|^2 dq]^{1/2}$ denotes the usual L^2-norm of Ψ. Among other things, this also means that the domains of definition of the operators $\hat{q}$ and $\hat{p}$ are not all of $\mathcal{H}$, but *dense subspaces* of $\mathcal{H}$. The commutation relation (5) makes sense on a still smaller dense domain that is invariant under the actions of both $\hat{q}$ and $\hat{p}$. In contrast, unitary operators are bounded operators, that are defined on all of $\mathcal{H}$.

A unitary representation $U(g)$ of a Lie group G in a (finite- or infinite-dimensional) Hilbert space $\mathcal{H}$ is (weakly) *continuous* if the inner product $(\Phi, U(g)\Psi)$ is continuous in g $(\forall\, \Phi, \Psi \in \mathcal{H})$. Suppose that we have such a representation. Let $\exp(aA)$, $a \in \mathbb{R}$, be the 1-parameter subgroup of G obtained from the fixed Lie algebra element A. Then $U[\exp(aA)]$ is just a continuous 1-parameter unitary group acting in $\mathcal{H}$. There now exists a (not necessarily bounded) self-adjoint operator $\sigma(A)$, defined by

$$\sigma(A)\Psi = \lim_{a \to 0} \frac{1}{ia}\{U[\exp(aA)]\Psi - \Psi\}, \tag{7}$$

where the domain of definition of $\sigma(A)$ consists of those vectors $\Psi \in \mathcal{H}$ for which the limit in Eq. (7) exists (with respect to the Hilbert space norm). From Eq. (7) we typically obtain from U (under the right domain conditions) self-adjoint operators giving us a representation σ of the Lie algebra of G.

Conversely, given a (not necessarily bounded) self-adjoint operator A in a Hilbert space $\mathcal{H}$, there exists a continuous 1-parameter unitary group $U(a) = \exp(iaA)$ from which A can be recovered by means of Eq. (7). Again under the right domain conditions, we can exponentiate the self-adjoint representation of the Lie algebra to a unitary representation of the corresponding Lie group.

As an illustration, let us exponentiate $\hat{q}$ and $\hat{p}$ in Eqs. (6) to obtain 1-parameter groups of unitary operators. For $a \in \mathbb{R}$, define $[U(a)\Psi](q) = [\exp(-ia\hat{q})\Psi](q)$, which is just the product function $\exp(-iaq)\Psi(q)$. (Recall that $\hat{q}$ is an operator, while q is a real variable. Here the minus sign is a convenient choice.) Similarly, for $b \in \mathbb{R}$, we define $[V(b)\Psi](q) = [\exp(-ib\hat{p})\Psi](q)$; this is just the translated function $\Psi(q - b\hbar)$ using Taylor's formula. Notice that the power series expansion of $\exp[-b\hbar\,(d/dq)]\Psi$ converges only when Ψ is analytic in q, which is a very restrictive condition; but the resulting formula $\Psi(q - b\hbar)$ for the limit of this power series *extends* (by continuity in $\mathcal{H}$) to *all* square-integrable functions (even those that are not differentiable).

We can now calculate what the group operation should be. Since $V(b)U(a) = \exp(i\hbar ab)\, U(a)V(b)$, it is natural to write a group element as a triple (α, a, b), where α is a complex number of modulus one, $a, b \in \mathbb{R}$, and $W(\alpha, a, b) = \alpha\, U(a)V(b)$ is to be a unitary group representation. Then

it is easy to see that

$$(\alpha_1, a_1, b_1)(\alpha_2, a_2, b_2) = (\alpha_1\alpha_2 \exp[i\hbar a_2 b_1],\, a_1 + a_2,\, b_1 + b_2)\,, \tag{8}$$

which is one form of the *Heisenberg group*.

More generally, a useful equation that often permits direct calculation of a Lie group operation from the Lie algebra bracket, or the product of exponentiated linear operators from the commutator bracket, is the Baker–Campbell–Hausdorff formula

$$e^{A}\, B\, e^{-A} = \sum_{n=1}^{\infty} \frac{1}{n!}\,(ad\ A)^n B\,, \tag{9}$$

where $(ad\ A)B = [A, B]$. Using Eq. (9), we have immediately from Eq. (5) that $\exp(-ib\hat{p})\,\hat{q}\,\exp(ib\hat{p}) = \hat{q} - ib[\hat{p}, \hat{q}] = \hat{q} - b\hbar I$, from which the Heisenberg group law of Eq. (8) follows.

In a representation of Eqs. (3) or (5), a vector Ψ corresponds to a quantum state. To describe the dynamical time-evolution, we let Ψ depend on the time t with $i\hbar\,\partial\Psi/\partial t = H\Psi$, where H is the self-adjoint Hamiltonian operator. Thus H also generates a 1-parameter unitary group acting in the Hilbert space. When the time-evolution respects the rotational symmetry of $\mathbb{R}^3$, *i.e.* when H commutes with all the unitary operators representing $SO(3)$, the commutators of the angular momentum operators $\hat{L}_j$ with H are zero and angular momentum is conserved. Likewise when H commutes with the unitary operators representing translations in $\mathbb{R}^3$, linear momentum is conserved. But the description of angular or linear momentum by means of self-adjoint generators of unitary group representations does not make use of the particular choice of Hamiltonian. It is important to note that the description of the quantum *kinematics* works even when the *dynamical* equation of motion does not respect the kinematical symmetry.

The Lie groups we have discussed in this subsection are all *finite-dimensional* as manifolds. This means that even if they are not compact, they are *locally compact* — every element has an open neighborhood whose closure is compact. The Lie algebras of finite-dimensional Lie groups are finite-dimensional as vector spaces.

Finite-dimensional Lie groups come equipped with natural measures on the group manifold invariant under the group operation, called (left or right) Haar measures: if E is any measurable subset of the Lie group G, left Haar measure μ_ℓ (for instance) satisfies $\mu_\ell(gE) = \mu_\ell(E)$. When the group is compact the Haar measure is finite, so that we can also choose to set $\mu_\ell(G) = 1$. Haar measures are extremely useful in the theory of unitary representations of Lie groups, and one of the difficulties in treating infinite-dimensional groups is their absence.

1.3. *Infinite-Dimensional Algebras and Groups*

The study of gauge symmetry, among other topics in physics, brings us to the study of *infinite-dimensional* groups and algebras. Suppose that G is a compact Lie group such as $U(1)$ or $SU(2)$, and let $\mathcal{M}$ be the space-time manifold. Then it is natural to consider the *gauge group* whose elements are smooth mappings from $\mathcal{M}$ to G, denoted by $Map\,(\mathcal{M}, G)$. The group operation in G is then applied pointwise to define the group operation in $Map\,(\mathcal{M}, G)$; that is, for a pair of mappings $g_1 : \mathcal{M} \to G$ and $g_2 : \mathcal{M} \to G$, we define $(g_1 g_2)(t, \mathrm{x}) = g_1(t, \mathrm{x}) g_2(t, \mathrm{x})$.

A *loop group* is a map group whose elements take the circle S^1 to G. We can think of $Map\,(S^1, G)$ in either of two ways — as a rule associating an element of G to every point in S^1, or as a parameterized image of S^1 seen as a subset of the target space G.

The Lie algebra associated with the group $Map\,(\mathcal{M}, G)$ [or, respectively, with $Map\,(S^1, G)$] consists of maps from $\mathcal{M}$ [respectively, S^1] to the Lie algebra $\mathcal{G}$ of G, with the Lie bracket defined pointwise. We shall write this Lie algebra $map\,(\mathcal{M}, \mathcal{G})$ [respectively, $map\,(S^1, \mathcal{G})$], using a lower-case letter m. Now there is a unique extension of $map\,(S^1, \mathcal{G})$ by one additional dimension, such that the new elements commute with all the original elements of $map\,(S^1, \mathcal{G})$. This 1-dimensional *central extension* is called an *affine Kac–Moody algebra* and the corresponding group is a *Kac–Moody* group. Kac–Moody groups and algebras find application to conformal-invariant quantum field theory, to nonperturbative string theory, and in many other physical and mathematical contexts. They are naturally related to another infinite-dimensional Lie algebra, the Virasoro algebra, which we shall introduce in the next subsection.

Let us mention here still another infinite-dimensional group, one that entered mathematical physics relatively early — the Heisenberg–Weyl group of canonical quantum field theory. Consider the field $\phi\,(t, \mathrm{x})$ and its canonical conjugate $\pi(t, \mathrm{x}) = \partial_t \phi(t, \mathrm{x})$, satisfying the equal-time commutation relations

$$[\phi(t, \mathrm{x}), \phi(t, \mathrm{y})] = [\pi(t, \mathrm{x}), \pi(t, \mathrm{y})] = 0\,,$$

$$[\phi(t, \mathrm{x}), \pi(t, \mathrm{y})] = i\delta^{(3)}(\mathrm{x} - \mathrm{y})\, I\,, \qquad (10)$$

where I is the identity operator. Equations (10) are a kind of generalization of Eq. (5) to describe infinitely many degrees of freedom. Here ϕ and π act linearly in a Hilbert space $\mathcal{H}$, but they are not *bona fide* operators in $\mathcal{H}$. Technically they are *operator-valued distributions* — that is, they are linear maps from a *test function* space of smooth, real-valued functions on $\mathcal{M}$ to the self-adjoint operators on $\mathcal{H}$. For fixed t, we may interpret Eqs. (10) by choosing test functions f and g that depend on the spatial variable x only. Then (suppressing the time coordinate) we write formally the "smeared fields" $\phi(f) = \int_{\mathbb{R}^3} \phi(\mathrm{x}) f(\mathrm{x}) d^3x$ and $\pi(g) = \int_{\mathbb{R}^3} \pi(\mathrm{x}) g(\mathrm{x}) d^3x$, which are actual (unbounded) operators. From Eqs. (10), we easily obtain the fixed-time commutation relations

$$[\phi(f), \phi(g)] = [\pi(f), \pi(g)] = 0\,,$$

$$[\phi(f), \pi(g)] = i(f, g)\, I\,, \qquad (11)$$

where $(f, g) = \int_{\mathbb{R}^3} f(\mathrm{x}) g(\mathrm{x}) d^3x$ is the formula for the usual L^2 inner product of functions. Notice that the singular Dirac δ-function in (10) no longer appears. Equations (11) thus represent an infinite-dimensional Lie algebra, modeled on the test function space.

Let us exponentiate Eqs. (11), setting $U(af) = \exp[-ia\phi(f)]$ and $V(bg) = \exp[-ib\pi(g)]$. Using Eq. (9), we have that $V(g)\phi(f)V(g)^{-1} = \phi(f) - (f, g)\, I$, or $V(g)U(f)V(g)^{-1} = \exp[i(f, g)]\, U(f)$. We thus obtain the infinite-dimensional Heisenberg–Weyl group, likewise modeled on the test function space, whose elements are triples (α, f, g); where α is again a complex number of modulus 1, and f and g are test functions. The group law is now given by

$$(\alpha_1, f_1, g_1)(\alpha_2, f_2, g_2) = (\alpha_1 \alpha_2 \exp[i(f_2, g_1)],\, f_1 + f_2,\, g_1 + g_2)\,. \qquad (12)$$

This should be compared with Eq. (8), which defined the Heisenberg group as a 3-dimensional Lie group.

Equation (12) generalizes readily from the L^2-inner product to

$$(\alpha_1, f_1, g_1)(\alpha_2, f_2, g_2) = (\alpha_1 \alpha_2 \exp[iB(f_2, g_1)],\, f_1 + f_2,\, g_1 + g_2)\,, \qquad (13)$$

where B is a positive definite bilinear form on the space of test functions.

Let us close this subsection by mentioning an important extension of the theory of groups, whose importance developed from work by Drinfeld, Jimbo, Manin, and others in the 1980s — the study of so-named "quantum groups". A quantum group can be constructed from a mathematically natural deformation of the enveloping algebra of a simple Lie algebra by a real or complex parameter q. It can also be defined as a mathematical object in its own right, a Hopf algebra (or, possibly, a more general object, as the axiomatization of quantum groups is not really complete). Quantum groups describe a kind of *generalized* symmetry, where the notion of the inverse of an element is weakened. The word "quantum" here does not mean we have "quantized" a classical theory, since representations of ordinary non-commutative Lie groups and algebras already describe quantum-mechanical systems. But quantum groups have application in physics to conformal field theory, quantum inverse scattering, exactly solvable lattice models, exotic quantum statistics, and other domains.[17]

Later, when we discuss braid statistics, we shall have occasion to make use of q-deformed commutation relations, where the commutator $[A, B]$ of field operators is replaced by the q-commutator,

$$[A, B]_q = AB - qBA\,, \qquad q \in \mathbb{C}. \tag{14}$$

1.4. *Diffeomorphism Groups and Algebras of Vector Fields*

Next let us focus attention on a particular sort of infinite-dimensional group, the group of diffeomorphisms of a manifold. Let M and N be smooth, finite-dimensional Riemannian manifolds. A diffeomorphism is a (k-fold or infinitely) differentiable homeomorphism ϕ from M to N, whose inverse $\phi^{-1} : N \to M$ is likewise differentiable. For there to exist a diffeomorphism between M and N means that the two manifolds are, in the sense of differential geometry as well as topology, equivalent.

We now give attention to diffeomorphisms that map a manifold M to itself. Any two such diffeomorphisms ϕ_1 and ϕ_2, acting successively on M, give a third diffeomorphism $\phi_2 \circ \phi_1$ where $\circ$ denotes composition; *i.e.*, $[\phi_2 \circ \phi_1](\mathrm{x}) = \phi_2(\phi_1(\mathrm{x}))$. Since the operation is associative, since the identity map is automatically a diffeomorphism of M, and since the inverse of any diffeomorphism of M is again a diffeomorphism of M, we have a group under composition.

Here we have a choice as to convention. Suppose that a group G acts on a space M in a way that respects the group multiplication. For $g \in G$ and $\mathrm{x} \in M$, we write $(g, \mathrm{x}) \to L_g(\mathrm{x})$, and call the action a *left action*, when $(\forall g_1, g_2 \in G)\ L_{g_1 g_2} = L_{g_1} \circ L_{g_2}$. We call it a *right action* and write $(g, \mathrm{x}) \to R_g(\mathrm{x})$ when $(\forall g_1, g_2 \in G)\ R_{g_1 g_2} = R_{g_2} \circ R_{g_1}$. If the group product of ϕ_1 and ϕ_2 is defined to be simply $\phi_1 \circ \phi_2$, then the action of the diffeomorphism group on M becomes a left action.

But we shall shortly allow diffeomorphisms of M to act on the space

of C^∞ real-valued functions f defined on M, by moving the argument of f. To obtain a left action on the space of functions, we would then need to set $\phi \cdot f = f \circ \phi^{-1}$; this gives us $[\phi_1 \circ \phi_2] \cdot f = \phi_1 \cdot [\phi_2 \cdot f]$. Alternatively, we may define the product of two diffeomorphisms by setting

$$\phi_1 \phi_2 = \phi_2 \circ \phi_1\,, \tag{15}$$

so that the action of the diffeomorphism group on M is a *right* action. This is the convention we shall actually adopt throughout these lecture notes. Then we let $\phi f = f \circ \phi$ (without the inverse), so that we have a *left* action of the group on the function-space.

We also want the diffeomorphism group to be a well-behaved *topological* group, and this requires (in general) some additional restriction on the diffeomorphisms. Recall that the *support* of a real- or complex-valued continuous function f on a space M is the intersection of all closed sets $C \subseteq M$ such that for $\mathrm{x} \notin C$, $f(\mathrm{x}) \equiv 0$. Define then the support of a diffeomorphism ϕ of M to be the intersection of all closed sets $C \subseteq M$ such that for $\mathrm{x} \in M - C$, $\phi(\mathrm{x}) \equiv \mathrm{x}$. Note that if $f \in C^\infty(M)$ has compact support, then for $\phi \in \mathit{Diff}^{\,c}(M)$, $f \circ \phi$ also has compact support.

Now the set of C^∞ diffeomorphisms of M having compact (but arbitrary) support forms a group under composition. We call this group $\mathit{Diff}^{\,c}(M)$, where the superscript c stands for "compact". $\mathit{Diff}^{\,c}(M)$ becomes a topological group when it is endowed with the topology of uniform convergence in all derivatives in compact sets. Of course if the manifold M itself is compact, $\mathit{Diff}^{\,c}(M)$ is just the full group of C^∞ diffeomorphisms $\mathit{Diff}(M)$. It is an infinite-dimensional group, whose continuous unitary representations (CURs) are of great interest for both mathematics and physics.

Associated with $\mathit{Diff}^{\,c}(M)$ is the infinite-dimensional Lie algebra $\mathit{vect}^{\,c}(M)$, consisting of the C^∞ (tangent) vector fields on M having compact support (*i.e.*, vanishing outside some compact set), endowed with the Lie bracket (cf. Eq. (21) below). Let us consider the relation of the Lie algebra $\mathit{vect}^{\,c}(M)$ to the group $\mathit{Diff}^{\,c}(M)$, by exploring the exponentiation of vector fields.

Integral curves of vector fields

Suppose that $\mathrm{v}(\mathrm{x})$ is an arbitrary (not necessarily compactly supported) C^∞ vector field on M (or, more generally, some open region M_0 of M). Then v generates *integral curves* $\mathrm{x}(a)$, where a is a real parameter. That is, for each $\mathrm{x} \in M_0$, there is an interval $I_\mathrm{x} \subset \mathbb{R}$ containing $a = 0$, such that for $a \in I_\mathrm{x}$, the function $\mathrm{x}(a)$ solves the ordinary differential equation

$\partial_a \mathrm{x}(a) = \mathrm{v}(\mathrm{x}(a))$ combined with the initial condition $\mathrm{x}(0) = \mathrm{x}$. For each initial value $\mathrm{x} \in M_0$ and each parameter value $a \in I_\mathrm{x}$, define $\phi_a^\mathrm{v}(\mathrm{x}) = \mathrm{x}(a)$; so that in particular, $\phi_{a=0}^\mathrm{v}(\mathrm{x}) = \mathrm{x}$ $(\forall \mathrm{x} \in M_0)$.

Then, where it exists, $\phi_a^\mathrm{v}(\mathrm{x})$ is actually C^∞ in both x and a. Moreover, when a, a', $a + a' \in I_\mathrm{x}$, we have as expected the composition law

$$\phi_a^\mathrm{v}(\phi_{a'}^\mathrm{v}(\mathrm{x})) = \phi_{a+a'}^\mathrm{v}(\mathrm{x}). \tag{16}$$

Thus one may visualize a fluid filling the region M_0, with $\mathrm{v}(\mathrm{x})$ describing the magnitude and direction of the fluid's velocity at x (taken as if unchanging in time). Then $\mathrm{x}(a)$ is simply the trajectory of a mote of dust suspended in the fluid, carried along by the velocity field.

Furthermore, we have

$$\exp[\, a\, \mathrm{v}(\mathrm{x}) \cdot \nabla]\, f(\mathrm{x}) \;=\; f(\phi_a^\mathrm{v}(\mathrm{x}))\,, \tag{17}$$

as long as the infinite series expansion of the left-hand side is defined and convergent. However, even when $\mathrm{v}(\mathrm{x})$ is a C^∞ vector field defined on the whole manifold M, it may well be that I_x depends on x in such a way that there is *no* fixed interval of values for a on which $\phi_a^\mathrm{v}(\mathrm{x})$ exists for all $\mathrm{x} \in M$. That is, while an arbitrary smooth vector field on a noncompact manifold M can be exponentiated locally, it does not necessarily generate a one-parameter group of diffeomorphisms of M.[16]

Examples for $M = \mathbb{R}$

Let us look concretely at how these things may happen in the special case $M = \mathbb{R}$, where integral curves can be calculated explicitly.[18] A vector field on a domain in $\mathbb{R}$ is given by a smooth, real-valued function $g(x)$, since the tangent vectors are 1-dimensional. Suppose that $g(x)$ is C^∞ and has no zeroes in a certain open interval $\mathcal{I} = (x_1, x_2) \subset \mathbb{R}$ (we may allow $x_1 = -\infty$ and/or $x_2 = \infty$). For specificity, take $g(x) > 0$ on the interval. Fix $x_0 \in \mathcal{I}$, and define

$$G(x) = \int_{x_0}^{x} \frac{dx'}{g(x')}\,. \tag{18}$$

for $x \in \mathcal{I}$. Then the function $y = G(x)$ is C^∞ and strictly monotonic in x (increasing, when g is taken to be positive). We have $G(x_0) = 0$ and $G'(x) = 1/g(x)$, where "$\,'\,$" stands for the first derivative. Denote the inverse function by $x = G^{-1}(y)$. It is defined on the range of G, which contains the region about $y = 0$ bounded by $G(x_1)$ and $G(x_2)$ (which may possibly be $-\infty$ or ∞); with g positive on $\mathcal{I}$, we have $G(x_1) < y < G(x_2)$.

Since $G^{-1}[G(x)] = x$, we calculate $(G^{-1})'[G(x)]G'(x) = 1$, so that $(G^{-1})'(y) = g[G^{-1}(y)]$. Now let a be a real parameter. It is straightforward to verify that the function

$$\phi_a^g(x) = G^{-1}[G(x) + a] \tag{19}$$

gives the desired integral curves of g, where $\phi_a^g(x)$ is defined at least on the domain of values $\{(x,a) \,|\, x \in (x_1, x_2),\ G(x_1) - G(x) < a < G(x_2) - G(x)\}$. On this domain, we indeed have $\partial_a \phi_a^g(x) = g(\phi_a^g(x))$ with $\phi_{a=0}^g(x) = x$, as well as the composition law in Eq. (16). Notice too that the formula in Eq. (19) leads to an answer that is *independent* of the choice of x_0 used to define $G(x)$; in fact, replacing $G(x)$ by $G(x) + C$ (where C is a constant) means that $G^{-1}(y)$ is replaced by $G^{-1}(y - C)$, so that $\phi_a^g(x)$ is invariant with respect to the choice of C. Equations (18)–(19) are thus elementary, concrete formulas that allow us to calculate $\phi_a^g(x)$ explicitly.

Furthermore, we recover Eq. (17) by observing that under the change of variable $y = G(x)$, we have

$$\exp[\, a\, g(x) \frac{d}{dx}]\, f(x) = \exp[\, a \frac{d}{dy}][\, f \circ G^{-1}](y) = [\, f \circ G^{-1}](y + a)\,. \tag{20}$$

We also note the general possibility of *translating* the vector field; *i.e.*, of replacing $g(x)$ by $h(x) = g(x - b)$ and working on the translated interval $x \in \mathcal{I} + b$, for any fixed $b \in \mathbb{R}$. The consequence is that $G(x)$ is replaced by $H(x) = G(x - b)$ with $H(x_0 + b) = 0$; and $H^{-1}(y) = G^{-1}(y) + b$. Then the integral curves $\phi_a^h(x)$ for $x \in \mathcal{I} + b$ are given by $G^{-1}[G(x - b) + a] + b$, which again is identically equal to x when $a = 0$.

Let us apply our formulas to some special cases of vector fields. First suppose $g(x) = x^3$, which is certainly C^∞ on the whole real line, but which grows large when $|x| \to \infty$. When $x = 0$ we have $g(0) = 0$, so that $\phi_a^g(0) = 0$ $(\forall a)$. Considering the region $x > 0$, we have

$$G(x) = -\frac{1}{2x^2} + C\,, \quad G^{-1}(y) = \frac{1}{\sqrt{-2(y - C)}}\,,$$

where $C > 0$ is fixed; and using Eq. (19),

$$\phi_a^g(x) = [\,\frac{1}{x^2} - 2a\,]^{-1/2} = \frac{x}{\sqrt{1 - 2ax^2}}\,.$$

We see that for any given initial value of x greater than 0, $\phi_a^g(x)$ grows without bound as a increases, becoming infinite while a is still finite. The interval I_x of values of a for which $\phi_a^g(x)$ exists is bounded above by $1/(2x)^2$. For $x < 0$ we obtain the same formula, so that $\phi_a^g(x)$ decreases without bound as a increases toward $1/(2x)^2$. Despite the continuity of $g(x)$, its rate of growth means there is no fixed interval containing $a = 0$ on which $\phi^g_a(x)$ is defined for all $x \in \mathbb{R}$.

On the other hand the choice $g(x) = x$, which grows more slowly as $|x| \to \infty$, gives us straightforwardly the one-parameter group of dilations $\phi^g_a(x) = xe^a$, defined for all values of x and a.

As an example where the vector field vanishes at ∞, consider $g(x) = 1/\cosh x = 2/(e^x + e^{-x})$. Here $g(x)$ is C^∞ and as $|x| \to \infty$ tends toward 0 (together with all its derivatives) faster than the reciprocal of any polynomial in x. Then $G(x) = \sinh x + C$, and $\phi^g_a(x) = \sinh^{-1}[\sinh x + a]$, which for all values of x is defined for all $a \in \mathbb{R}$. Thus we have a one-parameter group of diffeomorphisms of $\mathbb{R}$. When x is very large and positive, $\sinh x \approx e^x/2$, so that $\sinh^{-1}(y) \approx \ln 2y$. Then $\phi^g_a(x) \approx \ln(e^x + 2a)$, which grows without bound as $a \to \infty$. The integral curve through any initial point eventually reaches any point to the right as a increases, albeit very slowly as the growth with a is logarithmic.

In contrast, if we choose $g(x)$ to be a C^∞ compactly-supported vector field, then the one-parameter groups $\phi^g_a(x)$ give us integral curves that are bounded above and below.

Finally, some interesting formulas for integral curves of vector fields result even when the latter are only partially defined on $\mathbb{R}$. With $g(x) = x^r$ $(r \neq 1)$, we have formally $G(x) = x^{1-r}/(1-r) + C$, and $G^{-1}(y) = [(1-r)(y-C)]^{1/(1-r)}$. Then $\phi^g_a(x) = [x^{1-r} + (1-r)a]^{1/(1-r)}$. Suppose that $r = 1/2$; then $g(x) = \sqrt{x}$, which is defined, positive, and C^∞ for $x > 0$. The growth as $|x| \to \infty$ is moderate, since $\sqrt{x} < x$ and we have already exponentiated the vector field $g(x) = x$. In fact, we have $\phi^g_a(x) = [\sqrt{x} + a/2]^2 = (1/4)[4x + 4a\sqrt{x} + a^2]$, which at first glance appears to be defined for $x > 0$ and for $-\infty < a < \infty$, and which is identically x when $a = 0$. But here the appearance is somewhat deceptive. Notice that although the vector field $g(x)$ vanishes at $x = 0$, $\phi^g_a(0)$ is only zero to first order in a; so that $x = 0$ is not behaving like a stationary point of the flow. This is related to the fact that $g(x)$ is not differentiable at $x = 0$. Notice further that if $a/2 < -\sqrt{x}$, the application of $\phi^g_{a'}$ to $\phi^g_a(x)$ does not respect Eq. (16). Even if we augment the definition of g by setting $g(x) = 0$ for $x < 0$, so that $\phi^g_a(x) \equiv x$ for $x < 0$, the vector field g does not exponentiate to a 1-parameter group of diffeomorphisms.

Recently Duchamp and Penson found some interesting uses of Eqs. (18)–(19) in studying the combinatorics of orthogonal polynomials. Their work has motivated my inclusion of the above examples in these lecture notes.[19]

Lie algebras of vector fields

Returning to the case of a general manifold M, a *compactly-supported* C^∞ vector field $\mathbf{v}$ on M always exponentiates to a one-parameter group of C^∞ diffeomorphisms of M. That is, $\phi^{\mathbf{v}}_a(\mathbf{x})$ is defined for all $a \in \mathbb{R}$, and $\phi^{\mathbf{v}}_b(\phi^{\mathbf{v}}_a(\mathbf{x}) = \phi^{\mathbf{v}}_{a+b}(\mathbf{x})$. Conversely, let $a \to \phi_a$ $(a \in \mathbb{R})$ be a one-parameter group of diffeomorphisms of M, smooth in a. Such a group defines a vector field $\mathbf{v}$ on M, whose value at $\mathbf{x} \in M$ is just the tangent vector to the parameterized curve $\phi_a(\mathbf{x})$ at $a = 0$. Thus we have $\partial_a \phi_a(\mathbf{x}) = \mathbf{v}(\phi_a(\mathbf{x}))$, with $\phi_{a=0}(\mathbf{x}) = \mathbf{x}$, and $\phi_a = \phi^{\mathbf{v}}_a$. We call $\phi^{\mathbf{v}}_a$ the *flow* generated by the vector field $\mathbf{v}$. If the flow $\phi^{\mathbf{v}}_a$ has support in a compact region K, then $\mathbf{v}$ evidently vanishes outside K. The space of all such vector fields, under pointwise addition and multiplication by real scalars, is of course infinite-dimensional; this is the Lie algebra $vect^c(M)$.

As suggested by Eq. (2), if $\mathbf{v}_1$ and $\mathbf{v}_2$ belong to $vect^c(M)$, their *Lie bracket* $[\mathbf{v}_1, \mathbf{v}_2]$ is the vector field that corresponds to the (infinitesimal) outcome of flowing (infinitesimally) by each of the two vector fields in succession, and then flowing backward (infinitesimally) by each of the two vector fields. In local coordinates,

$$[\mathbf{v}_1, \mathbf{v}_2](\mathbf{x}) = \mathbf{v}_1(\mathbf{x}) \cdot \nabla \mathbf{v}_2(\mathbf{x}) - \mathbf{v}_2(\mathbf{x}) \cdot \nabla \mathbf{v}_1(\mathbf{x}). \tag{21}$$

The Lie bracket of two C^∞, compactly-supported vector fields on M is again a C^∞, compactly-supported vector field. The Lie bracket satisfies the Jacobi identity (1), and defines the Lie algebra structure on $vect^c(M)$.

In general, a Lie group G acts on its own Lie algebra $\mathcal{G}$ by the *adjoint representation,* defined as follows. For $A, B \in \mathcal{G}$, the adjoint action of $\mathcal{G}$ on itself is given by $(ad\ A)B = [A, B]$; see Eq. (9). At the group level, we may obtain $Ad(g)B$ (writing Ad with the capital letter A) by exponentiating $(ad\ A)B$. Alternatively, with $L_g(h) = gh$ denoting left multiplication in G, and $R_g(h) = hg$ denoting right multiplication, we may define the adjoint action of G on itself, $Ad_g : G \to G$, by the composition $Ad_g = R_{g^{-1}} \circ L_g$; *i.e.*, conjugation by g. We then have $Ad_{g_1 g_2}(h) = (Ad_{g_1} \circ Ad_{g_2})(h)$, so that Ad_g (like L_g) is a left action on G. Letting B be the infinitesimal generator associated with a curve h_a in G passing through the identity, we can then obtain $Ad(g)B$ by differentiating $Ad(g)h_a$ with respect to a and evaluating at $a = 0$. Thus $Ad(g) : \mathcal{G} \to \mathcal{G}$, for $g \in G$. The adjoint representation on the Lie algebra is also a left action, satisfying $Ad(g_1)Ad(g_2)B = Ad(g_1 g_2)B$.

Having established our convention for the group multiplication law for diffeomorphisms, $\phi_1\phi_2 = \phi_2 \circ \phi_1$, the adjoint representation is given by $Ad(\phi_1)\,\phi_2 = \phi_1\phi_2\phi_1^{-1} = \phi_1^{-1} \circ \phi_2 \circ \phi_1$. It is then a straightforward calculation to obtain the adjoint representation on the space of vector fields in local coordinates; it is

$$[Ad(\phi)\,\mathrm{v}]^j(\mathrm{x}) \;=\; \frac{\partial(\phi^{-1})^j}{\partial x^k}\,(\phi(\mathrm{x}))\,v^{\,k}(\phi(\mathrm{x})), \tag{22}$$

where our convention is to sum over the repeated index k. From Eq. (22), it is easy to check that $(ad\,\mathrm{v}_1)\,\mathrm{v}_2 = [\mathrm{v}_1, \mathrm{v}_2]$, in accordance with Eq. (21).

We remark further that the set of compactly-supported C^k diffeomorphisms of M (for $k = 0, 1, 2, 3, \ldots$) whose inverses are likewise C^k also form a group under composition. But the Lie bracket in Eq. (21) involves taking a derivative, so that in general the bracket of a pair of C^k vector fields is only C^{k-1}. Thus the requirement that we have a Lie algebra naturally restricts us to the C^∞ vector fields, for which the group elements should be C^∞ diffeomorphisms.

1.5. *Semidirect Products and Other Extensions*

Let $\mathcal{D}(M)$ be the set of C^∞ real-valued functions on M having compact support. Then defining addition pointwise, $\mathcal{D}(M)$ is an Abelian group. Endowed with its usual topology of uniform convergence in all derivatives in compact sets, it is a topological group. A diffeomorphism $\phi \in \mathit{Diff}^{\,c}(M)$ acts naturally on $\mathcal{D}(M)$ by transforming the argument of each function; *i.e.*, for $f \in \mathcal{D}(M)$, $\phi : f \to f \circ \phi$. Furthermore the map $(f, \phi) \to f \circ \phi$ is jointly continuous in f and ϕ. Then we have the natural semidirect product group $\mathcal{D}(M) \times \mathit{Diff}^{\,c}(M)$, with the semidirect product group law given by

$$(f_1, \phi_1)(f_2, \phi_2) = (f_1 + \phi_1 f_2\,,\; \phi_1\phi_2)\,; \tag{23}$$

where as noted above, $\phi_1 f_2 = f_2 \circ \phi_1$ and $\phi_1\phi_2 = \phi_2 \circ \phi_1$. At the level of the Lie algebra, we have a *semidirect sum* of the commutative Lie algebra of compactly-supported scalar functions on M with the Lie algebra $vect^c(M)$.

But $\mathcal{D}(M)$ is just the subgroup of $Map(M, \mathbb{R})$ consisting of the C^∞ compactly supported maps (regarding $\mathbb{R}$ as an additive Lie group). Thus it is useful to introduce more generally the subgroup $Map^c(M, G)$, of smooth maps that equal the identity element in G outside compact sets in M. A compactly-supported diffeomorphism ϕ of M then acts naturally on the compactly-supported maps $g : M \to G$ by $(\phi\, g)(\mathrm{x}) = g\,(\phi(\mathrm{x}))$, respecting the pointwise group operations. We have the natural semidirect product $Map^c(M, G) \times \mathit{Diff}^{\,c}(M)$ for a general Lie group G, with the group law $(g_1, \phi_1)(g_2, \phi_2) = (\,g_1(\phi_1\, g_2), \phi_1\phi_2)$, and the corresponding semidirect sum of the infinite-dimensional Lie algebras $map^c(M, \mathcal{G})$ and $vect^c(M)$.

Let us now consider the case $M = \mathbb{R}^d$, corresponding to d-dimensional physical space. Here an important subgroup of $\mathit{Diff}^{\,c}(\mathbb{R}^d)$ is the group of area- or volume-preserving diffeomorphisms $\mathit{SDiff}^{\,c}(\mathbb{R}^d)$, $d > 1$, where the prefix letter 'S' stands for "special". When $d = 2$, this subgroup coincides with the group of compactly supported symplectic diffeomorphisms of the plane. When $d = 1$, however, the group is trivial. The corresponding Lie subalgebra is $svect^c(\mathbb{R}^d)$, which is the algebra of *divergenceless* compactly-supported vector fields. Unitary representations of the group $\mathit{SDiff}^{\,c}(\mathbb{R}^d)$ and the algebra $svect^c(\mathbb{R}^d)$ are important to the quantum theory of an ideal, incompressible fluid in $\mathbb{R}^d$, $d > 1$.

The condition that diffeomorphisms be compactly supported can be weakened in various ways in $\mathbb{R}^d$, modifying the group topology appropriately while maintaining the correspondence between the resulting diffeomorphism group and a Lie algebra of C^∞ vector fields on $\mathbb{R}^d$ that generate global flows. For example, one possibility is to include diffeomorphisms that, in the limit as $|\mathrm{x}| \to \infty$, approach the identity map rapidly in all derivatives (here *rapidly* means faster than any polynomial). This group can be given the topology of uniform rapid convergence in all derivatives, and has been called $\mathcal{K}(\mathbb{R}^d)$. The natural corresponding Lie algebra consists of vector fields with components belonging to Schwartz' space $\mathcal{S}(\mathbb{R}^d)$, the space of real-valued C^∞ functions of rapid decrease in all derivatives, a property respected by the Lie bracket of Eq. (21). We saw an example in the preceding subsection in the vector field on $\mathbb{R}$ defined by $g(x) = 1/\cosh x$. In place of the semidirect product group $\mathcal{D}(\mathbb{R}^d) \times \mathit{Diff}^{\,c}(\mathbb{R}^d)$ we then have a semidirect product $\mathcal{S}(\mathbb{R}^d) \times \mathcal{K}(\mathbb{R}^d)$, whose elements satisfy Eq. (23).

Consider as a further alternative all C^∞ diffeomorphisms of $\mathbb{R}^d$ that coincide with some (uniform) translation outside of an arbitrary compact region $K \in \mathbb{R}^d$. These form a group that we may call $\mathit{Diff}^{\,\mathrm{trans}}(\mathbb{R}^d)$. We can obtain any such diffeomorphism by composing an element of $\mathit{Diff}^{\,c}(\mathbb{R}^d)$ with an element T of the translation group $\mathcal{T}(\mathbb{R}^d)$. Note further that there is a natural homomorphism from $\mathcal{T}(\mathbb{R}^d)$ to the group of automorphisms of $\mathit{Diff}^{\,c}(\mathbb{R}^d)$: for each translation T, we have the automorphism $\phi \to T \circ \phi \circ T^{-1}$. This lets us write $\mathit{Diff}^{\,\mathrm{trans}}(\mathbb{R}^d)$ as a semidirect product $\mathcal{T}(\mathbb{R}^d) \times \mathit{Diff}^{\,c}(\mathbb{R}^d)$. We can enlarge this group as well, to include diffeomorphisms which, in the limit as $|\mathrm{x}| \to \infty$, approach a translation rapidly in all derivatives.

Similarly we may define groups of diffeomorphisms that coincide with (outside compact sets), or rapidly approach (in the limit as $|\mathrm{x}| \to \infty$), the following: a rotation or a Euclidean transformation (for $d > 1$), a dilation, or a linear or affine transformation. When we work with the area- or volume-preserving diffeomorphisms, it is natural to extend them by Euclidean, special linear or special affine transformations.

For each such extension of the diffeomorphism group we have a corresponding infinite-dimensional Lie algebra of vector fields on $\mathbb{R}^d$, where the vector fields coincide with (outside compact regions) or rapidly approach (as $|\mathbf{x}| \to \infty$), the infinitesimal generators of a finite-dimensional Lie group acting *globally* on $\mathbb{R}^d$.

An important special case occurs when we consider the Lie algebras of vector fields on the line $\mathbb{R}^1$ or on the circle S^1, and the corresponding diffeomorphism groups. In this situation of a one-dimensional manifold there is a natural, nontrivial one-dimensional extension of the Lie algebra called the *Virasoro algebra* and, correspondingly, we have the Virasoro group. For the example of the circle, it is natural to parameterize the manifold by $0 \leq \theta < 2\pi$, and to choose a basis of vector fields $g_{(n)}(\theta) = i\exp(in\theta)$, $n = 0, \pm 1, \pm 2, \ldots$ for the (complexified) Lie algebra. Then Eq. (21) becomes $[\, g_{(m)}, g_{(n)} \,] = (m-n) g_{(m+n)}$. Adjoining to the Lie algebra a central element I (that commutes with all the $g_{(n)}$), the extended bracket is given by the formula

$$[\, g_{(m)}, g_{(n)}] = (m-n) g_{(m+n)} + c\, \frac{(n+1)n(n-1)}{12}\, \delta_{m,-n}\, I\,, \tag{24}$$

where the coefficient c is called the *central charge*. It is straightforward to verify that Eq. (24) satisfies the Jacobi identity.

The Virasoro algebra and group are the natural analogues for $\mathit{Diff}(S^1)$ of the affine Kac–Moody algebras and groups for $\mathit{Map}(S^1, G)$. Its representations have important application to quantum field theories in $(1+1)$-dimensional space-time, to exactly solvable models in statistical mechanics, and to many other domains.

In the next section, we introduce some basic ideas from quantum field theory. This permits us to see how representations of algebras of vector fields (and, correspondingly, groups of diffeomorphisms) occur naturally within such a theory, representing local currents.

2. Local Quantum Fields and Fock Space

A profound idea that deeply influenced the development of particle physics is the notion that *fields* rather than particles are the fundamental physical quantities. The particles that we observe in nature are then actually *quanta* of fields. For instance we understand photons to be quanta of the electromagnetic field, heavy vector bosons to be quanta of the weak field, and gluons to be quanta of the field that binds quarks into baryons and mesons and accounts for the strong interactions.

Equation (10) can be regarded as a way to quantize a classical relativistic neutral scalar field $\phi(t, \mathbf{x})$, describing theoretically a neutral scalar boson with mass. After writing the commutation relations for the fields, the challenge is to represent them by self-adjoint operators in Hilbert space, to write other operators such as the Hamiltonian (describing interactions) in terms of the field operators, and to deduce the particle interpretation. This program is essentially complete for free (*i.e.*, noninteracting) quantum fields, while major challenges have been overcome and others remain for interacting theories.[20,21,22]

Let us discuss first some nonrelativistic quantum field theory.

2.1. *Canonical Nonrelativistic Fields*

Consider the simple harmonic oscillator Hamiltonian in quantum mechanics given by $H = \hat{p}^2/2m + (k/2)\,\hat{q}^2$. Using Eqs. (6), the action of H may be represented by $H\Psi(q) = -(\hbar^2/2m)\, d^2\Psi(q)/\, dq^2 + (k/2)\, q^2\Psi(q)$, where m is the particle mass. We recall from elementary quantum mechanics that the solutions to the time-independent Schrödinger equation $H\Psi_n(q) = E_n\Psi_n(q)$ are Hermite functions, with energy levels $E_n = (\,n + \frac{1}{2}\,)\,\hbar\omega$, $n = 0, 1, 2, \ldots$, where $\omega = \sqrt{k/m}$. Appropriately normalized, these give us an orthonormal basis for the Hilbert space $L^2_{dq}(\mathbb{R})$, so that $(\Psi_m, \Psi_n) = \delta_{mn}$. Defining the raising and lowering operators in the usual way, $a^* = (2m\hbar\omega)^{-1/2}\,[\,m\omega\hat{q} - i\hat{p}\,]$ and $a = (2m\hbar\omega)^{-1/2}\,[\,m\omega\hat{q} + i\hat{p}\,]$, we find easily from the Heisenberg bracket of Eq. (5) that a and a^* obey the commutation relation

$$[\,a, a^*]_- = a\,a^* - a^*a = I\,, \tag{25}$$

while

$$H = (a^*a + \frac{1}{2})\,\hbar\omega. \tag{26}$$

Thus $a^*a\,\Psi_n = n\Psi_n$. It is straightforward to see that $a\,\Psi_n = n^{1/2}\Psi_{n-1}$, with $a\,\Psi_0 = 0$; while $a^*\,\Psi_n = (n+1)^{1/2}\Psi_{n+1}$. Thus we have a representation of Eq. (25) by linear (unbounded) operators in $L^2_{dq}(\mathbb{R})$.

Such a representation has another possible interpretation. Instead of thinking about the energy levels of the oscillator, we can think of n as describing the number of Bose particles in a given quantum state (the occupation number). Then Ψ_0, the lowest energy state, is the vacuum; a is the particle annihilation operator, a^* is the particle creation operator, and a^*a is the number operator. There is of course no limit to the number of bosons that can occupy the same quantum state, so the number operator is unbounded (as are a and a^*).

If we want to describe Fermi particles that obey the Pauli exclusion principle, however, we must restrict the permitted occupation numbers to be only 0 or 1. Such a system is obtained by replacing the commutation relation in Eq. (25) by the *anticommutation* relation.

$$[a, a^*]_+ = a a^* + a^* a = I, \tag{27}$$

where again the number operator is a^*a. Now a representation is given by $a\Psi_0 = 0$, $a\Psi_1 = \Psi_0$, $a^*\Psi_0 = \Psi_1$, and $a^*\Psi_1 = 0$.

To describe Bose or Fermi quantum particles occupying a family of distinct states indexed by the subscript α, with occupation numbers n_α, we can write

$$[a_\alpha, a_\beta]_\pm = [a^*_\alpha, a^*_\beta]_\pm = 0,$$

$$[a_\alpha, a^*_\beta]_\pm = \delta_{\alpha\beta} I. \tag{28}$$

The number operator with eigenvalues n_α is then $a^*_\alpha a_\alpha$.

In nonrelativistic quantum field theory, we posit the field operator $\psi(t, \mathrm{x})$ and its adjoint $\psi^*(t, \mathrm{x})$, obeying fixed-time canonical commutation (-) or anticommutation (+) relations, given by (suppressing t),

$$[\psi(\mathrm{x}), \psi(\mathrm{y})]_\pm = [\psi^*(\mathrm{x}), \psi^*(\mathrm{y})]_\pm = 0,$$

$$[\psi(\mathrm{x}), \psi^*(\mathrm{y})]_\pm = \delta^{(3)}(\mathrm{x} - \mathrm{y})\, I. \tag{29}$$

These equations are sometimes interpreted as a *second quantization* of the Schrödinger wave function ψ. Notice how they may be regarded as generalizations of Eqs. (28), with the discrete index α replaced by the continuous spatial coordinate x.

One representation of Eqs. (29) is the *Fock representation* or particle-number representation, which we introduce using positional coordinates. (We disregard here the possibility of particle spin.) Let us define the N-particle Hilbert space $\mathcal{H}_N$, $N = 0, 1, 2, \ldots$, as follows. For $N = 0$, we have a one-dimensional Hilbert space $\mathcal{H}_0 = \mathbb{C}$, which we interpret as the ray corresponding to the vacuum state. For $N \geq 1$, $\mathcal{H}_N$ consists of complex-valued wave functions Ψ_N which are square-integrable functions of $\mathrm{x}_1, \ldots \mathrm{x}_N$, with

$\mathrm{x}_j \in \mathbb{R}^3$. We have the standard L^2 inner product $(\ ,\ )_N$ in $\mathcal{H}_N$, given in $\mathcal{H}_0$ by $\overline{\Psi}_0\Psi_0$, and for $N \geq 1$ by

$$(\Phi_N, \Psi_N)_N = \int_{\mathbb{R}^{3N}} \overline{\Phi_N(\mathrm{x}_1, \ldots, \mathrm{x}_N)}\,\Psi_N(\mathrm{x}_1, \ldots, \mathrm{x}_N)\, d^3x_1 \cdots d^3x_N. \tag{30}$$

As usual in quantum mechanics, we are often interested in wave functions that satisfy specified exchange statistics. We may let the symmetric group S_N act on $\mathcal{H}_N$ by permuting the N indices labeling particle coordinates (in Sec. 4.2 we shall discuss the action of S_N in greater depth). For $\sigma \in S_N$, set $\sigma : (1, \ldots, N) \to (\sigma[1], \ldots \sigma[N])$. Let $\zeta(\sigma) = 1$ if σ is an even permutation, and $\zeta(\sigma) = -1$ if σ is odd. The Hilbert space $\mathcal{H}_N^{(s)}$ consists of wave functions $\Psi_N^{(s)}$ that are *symmetric* under exchange of particle coordinates; *i.e.*, they obey the condition

$$\Psi_N^{(s)}(\mathrm{x}_1, \ldots, \mathrm{x}_N) = \Psi_N^{(s)}(\mathrm{x}_{\sigma[1]}, \ldots, \mathrm{x}_{\sigma[N]}) \tag{31}$$

for all $\sigma \in S_N$. For notational convenience, we take $\mathcal{H}_0^{(s)} = \mathcal{H}_0$ and $\mathcal{H}_1^{(s)} = \mathcal{H}_1$. As usual, we can obtain $\mathcal{H}_N^{(s)}$ as the *symmetric tensor product* of N copies of the 1-particle Hilbert space: $\mathcal{H}_N^{(s)} = \mathcal{H}_1^{\otimes_s N}$. Alternatively, the Hilbert space $\mathcal{H}_N^{(a)}$ consists of wave functions $\Psi_N^{(a)}$ that are *antisymmetric* under coordinate exchange, so that

$$\Psi_N^{(a)}(\mathrm{x}_1, \ldots, \mathrm{x}_N) = (-1)^{\zeta(\sigma)}\Psi_N^{(a)}(\mathrm{x}_{\sigma[1]}, \ldots, \mathrm{x}_{\sigma[N]}) \tag{32}$$

for all $\sigma \in S_N$, and we write $\mathcal{H}_N^{(a)} = \mathcal{H}_1^{\otimes_a N}$.

The Fock Hilbert space is then the infinite direct sum $\mathcal{H} = \bigoplus_{N=0}^{\infty} \mathcal{H}_N$, or in the fixed-symmetry cases, $\mathcal{H}^{(s),(a)} = \bigoplus_{N=0}^{\infty} \mathcal{H}_N^{(s),(a)}$. That is, we can identify a vector $\Psi \in \mathcal{H}$ with an infinite sequence $\Psi = (\Psi_N)$, $N = 0, 1, 2, \ldots$, such that the infinite series $\sum_{N=0}^{\infty}(\Psi_N, \Psi_N)_N$ converges; and likewise for $\Psi^{(s),(a)} \in H^{(s),(a)}$. The inner product in $\mathcal{H}$ is given by

$$(\Phi, \Psi) = \sum_{N=0}^{\infty} (\Phi_N, \Psi_N)_N < \infty. \tag{33}$$

and similarly for the inner product in $\mathcal{H}^{(s),(a)}$.

Now we are ready to write representations of the fields satisfying Eqs. (29), acting in the appropriate Fock spaces. We follow (with small modifications) the notation in Schweber's book.[20]

Fields satisfying fixed-time canonical commutation relations are represented in $\mathcal{H}^{(s)}$ by:

$$
\begin{aligned}
[\psi(\mathbf{x})\Psi^{(s)}]_N(\mathbf{x}_1,\ldots\mathbf{x}_N) &= (N+1)^{1/2}\,\Psi^{(s)}_{N+1}(\mathbf{x}_1,\ldots,\mathbf{x}_N,\mathbf{x}),\\
[\psi^*(\mathbf{x})\Psi^{(s)}]_N(\mathbf{x}_1,\ldots\mathbf{x}_N) &= \\
&= N^{-1/2}\sum_{j=1}^{N}\delta^{(3)}(\mathbf{x}-\mathbf{x}_j)\,\Psi^{(s)}_{N-1}(\mathbf{x}_1,\ldots,\hat{\mathbf{x}}_j,\ldots,\mathbf{x}_N), \quad (34)
\end{aligned}
$$

where the notation $\hat{\mathbf{x}}_j$ means that the particular triple of coordinates $\mathbf{x}_j$ is omitted, and where $[\psi^*(\mathbf{x})\Psi^{(s)}]_0 = 0$. Note that because of the Dirac δ-functions, we know immediately that these expressions define not operators but operator-valued distributions — like the relativistic fields in Eq. (10), they must be interpreted as mapping test functions to actual linear operators in the Hilbert space. Thus if $f_1(\mathbf{x})$ and $f_2(\mathbf{x})$ belong to $\mathcal{D}(\mathbb{R}^3)$ or $\mathcal{S}(\mathbb{R}^3)$ we write, just as we did in obtaining Eq. (11), $\psi(f_1) = \int_{\mathbb{R}^3}\psi(\mathbf{x})f_1(\mathbf{x})d^3x$ and $\psi^*(f_2) = \int_{\mathbb{R}^3}\psi^*(\mathbf{x})f_2(\mathbf{x})d^3x$. This gives us the Lie algebra of canonical nonrelativistic fields modeled on test-function space,

$$
[\psi(f_1),\psi^*(f_2)] = (f_1,f_2)\,I, \tag{35}
$$

where (f_1,f_2) is the L^2 inner product formula applied to the test functions. We often speak loosely of field "operators" rather than the more technically correct operator-valued distributions.

We can see that $\psi(\mathbf{x})$ is an annihilation operator, and $\psi^*(\mathbf{x})$ is a creation operator. If the initial vector $\Psi^{(s)}$ is, for example, a one-particle state $(0,\Psi_1^{(s)}(\mathbf{x}_1),0,0,0,\ldots)$, then $\psi(\mathbf{x})\Psi^{(s)}$ is just the zero-particle state $(\Psi_1^{(s)}(\mathbf{x}),0,0,0,0,\ldots)$ while $\psi^*(\mathbf{x})\Psi^{(s)}$ becomes the two-particle state $(0,0,\delta(\mathbf{x}-\mathbf{x}_1)\Psi_1^{(s)}(\mathbf{x}_2)+\delta(\mathbf{x}-\mathbf{x}_2)\Psi_1^{(s)}(\mathbf{x}_1),0,0,\ldots)$.

The smeared versions of Eqs. (34) are easily obtained,

$$
\begin{aligned}
[\psi(f)\Psi^{(s)}]_N(\mathbf{x}_1,\ldots\mathbf{x}_N) &= \\
&= (N+1)^{1/2}\int_{\mathbb{R}^3}\Psi^{(s)}_{N+1}(\mathbf{x}_1,\ldots,\mathbf{x}_N,\mathbf{x})\,f(\mathbf{x})d^3x,\\
[\psi^*(f)\Psi^{(s)}]_N(\mathbf{x}_1,\ldots\mathbf{x}_N) &= \\
&= N^{-1/2}\sum_{j=1}^{N}f(\mathbf{x}_j)\,\Psi^{(s)}_{N-1}(\mathbf{x}_1,\ldots,\hat{\mathbf{x}}_j,\ldots,\mathbf{x}_N), \quad (36)
\end{aligned}
$$

again with $[\psi^*(f)\Psi^{(s)}]_0 = 0$ as befits the interpretation of $\psi^*(f)$ as a creation field.

In the case of the canonical anticommutation relations, we have a representation in $\mathcal{H}^{(a)}$ given by:

$$
\begin{aligned}
[\psi(\mathbf{x})\Psi^{(a)}]_N(\mathbf{x}_1,\ldots\mathbf{x}_N) &= (N+1)^{1/2}\,\Psi^{(a)}_{N+1}(\mathbf{x}_1,\ldots,\mathbf{x}_N,\mathbf{x}),\\
[\psi^*(\mathbf{x})\Psi^{(a)}]_N(\mathbf{x}_1,\ldots\mathbf{x}_N) &= \\
&= (-1)^{N+1}N^{-1/2}\sum_{j=1}^{N}(-1)^{j+1}\,\delta^{(3)}(\mathbf{x}-\mathbf{x}_j)\,\Psi^{(a)}_{N-1}(\mathbf{x}_1,\ldots,\hat{\mathbf{x}}_j,\ldots,\mathbf{x}_N), \quad (37)
\end{aligned}
$$

with $[\psi^*(\mathbf{x})\Psi^{(a)}]_0 = 0$, and with of course a corresponding representation of the smeared fields.

A number operator in these representations may be written $N_{\rm op} = \int_{\mathbb{R}^3}\psi^*(\mathbf{x})\psi(\mathbf{x})\,d^3x$. The eigenfunctions of $N_{\rm op}$ have as eigenvalues the particle number N. The order in which ψ^* and ψ are written is very important in the definition of $N_{\rm op}$. But let us remark that because $\psi^*(\mathbf{x})$ and $\psi(\mathbf{x})$ are operator-valued distributions, there is no *a priori* general definition for their product at a point $\mathbf{x}$. Indeed there are well-known difficulties with interpreting pointwise products of field operators, or equal-time commutation relations of currents constructed from such pointwise products. Special techniques such as taking "normal ordered" products, or splitting the points in space-time, are needed in relativistic theories to make sense of the products or to correctly calculate commutation relations in a representation. Without such techniques, making only formal calculations, pointwise products of field operators typically come out to be infinite, while equal-time current commutators that *cannot* vanish (such as commutators of time- with space-components of local, covariant currents) are nevertheless found to be zero. The missing terms (which are restored with more careful procedures) are sometimes called *Schwinger terms.*

The interpretation of the expression for $N_{\rm op}$ thus needs to be checked carefully. Here, in the nonrelativistic context, it turns out there is no difficulty. We have just $N_{\rm op}(0,\ldots,0,\Psi_N,0,\ldots) = (0,\ldots,0,N\Psi_N,0,\ldots)$, $(\forall\Psi_N \in \mathcal{H}_N^{(s),(a)})$. Since each $\mathcal{H}_N^{(s),(a)}$, $N = 1,2,3,\ldots$, can be identified with a particular subspace of $\mathcal{H}^{(s),(a)}$ invariant under $N_{\rm op}$, we write for short $N_{\rm op}\Psi_N = N\Psi_N$.

Of course, we have constructed the Fock space so as to make explicit the particle number content; and we have written the field operators that explicitly create and annihilate particles. Nevertheless, let us reiterate our point of view that the quantized fields are the more fundamental entities. The particle number interpretation is viewed as a *consequence of the repre-*

sentation of the fields, with the particles themselves occurring as the *quanta* of the fields.

Let us make a further philosophical comment about the quantum-field-theoretic approach to physics. Many (though not all) physicists have held the intuition that the fundamental physical entities — *i.e.*, those that "really exist" as distinct from those that are merely auxiliary constructs — are *local* in space-time (with any apparent "action at a distance" being a consequence of local interactions). In classical physics, one imagines these could consist of *finitely many* quantities describing matter (such as the mass density, the charge density, and so on) and the dynamical state of the matter (such as the momentum density, the electric current density, and so on). As the description of electrodynamics by means of field strengths proved so powerful, we could imagine the fundamental local entities to consist of a finite set of observable field strengths, depending only on space-time points. Then quantities such as the mass density would be derivable (locally) from the gravitational field strength, and the charge density and the electric current density would be obtainable (locally) from the electric and magnetic field strengths. The dynamical equations (*i.e.*, the equations of motion) would then form a system of coupled partial differential equations, first-order in time, telling us the classical time-evolution of such a fundamental set of local, physical fields — a "classical theory of everything".

But we have learned that classical physics fails as subatomic phenomena are taken into account. In the Schrödinger description of quantum physics, the (complex-valued) wave function is an *essentially nonlocal* construct. It is *not* a function on physical space-time; rather, it is a time-dependent function defined on the *configuration-space* of a multiparticle system. Thus a quantum state described by the wave function is nonlocal in the special sense that it can encode information descriptive of *quantum correlations* among spatially separated particles.

To recover a picture in which entities defined *locally* in space-time are fundamental, the *quantum field operator* is introduced. The latter is indexed again by the coordinates $\mathbf{x}$ and t. But as an operator-valued distribution in an infinite-dimensional Hilbert space, the quantum field necessarily encodes *infinitely many* degrees of freedom. From these field operators, then, we hope to be able to construct *algebras of local observables* modeled on localized regions of space-time.

We then expect the *local causality* property of the observables to express itself through the property of *local commutativity*. That is, when two regions $\mathcal{O}_1$ and $\mathcal{O}_2$ of space-time are causally separated (*e.g.*, because one would need to travel faster than light to reach a point in $\mathcal{O}_1$ from a point in $\mathcal{O}_2$, and conversely), all the local observables associated with measurement in $\mathcal{O}_1$ commute with those associated with measurement in $\mathcal{O}_2$. In the nonrelativistic case, we can anticipate fixed-time observables associated with regions of physical space. Galilean local causality then takes the form of requiring that when two spatial regions do not intersect, the equal-time commutation relations between observables associated with measurement in the respective regions are zero.

2.2. *Local Current Algebras*

Neither the field operators $\psi(f)$ and $\psi^*(f)$, nor self-adjoint linear combinations of them, actually represent physical observables directly. For one thing, we always have the possibility of replacing ψ and ψ^* by $\exp(i\theta)\psi$ and $\exp(-i\theta)\psi^*$ respectively, where θ is a fixed real parameter, without changing the physics. This corresponds to a *gauge transformation of the first kind* in quantum mechanics, so that the field operators are gauge dependent. Perhaps more importantly, we do not expect to actually observe changes in the particle number in nonrelativistic physics, while the operators $\psi(f)$ and $\psi^*(f)$ implement annihilation and creation respectively.

These are initial motivations for the introduction by Dashen and Sharp of local density and current operators as descriptive of local, nonrelativistic quantum observables.[23] Another motivation refers back to the early success of current algebra in describing features of the electroweak interactions of relativistic hadrons — strongly interacting particles.[24,25] Let us discuss this relativistic background first, and then elaborate on the nonrelativistic current algebra.

Relativistic local current algebra

The famous "eightfold way" associated various families of hadrons (including octets of both baryons and mesons) with irreducible unitary representations of $SU(3)$, is the Lie group which is an approximate symmetry of the strong interactions.[26] Various parts of the weak and electromagnetic currents can be combined into an eight-component "vector octet" of currents $F^a_\mu(x)$ $(a = 1,2,\ldots,8)$, where $\mu = 0,1,2,3$ is a Lorentz index [as usual, we use here the 4-vector notation $x = (x^0, \mathbf{x})$, with $\mathbf{x} = (x^1, x^2, x^3)$]. The axial vector parts of the weak currents likewise form an "axial vector octet" $F^{5\,a}_\mu(x)$. Gell-Mann hypothesized the time components F_0 and F^5_0 of the vector and axial vector octet to satisfy equal-time commutation relations,[27] using the structure constants of f^{abd} of the Lie algebra $su(3)$ of $SU(3)$. More specifically,

$$[F_0^a(x^0,\mathrm{x}),\, F_0^b(y^0,\mathrm{y})]_{x^0=y^0} = i\delta^{(3)}(\mathrm{x}-\mathrm{y})\, f^{abd} F_0^d(x^0,\mathrm{x}),$$

$$[F_0^a(x^0,\mathrm{x}),\, F_0^{5\,b}(y^0,\mathrm{y})]_{x^0=y^0} = i\delta^{(3)}(\mathrm{x}-\mathrm{y})\, f^{abd} F_0^{5\,d}(x^0,\mathrm{x}),$$

$$[F_0^{5\,a}(x^0,\mathrm{x}),\, F_0^{5\,b}(y^0,\mathrm{y})]_{x^0=y^0} = i\delta^{(3)}(\mathrm{x}-\mathrm{y})\, f^{abd} F_0^d(x^0,\mathrm{x}). \qquad (38)$$

When the F_0^a are integrated with test functions $f^a(\mathrm{x})$ on $\mathbb{R}^3$ (for fixed x^0), and the $F_0^{5\,a}$ with test functions $g^a(\mathrm{x})$, Eqs. (38) represent an infinite-dimensional Lie algebra of the type $map(\mathbb{R}^3, \mathcal{G})$; where the mappings are given by $\mathrm{x} \to \Sigma_a [\, f^a(\mathrm{x})\, Q^a + g^a(\mathrm{x})\, Q^{5\,a}\,]$; here Q^a and $Q^{5\,a}$ are *charges* that belong to the finite-dimensional Lie algebra $\mathcal{G}$. Integrating out the spatial variables entirely thus leads to the equal-time algebra $\mathcal{G}$ of charges Q at x^0, used (for example) in obtaining the Adler–Weisberger relation.[28]

It is natural, then, to try to extend the idea from Eqs. (38) to a Lie algebra that would also include the *spatial* components F_k^a of local currents at equal times. Further, since Eqs. (38) are not explicitly dependent on how the currents might be constructed from underlying canonical fields, one can imagine the possibility of expressing the Hamiltonian operator directly in terms of such local currents, bypassing the field operators entirely.

In 1+1 dimensions, we have available the Kac–Moody and Virasoro algebras, where the (finite) central extension plays the role of a Schwinger term. But difficulties occur with this idea in relativistic models in Minkowskian space-times of dimension higher than $1+1$. Here the Schwinger terms for local currents that are defined from canonical fields are typically infinite, suggesting that equal-time current algebras — if they can be used at all — need to be written down independently of any underlying fields.

In $d+1$ dimensions, $d \geq 1$, relativistic models with finite central or non-central Schwinger terms were proposed by Sugawara and by others.[29,30,31,32] The Sugawara model, which turned out to be perhaps the most influential of those proposed in the late 1960s and early 1970s, is based on the following infinite-dimensional Lie algebra: at the fixed time $x^0 = y^0$:

$$[\, J_0^a(\mathrm{x}),\, J_0^b(\mathrm{y}) = i\,\delta^{(3)}(\mathrm{x}-\mathrm{y})\, f^{abd}\, J_0^d(\mathrm{x}),$$

$$[\, J_0^a(\mathrm{x}),\, J_k^b(\mathrm{y})\,] = i\,\delta^{(3)}(\mathrm{x}-\mathrm{y})\, f^{abd}\, J_k^d(\mathrm{x}) + ic\,\delta^{ab}\,\frac{\partial}{\partial x^k}\,\delta^{(3)}(\mathrm{x}-\mathrm{y}),$$

$$[\, J_k^a(\mathrm{x}),\, J_\ell^b(\mathrm{y}) = 0, \qquad (39)$$

where $J_\mu^a = (J_0^a,\, J_k^a)$, $k = 1,2,3$ is again a 4-vector, the f^{abd} are the structure constants of the Lie algebra for an arbitrary compact internal symmetry group G, and c is a constant. The Sugawara Hamiltonian is given in terms of the local currents by the formal expression,

$$H = \int_{\mathbb{R}^3} \frac{1}{2c} \sum_a [\, J_0^a(\mathrm{x})^2 + \sum_{k=1}^{3} J_k^a(\mathrm{x})^2\,]\, d^3x. \qquad (40)$$

An excellent discussion of Kac–Moody and Virasoro algebras, as well as the Sugawara model, is provided by Goddard and Olive.[33]

Nonrelativistic local current algebra

The problem of Schwinger terms does not arise for the nonrelativistic current algebra defined in terms of the canonical fields satisfying Eqs. (29). Let us define the mass density operator $\rho(\mathrm{x})$ and the momentum density operator $\mathbf{J}(\mathrm{x})$ at fixed time t by

$$\rho(\mathrm{x}) = m\psi^*(\mathrm{x})\psi(\mathrm{x}),$$

$$\mathbf{J}(\mathrm{x}) = \frac{\hbar}{2i}\{\psi^*(\mathrm{x})\nabla\psi(\mathrm{x}) - [\nabla\psi^*(\mathrm{x})]\psi(\mathrm{x})\}, \qquad (41)$$

where it is understood that the products of field operators at a point must be interpreted within a specific representation of the canonical fields. But in the Fock representations, these products do have unambiguous and satisfactory meanings when the fields act in $\mathcal{H}^{(s)}$ or $\mathcal{H}^{(a)}$ according to Eqs. (34) or (37). We observe immediately that here, $\int_{\mathbb{R}^3} \rho(\mathrm{x})\, d^3x = m\, N_{\mathrm{op}}$, which is consistent with the interpretation of $\rho(\mathrm{x})$ as the total mass density.

Note that before the second quantization, the above formula for $\rho(\mathrm{x})$ would just be the mass m times the usual expression for the 1-particle probability density in positional space, while the formula for $\mathbf{J}(\mathrm{x})$ would just be the mass times the probability flux density.

Using Eqs. (29) formally, we can calculate the (singular) fixed-time commutation relations for the current algebra from the commutator or anti-commutator algebra of fields. The result for ρ together with the spatial components J_k of $\mathbf{J}$ is,

$$[\,\rho(\mathrm{x}),\, \rho(\mathrm{y})\,] = 0, \quad [\,\rho(\mathrm{x}),\, J_k(\mathrm{y})\,] = -i\hbar\,\frac{\partial}{\partial x^k}\,[\,\delta^{(3)}(\mathrm{x}-\mathrm{y})\,\rho(\mathrm{x})], \qquad (42)$$

$$[\, J_k(\mathrm{x}),\, J_\ell(\mathrm{y})\,] = i\hbar\,\{\, \frac{\partial}{\partial y^k}\,[\,\delta^{(3)}(\mathrm{x}-\mathrm{y})\, J_\ell(\mathrm{y})] - \frac{\partial}{\partial x^\ell}\,[\,\delta^{(3)}(\mathrm{x}-\mathrm{y})\, J_k(\mathrm{x})]\,\},$$

where in the nonrelativistic theory we have no Schwinger terms. Furthermore the result is the same whether the canonical commutation or anti-commutation relations for ψ and ψ^* are taken. This is a very important fact. It means that whereas the information as to particle statistics (Bose or Fermi) is encoded in the algebra of fields, it is *not* encoded in the Lie algebra of local currents. Rather the choice of particle statistics is, in general, encoded in the choice of *representation* of the algebra (up to unitary equivalence), as we shall see.

To obtain a *bona fide* Lie algebra from Eqs. (42), the final step is to integrate ρ and J with test functions. Define $\rho(f) = \int_{\mathbb{R}^3} \rho(\mathbf{x}) f(\mathbf{x}) d^3x$ and $J(\mathbf{g}) = \int_{\mathbb{R}^3} \sum_{k=1}^3 J_k(\mathbf{x}) g^k(\mathbf{x}) d^3x$, where f and the components g^k of the vector field g belong to $\mathcal{D}(\mathbb{R}^3)$. Then

$$[\rho(f_1),\, \rho(f_2)] = 0\,, \quad [\rho(f),\, J(\mathbf{g})] = i\hbar\rho(\mathbf{g}\cdot\nabla f),$$

$$[J(\mathbf{g}_1),\, J(\mathbf{g}_2)] \;=\; -i\hbar J([\mathbf{g}_1,\, \mathbf{g}_2])\,. \tag{43}$$

Notice that what we now have is a representation by self-adjoint operators of the semidirect sum of the Abelian Lie algebra of compactly-supported C^∞ scalar functions on $\mathbb{R}^d$, with the compactly-supported C^∞ vector fields — precisely the Lie algebra that is associated with Eq. (23) above.

This is perhaps a good place to observe that the current algebra of Eqs. (43) respects nonrelativistic local causality — if the supports of f and g in $\mathbb{R}^d$ are disjoint, then the equal-time commutator of $\rho(f)$ with $J(\mathbf{g})$ is zero; and if the supports of $\mathbf{g}_1$ and $\mathbf{g}_2$ are disjoint, then the equal-time commutator of $J(\mathbf{g}_1)$ with $J(\mathbf{g}_2)$ is zero.

2.3. *N-Particle Representations of the Nonrelativistic Current Algebra*

Let us refer back now to the canonical fields in the Fock representation, given by Eqs. (34) or alternatively Eqs. (37). We may use these expressions to obtain — for each N, and for each choice of exchange statistics (s) or (a) when $N \geq 2$ — a self-adjoint representation of the local current algebra satisfying the commutation relations of Eqs. (43). These representations, which we shall be discussing from several different points of view, are given by the pair of equations,

$$[\rho_N^{(s),(a)}(f)\, \Psi_N^{(s),(a)}](\mathbf{x}_1,...,\mathbf{x}_N) \;=\; m\sum_{j=1}^{N}\, f(\mathbf{x}_j)\Psi_N^{(s),(a)}(\mathbf{x}_1,...,\mathbf{x}_N)\,, \tag{44}$$

and

$$[\,J_N^{(s),(a)}(\mathbf{g})\,\Psi_N^{(s),(a)}\,](\mathbf{x}_1,...,\mathbf{x}_N) \;=$$
$$= -i\hbar\sum_{j=1}^{N}\frac{1}{2}\{\,\mathbf{g}(\mathbf{x}_j)\cdot\nabla_j\,\Psi_N^{(s),(a)}(\mathbf{x}_1,...,\mathbf{x}_N) + \nabla_j\cdot[\,\mathbf{g}(\mathbf{x}_j)\Psi_N^{(s),(a)}(\mathbf{x}_1,...,\mathbf{x}_N)\,]\}\,. \tag{45}$$

Note that the N-particle subspaces $\mathcal{H}_N$ remain invariant under the actions of ρ_N and J_N in Eqs. (44)–(45). These operators are also *symmetric* in the N variables $\mathbf{x}_1,\ldots,\mathbf{x}_N$, and thus manifestly respect the permutation symmetry of $\Psi_N^{(s)}$ or $\Psi_N^{(a)}$. It turns out that the representations of Eqs. (44)–(45) for different N and fixed exchange symmetry are irreducible and unitarily inequivalent to each other, which is not surprising. It also turns out that for $d > 1$ and $N > 1$, the representations with the same N and different exchange symmetry are unitarily inequivalent.[18] We shall see shortly why this is so, and why the case $d = 1$ is special.

Let us ask how *distinguishable* particles should be described, which we no longer expect to be quanta of a single field operator. In terms of mass and momentum densities, the most straightforward way to tell the particles apart is by assigning to them distinct masses m_j. We then remove the superscripts (s) or (a), and work in the Hilbert space $\mathcal{H} = \bigoplus_{N=1}^{\infty}\mathcal{H}_N$. The action of $J_N(\mathbf{g})$ remains formally the same as in Eq. (45), so that the fixed-symmetry subspaces remain invariant subspaces for all the momentum density operators in these representations. However, the operator-valued distribution ρ becomes

$$[\rho_N^{\{m_1,\ldots,m_N\}}(f)\,\Psi_N\,](\mathbf{x}_1,...,\mathbf{x}_N) \;=\; \sum_{j=1}^{N} m_j\, f(\mathbf{x}_j)\Psi_N(\mathbf{x}_1,...,\mathbf{x}_N)\,, \tag{46}$$

breaking the permutation symmetry.

Looking at Eqs. (44)–(45) [or Eq. (46)], we can see how our interpretations of ρ as the mass density operator and J as the momentum density operator make sense for ordinary N-particle nonrelativistic quantum mechanics. As usual it is important to distinguish clearly between the space of *ordered N-particle configurations* $(\mathbf{x}_1,\ldots,\mathbf{x}_N)$ that are the arguments of the wave functions, and the *physical* space of points $\mathbf{x}$ (written here without a subscript), over which ρ and J are operator-valued distributions.

Consider first the representation where $N = 1$. Let the particle mass be m. Then the expectation value $(\Psi_1, \rho_{N=1}(f)\Psi_1)$ is given by $m\int_{\mathbb{R}^3} f(\mathbf{x}_1)|\Psi_1(\mathbf{x}_1)|^2\, d^3x_1$, which is consistent with the usual probability amplitude interpretation for $\Psi_1(\mathbf{x}_1)$. If $f(\mathbf{x})$ approximates an indicator function $\chi_B(\mathbf{x})$ for a Borel set $B \subseteq \mathbb{R}^3$, then $[\rho_{N=1}(f)\Psi_1](\mathbf{x}_1)$ approximates $m\,\chi_B(\mathbf{x}_1)\Psi_1(\mathbf{x}_1)$, which is the mass times the usual localization operator projecting Ψ_1 to the subspace of wave functions vanishing outside B. And if $f(\mathbf{x})$ is (approximately) $\delta(\mathbf{x}-\mathbf{x}_0)$ for a fixed point $\mathbf{x}_0$ in physical space, we obtain the mass times the probability density $|\Psi_1(\mathbf{x}_0)|^2$ for the expectation value of $\rho_{N=1}(f)$. When $\mathbf{g}$ tends toward a constant vector field in some spatial direction (let us say, in the x^1-direction), then $[J_{N=1}(\mathbf{g})\Psi_1](\mathbf{x}_1)$ tends toward $-i\hbar\,\partial\Psi_1(\mathbf{x}_1)/\partial x_1^1$, which is just the application to Ψ_1 of the usual operator for the (total) 1-particle momentum in the x^1-direction.

When $N=2$ or more, the expectation values are sums. For example, let B be again a Borel set in $\mathbb{R}^3$ and consider $f \approx \chi_B$, which gives us

$$(\Psi_N, \rho_N(f)\Psi_N) \approx$$

$$\approx \sum_{j=1}^{N} m_j \int_{\mathbf{x}_j\in B}\int_{\mathbf{x}_k\in\mathbb{R}^3\,(k\neq j)} |\Psi_N(\mathbf{x}_1,\ldots,\mathbf{x}_N)|^2 \prod_{k=1}^{N} d^3x_k\,. \tag{47}$$

Here the j th term in the sum is the j th particle mass times the (marginal) probability that an idealized measurement detects particle j in region B. In Eq. (47) we have integrated over all possible values of the positional coordinates of all the other particles. The expectation value of $\rho_N(f)$ as $f(\mathbf{x})$ tends toward $\delta(\mathbf{x}-\mathbf{x}_0)$ becomes a sum over j of the expected mass density for particle j at the point $\mathbf{x}_0$, similarly calculated.

Of course (as is indicated by our earlier discussion of the need in quantum mechanics for infinitely many local degrees of freedom), not all the physical information about Ψ_N when $N>1$ can be contained in its expectation values with respect to $\rho_N(\mathbf{x})$ and $\mathbf{J}_N(\mathbf{x})$. One must also consider the *correlation functionals* among multiple points in physical space. We need, for instance, the 2-point functionals defined by $(\Psi_N, \rho_N(\mathbf{x})\rho_N(\mathbf{y})\Psi_N)$, $(\Psi_N, \rho_N(\mathbf{x})J_N(\mathbf{y})\Psi_N)$, and $(\Psi_N, J_N(\mathbf{x})J_N(\mathbf{y})\Psi_N)$, for $\mathbf{x},\mathbf{y}\in\mathbb{R}^3$; and likewise the higher correlations.

The N-particle representations are then characterized by systems of identities among the N-point functionals of ρ. For example, in a single-particle representation, ρ_1 satisfies the 1-particle identity,

$$\rho_1(\mathbf{x})\rho_1(\mathbf{y}) = m\,\delta(\mathbf{x}-\mathbf{y})\rho_1(\mathbf{y})\,. \tag{48}$$

Some time ago Grodnik and Sharp, who considered such identities, also introduced the discretized local current algebra in momentum space, with respect to which the N-particle identities characterizing Eqs. (44)–(45) take a nice form.[34] Let $\mathbf{n}=(n_1,n_2,n_3)$ be a triple of integers, and write the formal operator Fourier coefficients

$$\rho_{(\mathbf{n})} = \int \exp(-i\mathbf{n}\cdot\mathbf{x})\,\rho(\mathbf{x})\,d^3x\,,$$

$$J_{k\,(\mathbf{n})} = \int \exp(-i\mathbf{n}\cdot\mathbf{x})\,J_k(\mathbf{x})\,d^3x \quad (k=1,2,3)\,, \tag{49}$$

where we consider the integrals to be over a 3-torus of linear dimension 2π (that is, a "box" with periodic boundary conditions). Taking $\rho(\mathbf{x})$ and $J_k(\mathbf{x})$ to obey Eq. (42), we have a Lie algebra modeled on an integer lattice in momentum space:

$$[\rho_{(\mathbf{n})}, \rho_{(\mathbf{n}')}] = 0\,,$$

$$[\rho_{(\mathbf{n})}, J_{k\,(\mathbf{n}')}] = n_k\,\rho_{(\mathbf{n}+\mathbf{n}')}\,,$$

$$[J_{k\,(\mathbf{n})}, J_{\ell\,(\mathbf{n}')}] = n_\ell\,J_{k\,(\mathbf{n}+\mathbf{n}')} - n'_k\,J_{\ell\,(\mathbf{n}+\mathbf{n}')}\,. \tag{50}$$

In one dimension, this algebra is just the Virasoro algebra with central charge zero, and is sometimes called the "Witt algebra".

A formal representation of Eqs. (50) may be written

$$\rho_{(\mathbf{n})} = m\,z_{(\mathbf{n})}\,,$$

$$J_{k\,(\mathbf{n})} = \sum_{\mathbf{n}'} n_k\,z_{(\mathbf{n}+\mathbf{n}')}\frac{\partial}{\partial z_{(\mathbf{n}')}}\,, \tag{51}$$

where the operators act on a space of functions of infinitely many complex variables $z_{(\mathbf{n})}$. The expressions here can all be given rigorous meaning for quite general representations of the local current algebra, including the N-particle (s) and (a) representations discussed above.[18]

In terms of the operators $\rho_{(\mathbf{n})}$, the 1-particle identity given by Eq. (48) becomes

$$\rho_{(\mathbf{n})}\rho_{(\mathbf{n}')} = m\rho_{(\mathbf{n}+\mathbf{n}')}\,, \tag{52}$$

while the 2-particle identity for identical particles is

$$\rho_{(\mathbf{n})}\rho_{(\mathbf{n}')}\rho_{(\mathbf{n}'')} = m\,[\rho_{(\mathbf{n})}\rho_{(\mathbf{n}'+\mathbf{n}'')} + \rho_{(\mathbf{n}')}\rho_{(\mathbf{n}+\mathbf{n}'')} + \rho_{(\mathbf{n}'')}\rho_{(\mathbf{n}+\mathbf{n}')}] - 2m^2\rho_{(\mathbf{n}+\mathbf{n}'+\mathbf{n}'')}\,. \tag{53}$$

When the N-particle identity is satisfied, all of the N'-particle identities for $N' > N$ are necessarily also satisfied; but not conversely.

We have obtained the above "N-particle representations" as descriptive of the kinematics of nonrelativistic quantum systems. But it is worth noting that creation and annihilation field operators obeying equal-time canonical commutation relations, together with a corresponding Lie algebra of currents, exist within relativistic quantum field theories too. Let us take a moment to see how this occurs.

To write the relativistic Fock space representation of a neutral scalar field obeying Eqs. (10) at a fixed time t we proceed in the following standard manner.[20] As usual, we write 4-vectors $x = (x^0, \mathbf{x})$ and $k = (k_0, \mathbf{k})$, with $k_0 = \omega_\mathbf{k} = [\mathbf{k}^2 + m^2]^{1/2} > 0$; and $kx = k_\mu x^\mu = k_0 x^0 - \mathbf{k}\cdot\mathbf{x}$.

Let $a_\mathbf{k}$ and $a^*_\mathbf{k}$ be annihilation and creation operators for states of 4-momentum $(\omega_\mathbf{k}, \mathbf{k})$, satisfying the relativistic commutation relations

$$[a_{\mathbf{k}_1}, a_{\mathbf{k}_2}] = [a^*_{\mathbf{k}_1}, a^*_{\mathbf{k}_2}] = 0\,,$$

$$[a_{\mathbf{k}_1}, a^*_{\mathbf{k}_2}] = \omega_\mathbf{k}\, \delta^{(3)}(\mathbf{k}_1 - \mathbf{k}_2)\,. \tag{54}$$

The Fock space carrying a representation of Eqs. (54) can again be written as the direct sum of N-particle spaces: formally, $|\mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_N\rangle = (N!)^{-1/2} a^*_{\mathbf{k}_1} a^*_{\mathbf{k}_2} \cdots a^*_{\mathbf{k}_N} |0\rangle$, where $|0\rangle$ is the vacuum state. The normalization is established so that $\langle 0|0\rangle = 1$, while

$$\langle \mathbf{k}'_1, \mathbf{k}'_2, \ldots, \mathbf{k}'_M | \mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_N\rangle =$$

$$= (\delta_{MN}/N!) \sum_{\sigma \in S_N} \omega_{\mathbf{k}_1} \delta^{(3)}(\mathbf{k}_1 - \mathbf{k}'_{\sigma(1)}) \cdots \omega_{\mathbf{k}_N} \delta^{(3)}(\mathbf{k}_N - \mathbf{k}'_{\sigma(N)})\,. \tag{55}$$

Next write the so-called positive and negative frequency parts of the field operator:

$$\phi^{(+)}(x) = [2(2\pi)^3]^{-1/2} \int \frac{d^3k}{\omega_\mathbf{k}} \exp(-ikx)\, a_\mathbf{k}\,, \tag{56}$$

and $\phi^{(-)}(x) = \phi^{(+)}(x)^*$. Then with $\phi(x) = \phi^{(+)}(x) + \phi^{(-)}(x)$, we have

$$[\phi(x), \phi(y)] = i\Delta(x-y)\,, \tag{57}$$

where $\Delta(x)$ is the famous invariant distributional solution of the Klein–Gordon equation with initial conditions $\Delta(0, \mathbf{x}) = 0$, $(\partial\Delta/\partial x^0)|_{x^0=0} = -\delta^{(3)}(\mathbf{x})$. Defining the operator-valued distribution $\pi(x) = \partial\phi(x)/\partial x^0$, we obtain Eqs. (10) from Eq. (57) at equal times $x^0 = y^0$.

If we now introduce the positional annihilation operator

$$\phi_1(x) = (2\pi)^{-3/2} \int \frac{d^3k}{\omega_\mathbf{k}}\, \omega_\mathbf{k}^{1/2} \exp(ikx)\, a_\mathbf{k}\,, \tag{58}$$

and its adjoint, we have that ϕ_1 and ϕ_1^* satisfy the equal-time canonical commutation relations in Eqs. (29) — the same as we wrote for nonrelativistic second-quantized fields. Then $\phi_1^*(t,\mathbf{x})\phi_1(t,\mathbf{x})$ has the interpretation of a particle number density in position-space at time t, and $m\phi_1^*(t,\mathbf{x})\phi_1(t,\mathbf{x})$ is the mass density. However, compare the form of Eq. (58) to that of Eq. (56) and note the extra factor of $\omega_\mathbf{k}^{1/2}$ (so that we are no longer respecting the Lorentz covariance).

Likewise we can define a 3-momentum density operator in terms of ϕ_1 and ϕ_1^* at a fixed time. Thus we have obtained from these operators a representation in the *relativistic* Fock space of the same local current algebra in Eq. (43), which decomposes as before into N-particle Bose representations. However in Minkowskian space-time the resulting operators are *nonlocal* and *noncovariant*. The current algebra extended to commutation relations at unequal times leads to operators that do not commute at spacelike separations, nor are ρ and $\mathbf{J}$ the components of a 4-vector — that is, $(\rho, \mathbf{J})$ do not transform covariantly under the Lorentz group.

Nevertheless the occurrence of a representation of the equal-time Lie algebra of currents modeled on vector fields in relativistic quantum field theory is significant. Although the physical world is relativistic, we know that nonrelativistic quantum mechanics provides good approximations to observations at low velocities. While local, relativistic algebras of observables necessarily connect subspaces in the Hilbert space corresponding to different numbers of particles, if the local particle number makes sense there should exist mathematically (in a given reference frame) a system of operators for measuring the spatial locations of the particles and the flux of the particles. Here we see this is indeed the case — the "nonrelativistic" local current algebra *can* exist at a fixed time even in relativistic models, and generally does. At low energies in particle theories, it is this current algebra that (approximately) describes the kinematics.

We have shown, then, that a family of self-adjoint representations $\{\rho_N^{(s),(a)}, J_N^{(s),(a)} \,|\, N = 0, 1, 2, \ldots\}$ of the local current algebra can describe the kinematics of distinct systems of quantum particles.

But we can go somewhat further — we can also express a nonrelativistic Hamiltonian operator $H = H_0 + V$, with kinetic energy H_0 and potential energy V derived from a 2-body potential, in terms of the local currents.[35] To do this, we begin with the kinetic energy expression $H_0 = (\hbar^2/2m)\int_{\mathbb{R}^3} \nabla\psi^*(\mathbf{x})\cdot\nabla\psi(\mathbf{x})\,d^3x$. Then, using Eqs. (41), we can rewrite H_0 formally:

$$H_0 = \frac{1}{8}\int_{\mathbb{R}^3}[\frac{\hbar}{m}\nabla\rho(\mathbf{x}) - 2i\mathbf{J}(\mathbf{x})]\cdot\frac{1}{\rho(\mathbf{x})}[\frac{\hbar}{m}\nabla\rho(\mathbf{x}) + 2i\mathbf{J}(\mathbf{x})]\,d^3x\,, \quad (59)$$

while

$$V = \frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3}\rho(\mathbf{x})\,\frac{v(\mathbf{x}-\mathbf{y})}{m^2}\,\rho(\mathbf{y})\,d^3x\,d^3y\,. \quad (60)$$

The mathematical interpretation of Eq. (59) requires treating not merely the product of operator-valued distributions at a point, as in the Sugawara model, but the reciprocal of an operator-valued distribution. Despite its singular appearance, it is nevertheless possible to make sense of this expression as a bilinear form on an appropriate domain of vector-valued distributions in the N-particle Hilbert space.

Having reached this point, we shall want to reformulate the theory so as to think of the self-adjoint representations of the Lie algebra of local currents as the fundamental entities, and the field operators as a kind of auxiliary construct derived from and relating these representations. We do this in Sec. 4, after developing more about unitary representations of diffeomorphism groups.

2.4. *N-Particle Representations of Diffeomorphism Groups*

Now the algebra of scalar functions and vector fields entering Eqs. (43) exponentiates to the semidirect product group $\mathcal{D}(\mathbb{R}^3)\times$ $\mathit{Diff}^{\,c}(\mathbb{R}^3)$ or $\mathcal{S}((\mathbb{R}^3)\times\mathcal{K}(\mathbb{R}^3)$, according to the choice of limiting condition as $|\mathbf{x}|\to\infty$. So it is natural to write the unitary group representations of Eq. (23) that correspond to Eqs. (44)–(45). In general we shall write such unitary representations in the form $(f,\phi)\to U(f)V(\phi)$, so that $U(f)$ represents the subgroup (f,e) where $e(\mathbf{x})\equiv\mathbf{x}$ is the identity diffeomorphism, and $V(\phi)$ represents the subgroup $(0,\phi)$.

The group laws become

$$U(f_1)U(f_2) = U(f_1+f_2), \quad V(\phi_1)V(\phi_2) = V(\phi_1\phi_2),$$
$$V(\phi)U(f) = U(f\circ\phi)V(\phi)\,, \quad (61)$$

or equivalently

$$U(f_1)V(\phi_1)U(f_2)V(\phi_2) = U(f_1+\phi_1 f_2)V(\phi_1\phi_2)\,, \quad (62)$$

recalling our conventions $\phi_1 f_2 = f_2\circ\phi_1$ and $\phi_1\phi_2 = \phi_2\circ\phi_1$.

The corresponding N-particle unitary representations that satisfy Eq. (62) in $\mathcal{H}_N^{(s)}$ or $\mathcal{H}_N^{(a)}$ can be derived from Eqs. (44)–(45) as follows. For $a,b\in\mathbb{R}$, let us define continuous one-parameter unitary groups by exponentiating the self-adjoint density operators $\rho_N^{(s),(a)}(f)$ and currents $J_N^{(s),(a)}(\mathbf{g})$ respectively; thus:

$$U_N^{(s),(a)}(af) = \exp\,[\,(ia/m)\,\rho_N^{(s),(a)}(f)\,]\,,$$
$$V_N^{(s),(a)}(\phi_b^{\mathbf{g}}) = \exp\,[\,(ib/\hbar)\,J_N^{(s),(a)}(\mathbf{g})\,]\,. \quad (63)$$

Then we obtain

$$[U_N^{(s),(a)}(f)\Psi^{(s),(a)}](\mathbf{x}_1,\dots\mathbf{x}_N) = \left(\prod_{j=1}^{N}\exp\,[\,if(\mathbf{x}_j)\,]\right)\Psi^{(s),(a)}(\mathbf{x}_1,\dots\mathbf{x}_N),$$

$$[V_N^{(s),(a)}(\phi)\Psi^{(s,a)}](\mathbf{x}_1,\dots\mathbf{x}_N) = \Psi^{(s),(a)}(\phi(\mathbf{x}_1),\dots\phi(\mathbf{x}_N))\sqrt{\prod_{j=1}^{N}\mathcal{J}_\phi(\mathbf{x}_j)}\,, \quad (64)$$

where $\mathcal{J}_\phi(\mathbf{x})$ is the Jacobian of ϕ at $\mathbf{x}$. Notice how the square root of this Jacobian is just what is needed for Eq. (64) to give us a *unitary* representation of V_N — the change of variable $\mathbf{x}' = \phi(\mathbf{x})$ transforms the inner product (Φ_N,Ψ_N), expressed as an integral, to the inner product $(V_N\Psi_N, V_N\Psi_N)$.

Our perspective now is the following. Suppose we are given a continuous unitary representation (CUR) $U(f)V(\phi)$ of Eq. (23); for example, one of representations $U_N^{(s),(a)}, V_N^{(s),(a)}$, or some other CUR. We then have immediately the continuous 1-parameter unitary groups $U(af)$ and $V(\phi_b^{\mathbf{g}})$, $a,b\in\mathbb{R}$. Continuity of these unitary subgroups is a consequence of the continuity of the representation with respect to the topology of $\mathcal{D}(\mathbb{R}^3)\times$ $\mathit{Diff}^{\,c}(\mathbb{R}^3)$ or $\mathcal{S}((\mathbb{R}^3)\times\mathcal{K}(\mathbb{R}^3)$. The operators $\rho(f)$ and $J(\mathbf{g})$ can then be recovered in the representation as the self-adjoint generators of these 1-parameter unitary groups, using Eq. (7); *i.e.*,

$$\rho(f)\Psi = m\lim_{a\to 0}\frac{1}{ia}[U(af)\Psi - \Psi],$$
$$J(\mathbf{g})\Psi = \hbar\lim_{b\to 0}\frac{1}{ib}[V(\phi_b^{\mathbf{g}})\Psi - \Psi]\,. \quad (65)$$

The meaning of $\rho(f)$ as the spatially-averaged mass density observable, and $J(\mathbf{g})$ as the spatially-averaged momentum density observable, allows each such representation of the group to be interpreted physically. In particular, Eqs. (44) and (45) follow from Eqs. (64) using Eqs. (65).

Should the spectrum of $\rho(f)$, for $f(\mathrm{x}) \geq 0$, fail to be positive definite in the representation, we need not immediately discard the representation as unphysical. We reserve the possibility of modifying Eqs. (65), and changing our interpretation of the operators. For example, we can multiply the right-hand expressions by q/m, where q is the unit charge, and interpret the resulting operators $\rho(f)$, $J(\mathrm{g})$ as the spatially averaged charge density and the spatially averaged electric current density respectively. A situation where doing this is natural occurs in Sec. 3.3 below.

2.5. *Diffeomorphism Group Representations and Local Symmetry in Quantum Mechanics*

We have seen that the unitary representations of the diffeomorphism group are not unique, and that inequivalent representations can describe the kinematics of quantum systems that are physically distinct. Although the diffeomorphism group is infinite-dimensional, representations exist describing systems whose configuration-spaces are finite-dimensional. Later we shall obtain still other representations, with infinite-dimensional configuration spaces. Let us first digress briefly to discuss *why* the diffeomorphisms of $\mathbf{R}^d$, or those of a more general manifold M, should be fundamental for quantum mechanics.

From the point of view of symmetry, think first of a diffeomorphism ϕ of M as acting actively, taking whatever might be located in a neighborhood $\mathcal{O}$ of a point x_0, and moving it (while smoothly turning and distorting it) to a new neighborhood $\phi(\mathcal{O})$ containing $\phi(\mathrm{x}_0)$. Just as we identify the self-adjoint momentum operator $\hat{p}_1$ in quantum mechanics with the infinitesimal generator of the group of translations in the x-direction, or the self-adjoint angular momentum operator $\hat{L}_3$ with the infinitesimal generator of the group of rotations about the z-axis, we have interpreted the self-adjoint operator $J(\mathrm{g})$ as the infinitesimal generator of the flow generated by the vector field g — a "local symmetry" of physical space.

This identification is also *kinematical.* Just as the self-adjoint operators generating translations or rotations (as group actions on the spatial manifold) describe linear or angular momentum respectively, and do not depend on the Hamiltonian operator H being translation- or rotation-invariant, so do the self-adjoint generators of the flows describe local currents for N particles, independent of the particular dynamics. The description depends

only on the fact that the diffeomorphisms act smoothly as a group on the physical space (along with appropriate technical properties of the continuous unitary group representation).

Alternatively we can think of a diffeomorphism ϕ as acting passively, defining a *general coordinate transformation* that provides a smooth way to modify our description of the locations of objects in space at a particular time. The time-evolution operator (and consequently the Schrödinger equation) will not be invariant under such a transformation. But the probability amplitude for a system in state Ψ_1 to be observed in state Ψ_2, given as usual by the inner product (Ψ_2, Ψ_1), is understood as being defined at a fixed time; and the "collapse of the wave packet" is not itself a dynamical process. Then (Ψ_2, Ψ_1) should remain unchanged by such a change of description — *i.e.*, we expect the modification of coordinates to be implemented in $\mathcal{H}$ by a unitary operator $V(\phi)$. And we plausibly expect the correspondence $\phi \to V(\phi)$ to be smooth and to respect the composition of diffeomorphisms, providing a continuous unitary representation (CUR), or at least a projective representation, of $\mathit{Diff}^c(\mathbf{R}^d)$ in $\mathcal{H}$. But we cannot expect the expression for the Hamiltonian to be invariant under such general coordinate transformations — at least, outside the context of theories (such as some possible descriptions of quantum gravity) that are wholly independent of a background metric.

We begin to see the generality of the diffeomorphism group approach to quantum theory. Still another aspect of this description is that because we have a *local* symmetry group, we are not restricted to $\mathbf{R}^3$ as the spatial manifold. We can easily consider the group of compactly-supported diffeomorphisms of a manifold that lacks global translation- or rotation-invariance, one that is not simply-connected, and so on. Suppose we take physical space X to be a manifold M with boundary ∂M; taking X to be compact, the natural group consists of C^∞ invertible homeomorphisms whose inverse is C^∞; and these preserve ∂M (as a set). Thus, even when total momentum or angular momentum operators do not exist or are not uniquely specified, we have a natural way to describe the kinematics.

Consider next the general coordinate transformations of the space-time manifold $\mathbb{R}^{d+1}$, rather than just of $\mathbb{R}^d$. A natural group consists of diffeomorphisms $\hat{\phi} : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ that respect the causal structure; let us call such transformations *causal diffeomorphisms.*

In Galilean space-time, this means that the point $\hat{\phi}\,(t_1, \mathrm{x}_1)$ precedes the point $\hat{\phi}\,(t_2, \mathrm{x}_2)$ if and only if (t_1, x_1) precedes (t_2, x_2) (*i.e.*, $t_1 < t_2$); while $\hat{\phi}\,(t_1, \mathrm{x}_1)$ and $\hat{\phi}\,(t_2, \mathrm{x}_2)$ are simultaneous if and only if (t_1, x_1) and (t_2, x_2)

are simultaneous as well (*i.e.*, $t_1 = t_2$). The identity map is causal, and a diffeomorphism $\hat{\phi}$ is causal if and only if $\hat{\phi}^{-1}$ is causal; so we again have a group. A general Galilean causal diffeomorphism may be written,

$$(t', \mathrm{x}') = \hat{\phi}(t, \mathrm{x}) = (\tau(t), \phi_t(\mathrm{x}))\,, \tag{66}$$

where $\tau : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism of the time axis only, and $\phi_t(\mathrm{x})$ is a parameterized family of diffeomorphisms of $\mathbb{R}^d$ depending smoothly on t (not necessarily a flow, however). In effect, we consider $\mathbb{R}^{d+1}$ as a bundle over $\mathbb{R}$ (the time axis), and take the group of *bundle diffeomorphisms.* Evidently the Galilean boosts ($t' = t$, $\mathrm{x}' = \mathrm{x} - \mathrm{v}t$) belong to this group, as well as the time translations. There is also the natural embedding of $\mathit{Diff}^c(\mathbb{R}^d)$ in the larger group of causal diffeomorphisms of $\mathbb{R}^{d+1}$ given by $t' = t$, $\mathrm{x}' = \phi(\mathrm{x})$. Representation of this group of bundle diffeomorphisms may be interesting for the description of quantum mechanics in nonuniformly moving or accelerating reference frames.[36]

In Minkowskian space-time, there are four possible causal relations between two points x and y: (1) space-like [*i.e.*, $(x-y)_\mu(x-y)^\mu < 0$], (2) light-like [*i.e.*, $(x-y)_\mu(x-y)^\mu = 0$], (3) time-like with x preceding y [*i.e.*, $(x-y)_\mu(x-y)^\mu > 0$ and $x^0 < y^0$], or (4) time-like with x following y [*i.e.*, time-like with $x^0 > y^0$]. Causal diffeomorphisms must be such that the relation of $\hat{\phi}(ct_1, \mathrm{x}_1)$ to $\hat{\phi}(ct_2, \mathrm{x}_2)$ is the same as that of (ct_1, x_1) to (ct_2, x_2). In $(1+1)$-dimensional space-time, a diffeomorphism $\hat{\phi}$ of the Minkowskian plane with this property acts independently on *light cone coordinates.* This means that if we write a point (ct, x) in the form $(\chi_1, -\chi_1) + (\chi_2, \chi_2)$, where $\chi_1 = (ct - x)/2$ and $\chi_2 = (ct + x)/2$, there exists a pair of diffeomorphisms ϕ_1 and ϕ_2 of two different real lines (the left and the right light cone through the origin) such that with $\chi_1' = \phi_1(\chi_1)$ and $\chi_2' = \phi_2(\chi_2)$, $\hat{\phi}(ct, x) = (\chi_1', -\chi_1') + (\chi_2', \chi_2')$. We thus realize a certain group of causal diffeomorphisms of the Minkowskian plane as the direct product group $\mathit{Diff}^c(\mathbb{R}) \times \mathit{Diff}^c(\mathbb{R})$. Note, however, that even when ϕ_1 and ϕ_2 are compactly supported on $\mathbb{R}^1$, $\hat{\phi}$ is not compactly supported on $\mathbb{R}^2$.

The appropriate local currents here are light cone currents, not fixed-time currents. The appropriate representations are projective representations of the Lie algebra, accommodating Schwinger terms — so that we have not just two copies of the algebra of vector fields on $\mathbb{R}$, but two copies of the Virasoro algebra, leading into conformal field theory in $1+1$ dimensions. It is then possible (but nontrivial) to take a nonrelativistic limit, recovering the nonrelativistic local current algebra of Eqs. (43) in 1-dimensional space, and the corresponding group.

In Minkowskian space-time of greater than $1+1$ dimensions, the group of causal diffeomorphisms is finite-dimensional (as is the conformal group). We have Poincaré transformations that respect the time-direction, together with dilatations; but we can no longer deform the space-time *locally.* Special relativity in three or more space-time dimensions has a causal structure "too rigid" for the diffeomorphism group. But when we move from special to general relativity, the group of diffeomorphisms of a spacelike surface enters explicitly again. Here it plays the role of a gauge group, for instance in the superspace formulation of quantum gravity.[37,38]

3. Representation Theory for Diffeomorphism Groups

There are several approaches to studying unitary representations of diffeomorphism groups. In Sec. 3.1 we describe a very general picture, in which the group is represented in the Hilbert space of square-integrable functions on some configuration space. Then in Sec. 3.2 we consider various candidates for such spaces of configurations. In Sec. 3.3 we develop the "method of semidirect products," and realize the N-particle group representations that were described in Sec. 2.4 on particular orbits in a configuration space of distributions. We also introduce some additional representations that are associated with other orbits in the same space of distributions.

3.1. *Configuration Spaces, Measures, and Cocycles*

We shall see that the following picture provides a quite general framework. First consider a continuous unitary representation (CUR) $V(\phi)$ of $\mathit{Diff}^c(M)$. Typically $M = \mathbb{R}^d$, but more generally we can take M to be a C^∞, oriented Riemannian manifold that has all the desired topological properties — for example it is connected and locally simply connected (though it is *not* necessarily simply-connected); it is locally compact, σ-compact, second-countable, and metrizable (and therefore Hausdorff). Often one can then realize the representation $V(\phi)$ in a Hilbert space $\mathcal{H} = L^2_{d\mu}(\Delta, \mathcal{W})$, which is the space of functions $\Psi(\gamma)$ on a *configuration space* Δ taking values in an inner product space $\mathcal{W}$, square-integrable with respect to a measure μ on Δ. We write the inner product in $\mathcal{H}$ as

$$(\Phi, \Psi) = \int_\Delta \langle \Phi(\gamma), \Psi(\gamma) \rangle_{\mathcal{W}}\, d\mu(\gamma)\,, \tag{67}$$

where $\langle\ ,\ \rangle_{\mathcal{W}}$ denotes the inner product in $\mathcal{W}$. When $\mathcal{W} = \mathbb{C}$, Eq. (67) becomes $(\Phi, \Psi) = \int_\Delta \overline{\Phi(\gamma)}\Psi(\gamma)\, d\mu(\gamma)$.

For the inner product Eq. (67) to make sense, we require that Δ be a measurable space. That is, there must exist a σ-algebra $\mathcal{B}_\Delta$ of subsets of Δ (the "measurable" sets), closed under countable unions and intersections and under complements, that includes Δ itself. The measure μ is then a positive real-valued function on $\mathcal{B}_\Delta$ obeying the usual assumptions, including countable additivity.

We shall shortly see how to obtain some examples of the configuration space Δ. For any such example, there must be a natural group action by $Diff^c(M)$ on Δ; *i.e.*, a (continuous) map $Diff^c(M) \times \Delta \to \Delta$ respecting the composition of diffeomorphisms. We shall also write $\phi : \Delta \to \Delta$, or $\gamma \to \phi\gamma$, for $\gamma \in \Delta$ and $\phi \in Diff^c(M)$. We further require $\mathcal{B}_\Delta$ to be invariant under the action of $Diff^c(M)$, so that if $B \in \mathcal{B}_\Delta$, then $\phi B \in \mathcal{B}_\Delta$.

Then $V(\phi)$ is given by the important formula

$$[V(\phi)\Psi](\gamma) = \chi_\phi(\gamma)\Psi(\phi\gamma)\sqrt{\frac{d\mu_\phi}{d\mu}(\gamma)} \quad \text{a. e.}\,(\mu)\,, \tag{68}$$

whose meaning we shall now discuss.

First we remark that since we are working in an L^2-space over Δ with respect to the measure μ, functions on Δ are defined up to equivalence: two μ-square-integrable functions are "the same" if they differ only on a set of μ-measure zero. The abbreviation "a. e. (μ)" in Eq. (68) stands for "almost everywhere with respect to μ," and means that the equation may fail on some μ-measure zero set in Δ. Note that the failure set for Eq. (68) may depend on ϕ; it may even do so in such a way that there are no elements $\gamma \in \Delta$ where the equation holds for all ϕ.

Next observe that in order for the group representation property $V(\phi_1)V(\phi_2) = V(\phi_1\phi_2)$, as in Eq. (61), to be consistent with the factor $\Psi(\phi\gamma)$ in Eq. (68), the action of $Diff^c(M)$ on Δ should be defined as a *right* action; *i.e.*,

$$[\phi_1\phi_2]\gamma = \phi_2(\phi_1\gamma) \quad (\forall\, \phi_1, \phi_2 \in Diff^{\,c}(M), \gamma \in \Delta). \tag{69}$$

Now the transformed measure μ_ϕ occurring in Eq. (68) is defined by $\mu_\phi(B) = \mu(\phi B)$ for all $B \in \mathcal{B}_\Delta$. It is required that μ have the important property of *quasiinvariance* under the action of $Diff^c(M)$. This means that for all $\phi \in Diff^c(M)$ and for all $B \in \mathcal{B}_\Delta$, $\mu(B) = 0$ if and only if $\mu(\phi B) = 0$. Equivalently, B has positive measure if and only if ϕB has positive measure. This condition is necessary and sufficient for the existence for all ϕ of the *Radon–Nikodym derivative* in Eq. (68) — $d\mu_\phi/d\mu$ is a positive measurable function $\alpha_\phi(\gamma)$ defined for almost all $\gamma \in \Delta$, with $d\mu_\phi(\gamma) = \alpha_\phi(\gamma)d\mu(\gamma)$.

Note that the Radon–Nikodym derivative in Eq. (68) satisfies the "chain rule for derivatives"

$$\alpha_{\phi_1\phi_2}(\gamma) = \alpha_{\phi_2}(\phi_1\gamma)\alpha_{\phi_1}(\gamma) \tag{70}$$

almost everywhere in Δ. Eq. (70) is likewise satisfied by $\alpha_\phi(\gamma)^{\frac{1}{2}}$, making Eq. (68) consistent with the group law. Equation (70) is called a *cocycle* equation, and we say that $\alpha_\phi(\gamma)$ thus defines a real 1-cocycle.

In Eq. (68), we have $\Psi(\gamma) \in \mathcal{W}$. Then $\chi_\phi(\gamma) : \mathcal{W} \to \mathcal{W}$ is a system of unitary operators acting on $\mathcal{W}$ for $\gamma \in \Delta$, defined a.e. (μ). Unlike the real-valued cocycle $\alpha_\phi(\gamma)$, the operators $\chi_\phi(\gamma)$ do not in general commute with each other; so it is important to write the order of operators carefully in the cocycle equation they satisfy. In order that $V(\phi_1)V(\phi_2) = V(\phi_1\phi_2)$ we need $[V(\phi_1)[V(\phi_2)\Psi]](\gamma) = [V(\phi_1\phi_2)\Psi](\gamma)$; then Eq. (68) implies the cocycle equation for $\chi_\phi(\gamma)$,

$$\chi_{\phi_1\phi_2}(\gamma) = \chi_{\phi_1}(\gamma)\chi_{\phi_2}(\phi_1\gamma) \;\; \text{a. e.}\,(\mu)\,. \tag{71}$$

Equation (71) is permitted to fail on a set of μ-measure zero that can depend on ϕ_1 and ϕ_2; again, there may even be no elements of Δ where the equation holds for all diffeomorphisms.

Given the quasiinvariant measure μ on Δ, we can always choose $\mathcal{W} = \mathbb{C}$ and $\chi_\phi(\gamma) \equiv 1$, so that Eq. (68) already defines at least one unitary group representation. When $\mathcal{W} = \mathbb{C}$ we have complex-valued wave functions, and in that case χ is a 1-cocycle of complex numbers of modulus one. We can in fact obtain additional, nontrivial complex cocycles by setting $\chi_\phi(\gamma) = \alpha_\phi(\gamma)^{i\lambda} = \exp[\,i\lambda \ln \alpha_\phi(\gamma)\,]$, for arbitrary $\lambda \in \mathbb{R}$. Note that because of the square root of the Radon–Nikodym derivative in Eq. (68), evaluation of the inner product $(V(\phi)\Phi, V(\phi)\Psi)$ using Eq. (67) gives precisely (Φ, Ψ), by making the change of variable $\gamma' = \phi\gamma$ in Δ.

Thus we picture CURs of $Diff^c(M)$ as described by quasiinvariant measures on configuration spaces, together with unitary 1-cocycles. To have an irreducible representation, it is necessary that μ be *ergodic* in a certain sense for the action of $Diff^c(M)$ on Δ: namely, given any measurable set $B \in \mathcal{B}_\Delta$ that is invariant under all diffeomorphisms, either $\mu(B) = 0$ or $\mu(\Delta - B) = 0$. Indeed, if there exists an invariant set $B \in \mathcal{B}_\Delta$ with $\mu(B) > 0$ and $\mu(\Delta - B) > 0$, then the set of functions in $\mathcal{H}$ vanishing on B is a nontrivial invariant subspace for the representation. But in Sec. 3.2, we demonstrate a more precise result.

Imagine now that Δ is a subset of a larger measurable space, some "universal" space Π of *all possible* configurations in a class of theories, equipped with the σ-algebra $\mathcal{B}_\Pi$. It can be useful to have a topology on Π for which $\mathcal{B}_\Pi$ is generated by the open and closed sets (*i.e.*, for which $\mathcal{B}_\Pi$ is the Borel σ-algebra). The diffeomorphisms of M must act on Π with a right action in a natural way, so as to leave the σ-algebra $\mathcal{B}_\Pi$ invariant.

Taking this point of view, it is actually the quasiinvariant measure μ on Π that, in effect, singles out some class of configurations associated with the particular representation in Eq. (68). The configuration space $\Delta \subset \Pi$ is a set that *carries* the measure — it is invariant under the action of $Diff^c(M)$, and it is of full measure with respect to μ in the sense that the measure of its complement is zero.

We then distinguish two ways in which μ may be ergodic in the above sense for the action of $Diff^c(M)$ — either (1) Δ may be chosen so that the group acts *transitively* on it (so that Δ is a single orbit of $Diff^c(M)$ in Π), or else (2) Δ is an uncountable union of orbits, the measure of each of which is zero (in which case μ is called *strictly ergodic*). Both cases are important to physics. The single-orbit case is typically associated with finite-dimensional configuration spaces, and the strictly ergodic case with infinite-dimensional spaces.

Having chosen a quasiinvariant measure thus concentrated on a configuration space $\Delta \subset \Pi$, it turns out that the inequivalent choices of χ_ϕ for Eq. (68) — *i.e.*, the noncohomologous cocycles — are at least in some cases associated with nontrivial topological phase effects in quantum mechanics, and the quantum statistics of particles. Then the classification of the CURs of $Diff^c(M)$ by configuration space, quasiinvariant measure, and cocycle, allows us to predict or describe an extraordinarily wide variety of quantum systems within a single framework.

3.2. *Choices of Configuration Space*

There is no single, agreed-upon universal configuration space for the representation theory of $Diff^c(M)$ (or, for that matter, for the physics of systems having infinitely-many degrees of freedom). This can possibly be understood not just as an absence of consensus among physicists working in different domains, but as a gap in our present level of physical and mathematical understanding. Let us therefore survey several interesting choices that have been made, according to the physical context under discussion: (a) the space of locally finite point configurations, (b) the configuration space of closed subsets, (c) spaces of generalized functions (distributions), (d) the configuration space of countable subsets, (e) spaces of embeddings and immersions, (f) marked configuration spaces, and (g) configuration spaces derived from generalized vector fields. Each has its advantages, and allows the convenient description and interpretation of certain classes of unitary representations.[39]

The space of locally finite point configurations

The space that has played a preeminent role in statistical mechanics as well as quantum mechanics is the space Γ_M whose elements are *locally finite* subsets of M (where typically $M = \mathbb{R}^d$). That is, we let

$$\Gamma_M = \{\gamma \subset M \mid (\forall K \subset M,\, K \,\text{compact}) \, |\gamma \cap K| < \infty \}\,, \tag{72}$$

where $|\gamma \cap K|$ means the cardinality of $\gamma \cap K$. We can write

$$\Gamma_M = \bigsqcup_{N=0}^{\infty} \Gamma_M^{(N)} \bigsqcup \Gamma_M^{(\infty)}\,, \tag{73}$$

where $\Gamma_M^{(N)}$ consists of all N-point subsets of M, $\Gamma_M^{(\infty)}$ consists of all infinite but locally finite subsets, and $\sqcup$ is the disjoint union. For some purposes, it is useful to omit $\Gamma_M^{(0)}$, which contains just one element — the empty configuration.

For $\gamma = \{\mathrm{x}_j \mid j = 1, \dots, N \text{ or } j = 1, 2, \dots\}$, the natural action of a diffeomorphism ϕ of M is given by $\phi\gamma = \{\phi(\mathrm{x}_j)\}$. With our convention $\phi_1\phi_2 = \phi_2 \circ \phi_1$, this defines a right action (as desired). Note that the *physical* space M is naturally identified with (but is not the same as) the 1-particle *configuration* space, which is the class of 1-point subsets of M.

The space Γ_M may be topologized by the *vague* topology, which is the weakest topology such that for all continuous, compactly supported real-valued functions f on M, the functions from $\Gamma_M \to \mathbb{R}$ defined by $\gamma \to \sum_{\mathrm{x}\in\gamma} f(\mathrm{x})$ are all continuous. The corresponding Borel σ-algebra makes Γ_M a measurable space. In addition the Riemannian structure of M allows Γ_M to be given a natural differentiable structure, introduced and studied by Albeverio, Kondratiev, and Röckner. For $M = \mathbb{R}^d$, a measure on $\Gamma_M^{(N)}$ equivalent to (local) Lebesgue measure describes an N-particle system; so that $\Gamma_M^{(N)}$ is the N-particle configuration space (see below). Poisson and Gibbs measures on $\Gamma_M^{(\infty)}$ describe equilibrium states in statistical physics, or infinite gases in quantum theory.[18,35,40,41,42,43,44,45,46]

The configuration space of closed subsets

A much larger configuration space, introduced in early work by Ismagilov,[5,47,48,49] is the space Ω_M of all closed subsets of the manifold

M. As in earlier papers, one may for certain purposes want to omit the empty configuration. Then for $C \in \Omega_M$, we have that $\phi C = \{\phi(\mathbf{x}) \mid \mathbf{x} \in C\}$ also belongs to Ω_M, defining a right action of the diffeomorphism group. A σ-algebra for Ω_M, making it a measurable space, is generated by the family of sets in Ω_M consisting of all closed subsets of a given closed set. That is, for $C \subseteq \Omega_M$ closed, let $\Omega_C = \{C' \in \Omega_M \mid C' \subseteq C\}$; then let $\mathcal{B}_{\Omega_M}$ be the smallest σ-algebra containing the family of sets $\{\Omega_C\}_{C \subseteq M \text{ closed}}$. This σ-algebra can also be obtained as the algebra of Borel sets with respect to a topology on Ω_M, for which a subbase is the family of sets $\{C \mid C \cap \mathcal{O} \neq \emptyset\}_{\mathcal{O} \subseteq M \text{ open}}$.

Evidently any locally finite configuration $\gamma \in \Gamma_M$ is also a closed subset of M, so that in general $\Gamma_M \subset \Omega_M$.

Configuration spaces of generalized functions (distributions)

Still another choice, convenient to the method of semidirect products discussed below, is to take the dual space $\mathcal{D}'(M)$ to the space of C^∞ compactly-supported functions $\mathcal{D}(M)$. That is, a configuration $F \in \mathcal{D}'(M)$ is a continuous, linear, real-valued functional on $\mathcal{D}(M)$ — a *distribution* or *generalized function* on M. We shall write $\langle F, f\rangle$ for the value of F on the function $f \in \mathcal{D}(M)$. Diffeomorphisms act on $\mathcal{D}'(M)$ by the dual to their action on $\mathcal{D}(M)$; *i.e.*, ϕF is defined by $\langle \phi F, f\rangle = \langle F, f \circ \phi\rangle$ for all $f \in \mathcal{D}(M)$. With this definition and our earlier convention, $(\phi_1\phi_2)F = \phi_2(\phi_1 F)$, so that we have a right group action as desired. A σ-algebra in $\mathcal{D}'(M)$ may be built up directly from cylinder sets with Borel base,[50] or $\mathcal{D}'(M)$ can be endowed with the weak dual topology and measures constructed on the corresponding Borel σ-algebra.

When $M = \mathbb{R}^d$, it is also convenient to use the configuration space of *tempered* distributions $\mathcal{S}'(\mathbb{R}^d)$, dual to Schwartz' space $\mathcal{S}(\mathbb{R}^d)$. Since $\mathcal{D}(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)$, we have $\mathcal{S}'(\mathbb{R}^d) \subset \mathcal{D}'(\mathbb{R}^d)$. The somewhat smaller configuration space $\mathcal{S}'(\mathbb{R}^d)$ is convenient for representing the group $\mathcal{K}(\mathbb{R}^d)$ or the semidirect product group $\mathcal{S}(\mathbb{R}^d) \times \mathcal{K}(\mathbb{R}^d)$, as described in Sec. 1.4.

Evidently Γ_M, or more specifically $\Gamma_{\mathbb{R}^d}$, may be identified naturally with a subset of $\mathcal{D}'(M)$, or $\mathcal{S}'(\mathbb{R}^d)$, by the correspondence

$$\gamma \to \sum_{\mathbf{x} \in \gamma} \delta_{\mathbf{x}}, \tag{74}$$

where $\delta_{\mathbf{x}} \in \mathcal{D}'(M)$ or $\mathcal{S}'(\mathbb{R}^d)$ is the evaluation functional (*i.e.*, the Dirac δ-function) defined by $\langle \delta_{\mathbf{x}}, f\rangle = f(\mathbf{x})$. The vague topology in Γ_M is in fact the topology it inherits from the weak dual topology. While Γ_M is not a linear space, the larger spaces $\mathcal{D}'(M)$ and $\mathcal{S}'(\mathbb{R}^d)$ are. In addition to linear combinations of evaluation functionals (with possibly distinct real coefficients), $\mathcal{D}'(M)$ or $\mathcal{S}'(\mathbb{R}^d)$ contain other kinds of configurations of physical importance. For example, configurations may include terms that are derivatives of δ-functions, or generalized functions with support on embedded submanifolds of M (see below).

The configuration space of countable subsets

A natural configuration space still larger than Ω_M, that Moschella and I have found especially useful, consists of the space Σ_M of *all finite or countably infinite subsets of* M. We write

$$\Sigma_M = \bigsqcup_{N=0}^{\infty} \Gamma_M^{(N)} \bigsqcup \Sigma_M^{(\infty)}, \tag{75}$$

where $\Gamma_M^{(N)}$ is as above, and $\Sigma_M^{(\infty)}$ consists of all countably infinite subsets. Evidently $\Gamma_M \subset \Sigma_M$, but now there can also be *accumulation points* for configurations in $\Sigma_M^{(\infty)}$, giving us the possibility of fractals or of point-like approximations to manifolds embedded in M. Let us adopt the same convention for Γ_M, Σ_M, and Ω_M, of including the empty configuration. Since M is separable, the closure map $\pi : \Sigma_M \to \Omega_M$ is surjective.

The space Σ_M is of special interest because of its relation to random point processes in M. Let M^n denote the Cartesian product $M \times \cdots \times M$ (n times), and let M^∞ be the projective limit of M^n as $n \to \infty$; thus M^∞ is the space of infinite sequences $(\mathbf{x}_j)$, $j = 1, 2, 3, \ldots$ of elements of M. As usual in probability theory, M^∞ is endowed with the weak product topology, and thus also with the σ-algebra of Borel sets with respect to this topology. Define the map $p : M^\infty \to \Sigma_M$ to take the (ordered) sequence $(\mathbf{x}_j)$ to the (unordered) set $\{\mathbf{x}_j\}$. Then the natural right action of the diffeomorphism group on these spaces commutes with p; that is, for $\phi \in \mathit{Diff}^{\,c}(M)$, define $\tilde{\phi} : M^\infty \to M^\infty$ by $\tilde{\phi}[(\mathbf{x}_j)] = (\phi(\mathbf{x}_j))$, and define $\phi : \Sigma_M \to \Sigma_M$ by $\phi[\{\mathbf{x}_j\}] = \{\phi(\mathbf{x}_j)\}$, whence $p \circ \tilde{\phi} = \phi \circ p$.

Next we introduce in Σ_M the σ-algebra $\mathcal{B}_{\Sigma_M}$, defined to be the largest σ-algebra with the property that p is measurable; $\mathcal{B}_{\Sigma_M}$ is preserved by diffeomorphisms of M. Now probability measures on M^∞ project to probability measures on Σ_M which, for certain classes of self-similar random point processes, are quasiinvariant for the action of $\mathit{Diff}^{\,c}(M)$ on M^∞. This permits the construction of unitary group representations describing extended "clouds" of particles having a point of condensation.[39,51,52,53,54,55,56,57,58]

While the vague topology on Γ_M does not have an analogue on Σ_M, one may instead extend the topology described above on Ω_M to Σ_M. To do this, let a subbase be the family of sets $\{\sigma \in \Sigma_M \mid \sigma \cap \mathcal{O} \neq \emptyset\}_{\mathcal{O} \subseteq M \text{ open}}$. But the Borel σ-algebra for this topology is not large enough to allow us to measure the number of points in a given open set in M. A stronger topology of interest is the Vietoris topology. This is actually a topology defined on the space of *all* subsets of M, for which a subbase consists of the family of all sets $\{X \subset M \mid X \cap \mathcal{O} \neq \emptyset\}_{\mathcal{O} \subseteq M \text{ open}}$, together with all sets of the form $\{X \subset M \mid X \subseteq \mathcal{O}\}_{\mathcal{O} \subseteq M \text{ open}}$. Restricted to Σ_M, it provides a useful topology, whose Borel σ-algebra is contained in $\mathcal{B}_{\Sigma_M}$.

Configuration spaces of embeddings and immersions

Yet another way to approach the characterization of quantum configurations is to consider a given manifold or manifold with boundary L, together with the set of maps α from L to M obeying some specified regularity and continuity properties (for which there are numerous possible choices) that are respected by diffeomorphisms. Then L is the *parameter space* for a class of configurations, and M is the *target space*. For example, L might be the circle S^1, or the closed interval $[0, 2\pi]$, so that configurations are (respectively) closed strings (loops) or open strings (arcs) in M. Further possibilities include configurations that are ribbons, tubes, or higher-dimensional submanifolds of M.

When α is injective (so that self-intersection of the image of L in the target space is not permitted), we have a configuration space of *embeddings* $Emb(L, M)$; without this restriction, it is a space of *immersions* $Imm(L, M)$; so that $Emb(L, M) \subset Imm(L, M)$.

Note too that we may consider either *parameterized* or *unparameterized* configurations. A parameterized configuration is just the map $\alpha(\theta)$, $\theta \in L$. For $\phi \in \mathit{Diff}^c(M)$, $\phi : \alpha \to \phi \circ \alpha$ defines the (right) group action on the space of parameterized configurations $Imm(L, M)$, and this action leaves $Emb(L, M)$ invariant as a subset. But in addition, the group $\mathit{Diff}(L)$ acts on $Imm(L, M)$. It does so (as a left action) by *reparameterization*, so that for $\psi \in \mathit{Diff}(L)$, $\psi : \alpha \to \alpha \circ \psi$. An unparameterized configuration is just the image set $\alpha(L) \subset M$, where the parameterization has been disregarded. Alternatively, under the right conditions on α, we can obtain the set $\alpha(L)$ as an equivalence class of parameterized configurations *modulo* reparameterization; thus, $\alpha_1 \sim \alpha_2$ if and only if $\exists \psi \in \mathit{Diff}(L)$ such that $\alpha_1 \circ \psi = \alpha_2$. Observe that the configuration space of unparameterized immersions of L in M is a subset of the configuration space Ω_M that is invariant (as a set) under the action of $\mathit{Diff}^c(M)$. Thus this description allows us to refine Ω_M as sensitively as desired, according to the topological properties of extended configurations.

We shall come to see that reparameterization invariance has very nice consequences for quantum mechanics, when expressed in terms of diffeomorphism group representations. Note that we can consider the N-particle configuration space $\Gamma_M^{(N)}$ as a special case of $Emb(L, M)$ *modulo* reparameterization, with $L = \{1, \ldots, N\}$. The group $\mathit{Diff}(L)$ reduces in this case to the symmetric group S_N. Likewise the configuration space $\Sigma_M^{(\infty)}$ can be regarded as the special case in which $L = \mathbb{Z}$ (the integers), and $\mathit{Diff}(L)$ is the group S^∞ of all bijections of $\mathbb{Z}$.

Marked configuration spaces

As before, let M be the manifold of physical space. Let S be another manifold, the "internal space" or *mark space*, introduced to describe some possible internal degrees of freedom of the particles in a statistical theory or quantum theory. Frequently S will be a homogeneous space for some internal symmetry group. A single-particle configuration is then described by an element of a bundle space $\hat{M}$ over M equipped with a projection map $p : \hat{M} \to M$, with fibers $p^{-1}(\mathbf{x}) \cong S$.

In the most interesting applications M is non-compact, while S may or may not be compact. Restricting ourselves to the case of a trivial bundle, we take $\hat{M} = M \times S$ and $p(\mathbf{x}, s) = \mathbf{x}$. Naturally $M \times S$ is just another manifold, and we might consider the group $\mathit{Diff}^c(M \times S)$ acting on it. But a general diffeomorphism of $M \times S$ does not respect the assignment of a copy of S to each particle. Writing $(\mathbf{x}', s') = \hat{\phi}(\mathbf{x}, s)$, where $\hat{\phi}$ is a diffeomorphism of $M \times S$, the condition desired is that $\mathbf{x}' = \phi(\mathbf{x})$, while $s' = \psi(\mathbf{x}, s)$ — that is, $\mathbf{x}'$ depends only on $\mathbf{x}$ and not on s. Here ϕ is a diffeomorphism of M, while for each fixed $\mathbf{x}$ the condition $s' = \psi(\mathbf{x}, s)$ defines a diffeomorphism of S. Then $\hat{\phi}$ respects p and is a bundle diffeomorphism, in that $p \circ \hat{\phi}$ is well-defined and equals $\phi \circ p$. With $\hat{\phi}_3 = \hat{\phi}_1 \hat{\phi}_2$, we have $\hat{\phi}_3(\mathbf{x}, s) = [\hat{\phi}_1 \hat{\phi}_2](\mathbf{x}, s) = \hat{\phi}_2(\phi_1(\mathbf{x}), \psi_1(\mathbf{x}, s)) = (\phi_2(\phi_1(\mathbf{x})), \psi_2(\phi_1(\mathbf{x}), \psi_1(\mathbf{x}, s)))$, so that $\phi_3 = \phi_1 \phi_2$ while $s'' = \psi_3(\mathbf{x}, s) = \psi_2(\phi_1(\mathbf{x}), \psi_1(\mathbf{x}, s)))$.

We also require the support of $\hat{\phi}$ to be compact in M; *i.e.*, to be contained in a set $K \times S$ for some compact region $K \subset M$. This condition is stronger than requiring ϕ to be compactly supported; it means that for $\mathbf{x}$ outside the region K, we not only have $\mathbf{x}' = \mathbf{x}$ but also $s' = s$. As these constraints respect the composition of diffeomorphisms in $\mathit{Diff}^c(M \times S)$, they define a subgroup whereby S is treated differently from M with respect to the action of diffeomorphisms.

For some applications, one may impose additional conditions according to the particular situation. For instance, when S is a homogeneous space for a finite-dimensional internal symmetry group G we may restrict ourselves to diffeomorphisms $\hat{\phi}$ such that for all $\mathrm{x} \in M$, the diffeomorphism $\psi(\mathrm{x}, s)$ of S corresponds to the action of an element of G on S. The semidirect product $Map^c(M,G) \times Diff^c(M)$ introduced in Sec. 1.5 is realized naturally by setting S equal to the group manifold of G, so that it is a homogeneous space for the action of G on itself by right multiplication. With $g_1, g_2 \in Map^c(M,G)$ and $s \in G$, we then have $s' = sg_1(\mathrm{x}) = \psi_1(\mathrm{x}, s)$ and $s'' = sg_1(\mathrm{x})g_2(\phi_1(\mathrm{x})) = s[g_1(\phi_1 g_2)](\mathrm{x})$, consistent with the semidirect product group law $(g_1, \phi_1)(g_2, \phi_2) = (g_1(\phi_1 g_2), \phi_1\phi_2)$.

Now the space $\hat{\Gamma}_M$ of *locally finite marked configurations* is defined by

$$\hat{\Gamma}_M = \{\hat{\gamma} \in \Gamma_{M\times S} \mid (\forall \mathrm{x} \in M)\ |\hat{\gamma} \cap (\{\mathrm{x}\} \times S)| = 0 \text{ or } 1\}, \qquad (76)$$

where $\Gamma_{M\times S}$ is defined from Eq. (72). For $\hat{\gamma} \in \hat{\Gamma}_M$, there is a unique corresponding configuration $\gamma \in \Gamma_M$ given by $\{\mathrm{x} \mid |\hat{\gamma} \cap (\{\mathrm{x}\} \times S)| = 1\}$; and a *single* point $s \in S$ is associated with each $\mathrm{x} \in \gamma$, so that at most one particle can occupy a point in the physical space.

This framework is natural for describing various physical examples, such as a gas of hadrons with internal quantum numbers derived from $SU(3)$ symmetry. Other possibilities include letting S be a higher-dimensional sphere or torus, with $G = Diff(S)$, to model the compactified spatial dimensions in a critical string theory; or letting S be an infinite-dimensional space of pointed loops in a target space M', with $G = Diff^c(M')$.[59]

Configuration space derived from generalized vector fields

The final possibility we mention here is to make use of the *coadjoint representation* of the group $Diff^c(M)$, which leads to a configuration space that is natural in the geometric quantization framework.

Earlier we described the adjoint representation of a Lie group G, with $Ad(g) : \mathcal{G} \to \mathcal{G}$ for $g \in G$. Now let $\mathcal{G}'$ be the dual space to $\mathcal{G}$; that is, the space of continuous linear functionals on $\mathcal{G}$. For $\eta \in \mathcal{G}'$, let $\langle \eta, A\rangle$ denote the value of η at $A \in \mathcal{G}$. We next define the coadjoint representation of G as a right action on $\mathcal{G}'$, given by $\langle Coad(g)\eta, A\rangle = \langle \eta, Ad(g)A\rangle$. For finite-dimensional groups, $\mathcal{G}'$ is isomorphic as a vector space to $\mathcal{G}$; but when G is an infinite-dimensional group of the kind that we consider, then $\mathcal{G}'$ is in a sense larger than $\mathcal{G}$.

In particular, the dual space to $vect^c(M)$ is the space $vect^c(M)'$ of *generalized* vector fields; that is (intuitively speaking), vector fields whose *components* belong to $\mathcal{D}'(M)$. The coadjoint representation of $Diff^c(M)$ acts thus on generalized vector fields. It might seem, then, that we could just take $vect^c(M)'$ as our configuration space. However, from the point of view of geometric quantization, this space is not the configuration space but corresponds rather to a classical phase space. A bit more work is necessary to distinguish the "position-like" coordinates (that characterize the configuration space) from the "momentum-like" coordinates, in a way that is consistent with the group action. When we consider a *coadjoint orbit* under $Diff^c(M)$, configurations may be identified (under the right conditions) with *leaves in a foliation* of the orbit. Then *equivalence classes of generalized vector fields* define the elements of the configuration space.

Quantization on coadjoint orbits of the group of volume-preserving diffeomorphisms is especially useful in the description of quantized vortex configurations in ideal, incompressible superfluids. Further discussion of these topics is beyond the scope of the present lecture notes.[7,60,61,62,63,64,65,66]

3.3. *Orbits in $\mathcal{S}'(\mathbb{R}^d)$*

In this subsection, we consider how the "method of semidirect products" allows us to obtain measures on the space $\mathcal{S}'(\mathbb{R}^d)$, quasiinvariant under the group $Diff^c(\mathbb{R}^d)$ or the larger group $\mathcal{K}(\mathbb{R}^d)$ of diffeomorphisms that together with all derivatives become rapidly trivial at infinity. We obtain measures and corresponding irreducible representations carried by N-point configuration spaces $\Gamma^{(N)}_{\mathbb{R}^d}$, regarding each of these (for fixed N) as an orbit $\Delta^{(N)}_{\mathbb{R}^d}$ in $\mathcal{S}'(\mathbb{R}^d)$ under the diffeomorphism group. We also consider cocycles on these orbits corresponding to the N-particle Bose and Fermi representations given by Eqs. (64). These results enable us to understand the N-particle representations of the diffeomorphism group in the general framework described by Eq. (68). Finally we mention some other orbits and their possible physical interpretations.

Then in Sec. 4.1, we give a concise review of Mackey's theory of induced representations.[67,68,69] Section 4.2 makes use of ideas motivated by this theory to understand the inequivalent cocycles on the spaces $\Gamma^{(N)}_{\mathbb{R}^d}$. This leads to important insight into how representations of the symmetric group S_N enter the picture. In Sec. 5, we see how these ideas predict "topological" effects when the physical space itself is non-simply connected, "exotic statistics" for particles in two space dimensions.[70,71]

Measures on a space of distributions

For specificity let us work with the group $\mathcal{S}(\mathbb{R}^d) \times \mathcal{K}(\mathbb{R}^d)$, and see first in detail how to obtain Eq. (68) on the configuration space $\mathcal{S}'(\mathbb{R}^d)$.

The function space $\mathcal{S}(\mathbb{R}^d)$ has many useful technical properties as a topological space. In particular, it is a *nuclear space* in the sense defined in the important book by Gelfand and Vilenkin.[50] Following the discussion there, the (generally linear) complex-valued functional $L(f)$ on $\mathcal{S}(\mathbb{R}^d)$ is called *positive definite* if and only if

$$\sum_{j,k=1}^{m} \overline{\lambda}_k \lambda_j L(f_j - f_k) \geq 0 \quad (\forall f_1, \ldots, f_m \in \mathcal{S}(\mathbb{R}^d))\,(\forall \lambda_1, \ldots, \lambda_m \in \mathbb{C}). \tag{77}$$

Using the fact that $\mathcal{S}(\mathbb{R}^d)$ is a nuclear space, we then have the following theorem, which is the analog for nuclear spaces of Bochner's theorem: The functional $L(f)$ is the Fourier transform of a cylinder set probability measure μ on the configuration space $\mathcal{S}'(\mathbb{R}^d)$ if and only if $L(f)$ is positive definite, (sequentially) continuous, and $L(0) = 1$. In that case, we have

$$L(f) = \int_{F \in \mathcal{S}'(\mathbb{R}^d)} e^{i\langle F, f\rangle}\, d\mu(F)\,. \tag{78}$$

Suppose now that we have a CUR $U(f)$ of the additive group $\mathcal{S}(\mathbb{R}^d)$ in a Hilbert space $\mathcal{H}$. The representation is called *cyclic* if there is a vector $\Omega \in \mathcal{H}$ such that the set $\{U(f)\Omega \,|\, f \in \mathcal{S}(\mathbb{R}^d)\}$ spans a dense subspace $\mathcal{H}_\Omega$ of $\mathcal{H}$. Then Ω is called a *cyclic vector* for the representation. Given the CUR $U(f)$ with normalized cyclic vector Ω, the functional $L(f) = (\Omega, U(f)\Omega)$ satisfies the conditions of the preceding theorem, and is thus the Fourier transform of a measure μ on $\mathcal{S}'(\mathbb{R}^d)$. In this case, we can realize the Hilbert space $\mathcal{H}$ as $L^2_{d\mu}(\mathcal{S}'(\mathbb{R}^d), \mathbb{C})$. The cyclic vector is given by the function $\Omega(F) \equiv 1$, and the inner product is given by $(\Phi, \Psi) = \int_{\mathcal{S}'(\mathbb{R}^d)} \overline{\Phi(F)}\, \Psi(F) d\mu(F)$. The unitary operators $U(f)$ act by multiplication,

$$[U(f)\Psi](F) = e^{i\langle F, f\rangle} \Psi(F). \tag{79}$$

So we have simultaneously "diagonalized" all the operators $U(f)$, which are associated with the positional densities of the particle numbers or the particle masses.

Next suppose that $U(f)V(\phi)$ is a CUR of the semidirect product group $\mathcal{S}(\mathbb{R}^d) \times \mathcal{K}(\mathbb{R}^d)$ in $\mathcal{H}$. Then for any $\phi \in \mathcal{K}(\mathbb{R}^d)$, the vector $V(\phi)\Omega$ is likewise a cyclic vector for $U(f)$, and $L_\phi(f) = (V(\phi)\Omega, U(f)V(\phi)\Omega)$ is likewise the Fourier transform of a measure μ_ϕ. But by the semidirect product group law in Eq. (61), $L_\phi(f) = L(f \circ \phi^{-1})$; so that

$$L_\phi(f) = \int_{F \in \mathcal{S}'(\mathbb{R}^d)} e^{i\langle F, f\circ\phi^{-1}\rangle}\, d\mu(F) = \int_{F \in \mathcal{S}'(\mathbb{R}^d)} e^{i\langle \phi^{-1}F, f\rangle}\, d\mu(F)$$

$$= \int_{F \in \mathcal{S}'(\mathbb{R}^d)} e^{i\langle F, f\rangle}\, d\mu(\phi F)\,, \tag{80}$$

where the last step is just a change of variable. Hence $d\mu_\phi(F) = d\mu(\phi F)$. But from the inner product in the definition of $L_\phi(f)$, we also have

$$L_\phi(f) = \int_{F \in \mathcal{S}'(\mathbb{R}^d)} e^{i\langle F, f\rangle} |[V(\phi)\Omega](F)|^2 d\mu(F)\,. \tag{81}$$

Comparing the two Eqs. (80) and (81), we observe directly that $d\mu_\phi(F) = |[V(\phi)\Omega](F)|^2 d\mu(F)$. Hence μ_ϕ is absolutely continuous with respect to μ (meaning that any set of μ-measure zero is also of μ_ϕ-measure zero). The Radon–Nikodym derivative exists, and is given by

$$\frac{d\mu_\phi}{d\mu}(F) = |[V(\phi)\Omega](F)|^2, \tag{82}$$

and

$$[V(\phi)\Omega](F) = \chi_\phi(F)\sqrt{\frac{d\mu_\phi}{d\mu}(F)} \qquad \text{a.e. } (\mu), \tag{83}$$

where for ϕ given, $\chi_\phi(F)$ is a complex-valued function of modulus one on $\mathcal{S}'(\mathbb{R}^d)$, defined almost everywhere with respect to the measure μ.

Finally let us apply the unitary operator $V(\phi)$ to a general vector $\Psi \in \mathcal{H}_\Omega$. Writing $\Psi = \sum_{j=1}^N \lambda_j U(f_j)\Omega$ $(\lambda_j \in \mathbb{C})$, and using Eq. (61), we have

$$V(\phi)\Psi = \{V(\phi)[\sum_{j=1}^{N} \lambda_j U(f_j)]\, V(\phi)^{-1}\}\, V(\phi)\Omega = [\sum_{j=1}^{N} \lambda_j U(f_j \circ \phi)] V(\phi)\Omega. \tag{84}$$

As Ψ takes the general form $\Psi(F) = \sum_{j=1}^N \lambda_j\, e^{i\langle F, f\rangle}$ in $L^2_{d\mu}(\mathcal{S}'(\mathbb{R}^d), \mathbb{C})$, we then obtain from Eq. (83) the desired expression,

$$[V(\phi)\Psi](F) = \chi_\phi(F)\Psi(\phi F)\sqrt{\frac{d\mu_\phi}{d\mu}(F)} \qquad \text{a. e. } (\mu), \tag{85}$$

where the action of ϕ on F is given by $\langle \phi F, f\rangle = \langle F, f \circ \phi\rangle$. We have demonstrated the formula in Eq. (85) on $\mathcal{H}_\Omega$; but the continuity of $V(\phi)$ as a bounded, linear operator in $\mathcal{H}$ allows us to infer that the same formula holds on the closure of $\mathcal{H}_\Omega$, which is all of $\mathcal{H}$. The complex-valued function $\chi_\phi(F)$ in Eqs. (83) and (85) satisfies, for any pair of diffeomorphisms, the cocycle equation (71) almost everywhere (μ), with γ standing for the distribution F.

To sum up, we have realized an arbitrary CUR of $\mathcal{S}(\mathbb{R}^d) \times \mathcal{K}(\mathbb{R}^d)$, cyclic for the unitary representation of the Abelian subgroup $\mathcal{S}(\mathbb{R}^d)$, in the general form given by Eq. (68) but with the specialization $\mathcal{W} = \mathbb{C}$. We have done so using as our "universal configuration space" $\Pi = \mathcal{S}'(\mathbb{R}^d)$. The non-cyclic case, which we shall not discuss here (but see the discussion of induced representations in Sec. 4), requires that Ψ take values in a higher-dimensional space $\mathcal{W}$, and that χ_ϕ define a unitary operator-valued 1-cocycle acting in $\mathcal{W}$.

Ergodicity and irreducibility

Next we consider in a little more detail the relationship between irreducibility of the representation realized in this way, and ergodicity of the measure μ under the action of the group.

Suppose we have a configuration space Δ that is a measurable space, a group G acting measurably on Δ, and a measure μ on Δ that is quasi-invariant under the action of elements g of G. We call a measurable set $B \subseteq \Delta$ *almost surely invariant* (with respect to μ) if and only if $(\forall g \in G)$, $\mu(gB - B) = 0$. Evidently such sets include (but are not restricted to) invariant sets as well as arbitrary sets of μ-measure zero. But the measure zero set $gB - B$ may depend on the choice of g, and there is in general no guarantee that the union (over $g \in G$) of all such sets is of zero measure. Thus let us call μ *ergodic* for the action of G if any almost surely invariant B that is of positive measure is necessarily of full measure; that is, the measure of its complement $\Delta - B$ is zero. This is a logically stronger sense of ergodicity than that mentioned in the previous section.

Now we shall show that the representation $U(f)V(\phi)$ is irreducible if and only if μ is ergodic.

First, suppose μ is not ergodic. Then there exist measurable sets $B_1, B_2 \subset \mathcal{S}'(\mathbb{R}^d)$, with the properties that $\mu(B_1) > 0$ and $\mu(B_2) > 0$, and with $\mu(\phi B_1 - B_1) = 0$ and $\mu(\phi B_2 - B_2) = 0\ (\forall \phi \in \mathcal{K}(\mathbb{R}^d))$. Let $\mathcal{H}_1$ be the subspace of functions that vanish almost everywhere on $\mathcal{S}'(\mathbb{R}^d) - B_1$, and similarly define $\mathcal{H}_2$. These are non-empty, closed subspaces of $L^2_{d\mu}(\mathcal{S}'(\mathbb{R}^d), \mathbb{C})$. By Eqs. (79) and (85) they are invariant under all the operators $U(f)$ and $V(\phi)$, and the representation is reducible.

Conversely, suppose that the representation is not irreducible, so there exists a nontrivial, proper closed invariant subspace $\mathcal{H}_1 \subset \mathcal{H}$. Let P be the operator of orthogonal projection onto $\mathcal{H}_1$; then P commutes with all the operators $U(f)V(\phi)$. Because P commutes with the $U(f)$, it is easy to show that on the dense subspace $\mathcal{H}_\Omega$ it acts as a multiplication operator by the measurable function $P(F) = [P\Omega](F)$. This action extends to all of $\mathcal{H}$ by continuity of the operator P. Since P is a projection, $P(F)^2 = P(F)$ a.e. (μ), so that $P(F) = 1$ or $P(F) = 0$ a.e. (μ). And because $\mathcal{H}_1$ is non-empty and not all of $\mathcal{H}$, P is not the zero operator or the identity operator, so that $P(F)$ is not almost everywhere zero or almost everywhere one. There must exist disjoint sets $B_1, B_2 \subset \mathcal{S}'(\mathbb{R}^d)$, both having positive measure, with $P(F) = 1$ on B_1 and $P(F) = 0$ on B_2. The subspace $\mathcal{H}_1$ corresponds to the set of L^2 functions that vanish almost everywhere on $\mathcal{S}'(\mathbb{R}^d) - B_1$. Applying $V(\phi)$ for arbitrary ϕ to the vector $P\Omega \in \mathcal{H}_1$, we see from Eq. (85) that invariance of $\mathcal{H}_1$ implies $\mu(\phi B_1 - B_1) = 0$. Thus B_1 is an almost surely invariant set of less than full measure, and μ is not ergodic. This completes the argument.

Ergodicity, then, is necessary and sufficient for irreducibility in the cyclic case. But there are two rather different ways in which a measure μ on $\mathcal{S}'(\mathbb{R}^d)$ could be ergodic. First, it might be concentrated on a single *orbit* in $\mathcal{S}'(\mathbb{R}^d)$. Alternatively, the measure of every orbit could be zero, so that any invariant set of positive measure is an uncountable union of orbits. In this case, we call μ *strictly ergodic*.

Measures on N-particle orbits

Let us consider some examples of orbits. One approach is to take a particular element of $\mathcal{S}'(\mathbb{R}^d)$ and consider all elements obtained by applying diffeomorphisms to it.

Suppose we start with the evaluation functional $\delta_{\mathbf{y}} \in \mathcal{S}'(\mathbb{R}^d)$, (for fixed $\mathbf{y} \in \mathbb{R}^d$) defined by $\langle \delta_{\mathbf{y}}, f\rangle = f(\mathbf{y})\ (\forall f \in \mathcal{S}(\mathbb{R}^d))$. Then the orbit containing $\delta_{\mathbf{y}}$ is the set $\Delta^{(1)} = \{\phi\delta_{\mathbf{y}} \,|\, \phi \in \mathcal{K}(\mathbb{R}^d)\}$. But $\langle \phi\delta_{\mathbf{y}}, f\rangle = \langle \delta_{\mathbf{y}}, f \circ \phi\rangle = f(\phi(\mathbf{y})) = \langle \delta_{\phi(\mathbf{y})}, f\rangle$, whence $\phi\delta_{\mathbf{y}} = \delta_{\phi(\mathbf{y})}$. Since $(\forall\, \mathbf{x} \in \mathbb{R}^d)(\exists \phi \in \mathcal{K}(\mathbb{R}^d))$ such that $\mathbf{x} = \phi(\mathbf{y})$, we have the $\mathcal{K}$-orbit $\Delta^{(1)} = \{\delta_{\mathbf{x}} \,|\, \mathbf{x} \in \mathbb{R}^d\}$ under the natural action of the diffeomorphism group. The Lebesgue measure $d\mathbf{x} = dx^1 \cdots dx^d$ on $\mathbb{R}^d$, or any measure μ equivalent to it, immediately gives us a measure on $\mathcal{S}'(\mathbb{R}^d)$, concentrated on $\Delta^{(1)}$, that is quasiinvariant under diffeomorphisms.

Then we have the Hilbert space $\mathcal{H}_1 = L^2_{d\mu}(\Delta^{(1)}, \mathbb{C})$. With the choice $d\mu = d\mathbf{x}$, $\mathcal{H}_1$ is straightforwardly identified with the usual 1-particle Hilbert space $L^2_{d\mathbf{x}}(\mathbb{R}^d,\mathbb{C})$. For $\Psi \in \mathcal{H}_1$ we may write $\Psi = \Psi(\delta_\mathbf{x})$ a. e. (μ). Then

$$[U(f)\Psi](\delta_\mathbf{x}) = e^{\,i\langle\delta_\mathbf{x},f\rangle}\,\Psi(\delta_\mathbf{x}) = e^{\,if(\mathbf{x})}\,\Psi(\delta_\mathbf{x}). \tag{86}$$

Taking the cocycle in Eq. (85) to be identically one, as we may always do, we complete a representation on the 1-particle orbit by writing

$$[V(\phi)\Psi](\delta_\mathbf{x}) = \Psi(\delta_{\phi(\mathbf{x})})\sqrt{\frac{d\mu_\phi}{d\mu}(\delta_\mathbf{x})}\,. \tag{87}$$

Thus we recover the 1-particle representation of Eqs. (64), having simply replaced the particle coordinate $\mathbf{x}$ by the distribution $\delta_\mathbf{x}$. Thus $\Delta^{(1)}$ is identified with $\Gamma^{(1)}_{\mathbb{R}^d}$.

We remark that if $\Omega(\delta_\mathbf{x}) \equiv 1$ is to be a (normalizable) cyclic vector as in the immediately preceding discussion, so that $L(f) = (\Omega, U(f)\Omega)$ is the Fourier transform of a probability measure as in Eq. (78), then we need $\mu(\Delta^{(1)}) = 1$. The Lebesgue measure $d\mathbf{x}$ itself does not qualify, since it is infinite. Instead we must consider a normalized measure equivalent to it. Therefore, we may make the choice of writing $d\mu(\delta_\mathbf{x}) = |\Phi_0(\mathbf{x})|^2 d\mathbf{x}$, where $|\Phi_0(\mathbf{x})|^2$ is integrable, positive almost everywhere, and $\int|\Phi_0(\mathbf{x})|^2 d\mathbf{x} = 1$. Then $\Omega(\delta_\mathbf{x}) \in L^2_{d\mu}(\Delta^{(1)},\mathbb{C})$ is cyclic under the representation in Eq. (86).

A second, alternative choice is to replace $d\mu(\delta_\mathbf{x})$ by the Lebesgue measure $d\mathbf{x}$, so that Φ_0 is the vector cyclic under the representation U given by Eq. (86). With the first choice, the square root of the Radon–Nikodym derivative in Eq. (87) becomes

$$\sqrt{\frac{d\mu_\phi}{d\mu}(\delta_\mathbf{x})} = \frac{|\Phi_0(\phi(\mathbf{x}))|}{|\Phi_0(\mathbf{x})|}\sqrt{\mathcal{J}_\phi(\mathbf{x})}\,, \tag{88}$$

which is defined almost everywhere (μ) because $|\Phi_0(\mathbf{x})| > 0$ a. e. (μ). With the second choice, the square root of the Radon–Nikodym derivative is simply $\sqrt{\mathcal{J}_\phi(\mathbf{x})}$ as in Eqs. (64). It is natural to think of choosing the vector Φ_0 to be the lowest-energy state of a 1-particle Hamiltonian: $H\Phi_0 = E_0\Phi_0$.

More generally, let us obtain an orbit by starting with the distribution $\Sigma_{j=1}^N \delta_{\mathbf{y}_j} \in \mathcal{S}'(\mathbb{R}^d)$, for a given set of (distinct) points $\{\mathbf{y}_1,\ldots,\mathbf{y}_N\} \subset \mathbb{R}^d$. Because we have a *sum* of δ-functionals, such a distribution depends only on the *set* and not on the order in which its elements are listed. Applying a diffeomorphism ϕ to the distribution gives us $\Sigma_{j=1}^N \delta_{\phi(\mathbf{y}_j)}$. Moreover, for an arbitrary N-point subset $\{\mathbf{x}_1,\ldots,\mathbf{x}_N\} \subset \mathbb{R}^d$, there exist diffeomorphisms $\phi \in \mathcal{K}(\mathbb{R}^d)$ for which $\Sigma_{j=1}^N \delta_{\mathbf{x}_j} = \Sigma_{j=1}^N \delta_{\phi(\mathbf{y}_j)}$. The desired orbit is therefore $\Delta^{(N)}_{\mathbb{R}^d} = \{\Sigma_{j=1}^N \delta_{\mathbf{x}_j} \in \mathcal{S}'(\mathbb{R}^d)\,|\,\mathbf{x}_j \in \mathbb{R}^d, \mathbf{x}_j \neq \mathbf{x}_k \text{ for } j \neq k\}$, and we have a natural identification of the distributions in this orbit with the elements of the configuration space $\Gamma^{(N)}_{\mathbb{R}^d}$.

Again, any measure concentrated on $\Delta^{(N)}_{\mathbb{R}^d}$ that is equivalent (locally) to the Lebesgue measure $d\mathbf{x}_1\cdots d\mathbf{x}_N$ is quasiinvariant under diffeomorphisms. It is easiest just to use the Lebesgue measure, but it may be more instructive to use a normalized measure μ (see below). The former choice allows us to construct the Hilbert space $\mathcal{H}_N = L^2_{d\mathbf{x}_1\cdots d\mathbf{x}_N}(\Delta^N,\mathbb{C})$; then we write $\Psi \in \mathcal{H}_N$ as a square-integrable function on the configuration space $\Delta^{(N)}_{\mathbb{R}^d}$. We have

$$[U(f)\Psi](\Sigma_{j=1}^N \delta_{\mathbf{x}_j}) = e^{\,i\langle\Sigma_{j=1}^N \delta_{\mathbf{x}_j},f\rangle}\,\Psi(\Sigma_{j=1}^N \delta_{\mathbf{x}_j})$$

$$= e^{\,i\Sigma_{j=1}^N f(\mathbf{x}_j)}\,\Psi(\Sigma_{j=1}^N \delta_{\mathbf{x}_j}), \tag{89}$$

which should be compared with the first of Eqs. (64).

Next we take the important step of making an identification between $\mathcal{H}^{(\mathrm{s})}_N$ or $\mathcal{H}^{(\mathrm{a})}_N$ and $\mathcal{H}_N$ for $N \geq 2$ (which was not a problem for $N = 1$). Recall that totally symmetric or antisymmetric wave functions $\Psi^{(\mathrm{s})}_N$ or $\Psi^{(\mathrm{a})}_N$ are defined on the coordinate space of *ordered* N-tuples of points $(\mathbf{x}_1,\ldots,\mathbf{x}_N)$, with $\mathbf{x}_j \in \mathbb{R}^d$; but each satisfies a symmetry condition whereby its value at one such N-tuple actually determines its values at $N!$ points in the coordinate-space that are related by permutations. Let us therefore distinguish a *preferred sector* in the coordinate space. We shall do this conventionally for the manifold $\mathbb{R}^d$. Given two distinct points $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^d$, introduce the "lexicographical ordering" whereby $\mathbf{x} < \mathbf{y}$ if $x^1 < y^1$, or if when $x^j = y^j$ for $j \leq k$ then $x^{k+1} < y^{k+1}$ (from $k = 1$ up to $k = N-1$). The preferred sector will then be $\{(\mathbf{x}_1,\ldots,\mathbf{x}_N)\,|\,\mathbf{x}_1 < \ldots < \mathbf{x}_N\}$, whose points are in natural one-to-one correspondence with $\Delta^{(N)}_{\mathbb{R}^d}$.

Now either Hilbert space $\mathcal{H}^{(\mathrm{s})}_N$ or $\mathcal{H}^{(\mathrm{a})}_N$ $(N \geq 2)$ may be mapped unitarily to $\mathcal{H}_N = L^2_{d\mathbf{x}_1\cdots d\mathbf{x}_N}(\Delta^{(N)}_{\mathbb{R}^d},\mathbb{C})$, just by restricting the values of $\Psi^{(\mathrm{s})}$ or $\Psi^{(\mathrm{a})}$ to the preferred sector and normalizing the wave functions with a factor of $\sqrt{N!}$. Then it is easy to see that the representations $U^{(\mathrm{s})}_N$ and $U^{(\mathrm{a})}_N$ in Eqs. (64) are unitarily equivalent to each other — when written in $\mathcal{H}_N$, both representations act according to Eq. (89). The fermionic and bosonic N-particle representations of the nonrelativistic local current algebra (and the corresponding semidirect product group) are thus modeled on the *same* configuration space, obtained as the $\mathcal{K}$-orbit $\Delta^{(N)}_{\mathbb{R}^d}$ in $\mathcal{S}'(\mathbb{R}^d)$, equipped with a measure locally equivalent to the Lebesgue measure.

Cocycles on N-particle orbits

If these representations are to be unitarily inequivalent, they must be distinguished not by the multiplication operators constructed from the mass density operators $\rho_N(f)$, but by the associated representations $V_N^{(s)}$ and $V_N^{(a)}$ of the diffeomorphism group constructed from the momentum density operators $J_N(\mathbf{g})$. Mathematically, the different exchange statistics for finite particle systems will enter not through the measure on $\mathcal{S}'(\mathbb{R}^d)$ or, correspondingly, the choice of quantum configuration space $\Delta_{\mathbb{R}^d}^{(N)}$, but through a complex-valued (unitary) 1-cocycle for the action of the diffeomorphism group on that configuration space.

It remains to rewrite the representations $V_N^{(s)}$ and $V_N^{(a)}$ in Hilbert spaces of square-integrable functions on $\Delta_{\mathbb{R}^d}^{(N)}$. This is a convenient point at which to introduce normalized measures. Suppose that the totally antisymmetric N-particle wave function $\Phi_0^{(a)}(\mathbf{x}_1, \ldots, \mathbf{x}_N) \in \mathcal{H}_N^{(a)}$ is a cyclic vector for the operators $U_N^{(a)}$ in the fermionic (a) representation of Eqs. (64), with $||\Phi_0^{(a)}|| = 1$. Similarly let the totally symmetric wave function $\Phi_0^{(s)}(\mathbf{x}_1, \ldots, \mathbf{x}_N) \in \mathcal{H}_N^{(s)}$ be a normalized cyclic vector for the bosonic representation $U_N^{(s)}$. As these are cyclic vectors, they are nonvanishing a. e. (in the measure $d\mathbf{x}_1 \cdots d\mathbf{x}_N$). Note also that $|\Phi_0^{(a)}(\mathbf{x}_1, \ldots, \mathbf{x}_N)|^2$ and $|\Phi_0^{(s)}(\mathbf{x}_1, \ldots, \mathbf{x}_N)|^2$ are independent of the order of the arguments $\mathbf{x}_1, \ldots, \mathbf{x}_N$, and thus are fully specified by their values on the preferred sector. The measures

$$d\mu^{(a)}(\Sigma_{j=1}^N \delta_{\mathbf{x}_j}) = |\Phi_0^{(a)}(\mathbf{x}_1, \ldots, \mathbf{x}_N)|^2 d\mathbf{x}_1 \cdots d\mathbf{x}_N ,$$

$$d\mu^{(s)}(\Sigma_{j=1}^N \delta_{\mathbf{x}_j}) = |\Phi_0^{(s)}(\mathbf{x}_1, \ldots, \mathbf{x}_N)|^2 d\mathbf{x}_1 \cdots d\mathbf{x}_N ,$$

are equivalent to each other on $\Delta_{\mathbb{R}^d}^{(N)}$ (*i.e.*, they have the same class of measure zero sets).

Now define the linear operators $Q^{(a)} : \mathcal{H}_N^{(a)} \to L^2_{d\mu^{(a)}}(\Delta_{\mathbb{R}^d}^{(N)}, \mathbb{C})$ and $Q^{(s)} : \mathcal{H}_N^{(s)} \to L^2_{d\mu^{(s)}}(\Delta_{\mathbb{R}^d}^{(N)}, \mathbb{C})$, by:

$$[\, Q^{(a,s)}\, \Psi^{(a,s)}] (\Sigma_{j=1}^N \delta_{\mathbf{x}_j}) = \Psi^{(a,s)}(\mathbf{x}_1, \ldots, \mathbf{x}_N) \Phi_0^{(a,s)}(\mathbf{x}_1, \ldots, \mathbf{x}_N)^{-1} . \quad (90)$$

Since $\Psi^{(a)}$ and $\Phi_0^{(a)}$ (respectively, $\Psi^{(s)}$ and $\Phi_0^{(s)}$) have the same exchange symmetry, and $\Phi_0^{(a)}$ and $\Phi_0^{(s)}$ are almost everywhere nonzero, the right-hand side of Eq. (90) is independent of the order of the arguments and is a well-defined function of $\Sigma_{j=1}^N \delta_{\mathbf{x}_j}$ (a. e.). From the definitions of $d\mu^{(a)}$

and $d\mu^{(a)}$, it is easy to check that $Q^{(a)}$ and $Q^{(s)}$ are unitary. We also have $Q^{(a)}\Phi_0^{(a)} = \Omega^{(a)} \equiv 1$, and $Q^{(s)}\Phi_0^{(s)} = \Omega^{(s)} \equiv 1$.

It is straightforward to determine that the two unitary representations of the diffeomorphism group, $Q^{(a)} V_N^{(a)}(\phi)[Q^{(a)}]^{-1}$ and $Q^{(s)} V_N^{(s)}(\phi)[Q^{(s)}]^{-1}$, act (respectively) on vectors $\Psi \in L^2_{d\mu^{(a)}}(\Delta_{\mathbb{R}^d}^{(N)}, \mathbb{C})$ or $L^2_{d\mu^{(s)}}(\Delta_{\mathbb{R}^d}^{(N)}, \mathbb{C})$ as follows:

$$\left(Q^{(a,s)} V_N^{(a),(s)}(\phi)[Q^{(a,s)}]^{-1} \Psi \right) (\Sigma_{j=1}^N \delta_{\mathbf{x}_j}) =$$

$$= \frac{\Phi_0^{(a,s)}(\phi(\mathbf{x}_1), \ldots, \phi(\mathbf{x}_N))}{\Phi_0^{(a,s)}(\mathbf{x}_1, \ldots, \mathbf{x}_N)} \Psi(\Sigma_{j=1}^N \delta_{\phi(\mathbf{x}_j)}) \sqrt{\prod_{j=1}^N \mathcal{J}_\phi(\mathbf{x}_j)} \,. \quad (91)$$

Again because of the antisymmetry or symmetry of $\Phi_0^{(a),(s)}$ under permutations of the $\mathbf{x}_j$, the ratio of functions entering Eq. (91) is invariant under such permutations. It is thus well-defined a. e. by the specification of the functional $\Sigma_{j=1}^N \delta_{\mathbf{x}_j} \in \Delta_{\mathbb{R}^d}^{(N)}$. Comparing Eq. (91) with Eq. (85), we find the square root of the Radon–Nikodym derivative occurring in Eq. (85) to be the real-valued 1-cocycle,

$$\sqrt{\frac{d\mu_\phi^{(a,s)}}{d\mu^{(a,s)}} (\Sigma_{j=1}^N \delta_{\mathbf{x}_j})} =$$

$$= \frac{|\Phi_0^{(a,s)}(\phi(\mathbf{x}_1), \ldots, \phi(\mathbf{x}_N))|}{|\Phi_0^{(a,s)}(\mathbf{x}_1, \ldots, \mathbf{x}_N)|} \sqrt{\prod_{j=1}^N \mathcal{J}_\phi(\mathbf{x}_j)} \,, \quad (92)$$

while the unitary 1-cocycle in Eq. (85) is just

$$\chi_\phi(\Sigma_{j=1}^N \delta_{\mathbf{x}_j}) = \text{phase} \left[\frac{\Phi_0^{(a,s)}(\phi(\mathbf{x}_1), \ldots, \phi(\mathbf{x}_N))}{\Phi_0^{(a,s)}(\mathbf{x}_1, \ldots, \mathbf{x}_N)} \right] . \quad (93)$$

Let us look at Eqs. (92)–(93). Typically the wave function describing a stationary state in Schrödinger quantum mechanics belongs to the domain of a Hamiltonian that is expressed as a differential operator in the particle coordinates. Thus it is at least a continuously differentiable function of those coordinates. For such a smooth, almost everywhere nonvanishing function $\Phi_0^{(a)}$ or $\Phi_0^{(s)}$, the right-hand sides of the above equations are defined *except at the zeroes* of the wave function — that is, they are defined outside nodal surfaces in configuration space. In the totally symmetric case, it is typically possible to choose the ground state $\Phi_0^{(s)}$ so that it is *nowhere* vanishing; but this is not so for totally antisymmetric wave functions. Thus the existence of measure zero sets where, in particular, $\chi_\phi\,(\Sigma_{j=1}^N \delta_{x_j})$ is undefined, and where Eq. (71) for a 1-cocycle fails, is inevitable! However, we have an explicit handle on these sets — the "almost everywhere" qualification of Eq. (71) has turned out to be not merely an abstract, technical restriction, but a condition characterized by the nodal surfaces of fermionic ground state wave functions. Letting $\mathcal{Z} \subset \Delta_{\mathbb{R}^d}^{(N)}$ be the measure zero set where $\Phi_0^{(a)}$ vanishes and χ_ϕ is undefined, we see that Eq. (71) fails precisely when the configuration γ belongs to the measure zero set $\mathcal{Z} \cup \phi_1^{-1}\mathcal{Z}$. Thus we see that there does not exist a single set of measure zero outside of which Eq. (71) holds for all $\phi_1, \phi_2 \in \mathcal{K}(\mathbb{R}^d)$; indeed, there may be no elements of the configuration space where this is so.

To conclude this subsection, let us discuss the unitary equivalence or inequivalence of the representations defined by apparently distinct 1-cocycles, where the underlying quasiinvariant measures on the configuration space are equivalent.

First consider the N-particle symmetric (s) case of Eq. (91), where $N \geq 2$. Let us introduce the multiplication operator M_0, defined on $L^2_{d\mu^{(s)}}(\Delta_{\mathbb{R}^d}^{(N)}, \mathbb{C})$ as multiplication by $\Phi_0^{(s)}(\mathbf{x}_1, \ldots, \mathbf{x}_N)/|\Phi_0^{(s)}(\mathbf{x}_1, \ldots, \mathbf{x}_N)|$ $=$ phase$[\Phi_0^{(s)}(\mathbf{x}_1, \ldots, \mathbf{x}_N)]$. Then M_0 is a well-defined, unitary operator that commutes with all of the operators $U^{(s)}(f)$. From Eq. (91), it follows straightforwardly that the equivalent unitary representation $M_0\, Q^{s)} V_N^{(s)}(\phi)[Q^{(s)}]^{-1} M_0^{-1}$ is the representation associated with the trivial 1-cocycle, $\chi_\phi(\Sigma_{j=1}^N \delta_{x_j}) \equiv 1$.

Next consider the N-particle antisymmetric (a) case of Eq. (91), with $N \geq 2$. The unitary inequivalence between this representation and the N-particle symmetric representation depends in an essential way on the dimensionality d of the space. Intuitively, this is because diffeomorphisms that become trivial at infinity cannot implement an exchange of distinct particle coordinates on the real line, while they *can* do so in higher-dimensional Euclidean space.

For the case $d = 1$ we may, as in the symmetric case, let M_0 be the operator of multiplication by the phase of an almost everywhere nonvanishing wave function $\Phi_0^{(a)}(x_1, \ldots, x_N)$. We have dropped the bold face notation to remind us that with $d = 1$, the x_j are just real numbers. But here, we specify $(x_1, \ldots, x_N)$ to be in the *preferred sector* $x_1 < \ldots < x_N$ of the space of ordered N-tuples of distinct particle coordinates. The elements of $\mathcal{K}(\mathbb{R}^1)$, being order-preserving diffeomorphisms, act on this space of ordered N-tuples so as to *leave invariant* this preferred sector. Using this

fact, together with Eq. (91), we have the perhaps surprising observation that the N-particle antisymmetric and symmetric representations of the diffeomorphism group $\mathcal{K}(\mathbb{R}^d)$, and hence of the semidirect product group $\mathcal{S}(\mathbb{R}^d) \times \mathcal{K}(\mathbb{R}^d)$, are *unitarily equivalent in one space dimension.*

Thus, to distinguish bosons from fermions when $d = 1$, it is necessary to refer to observables that are not fully expressible in terms of the local current algebra. For example, consider the kinetic energy operator H_0, which acts on some domain in $L^2_{dx_1 dx_2 \cdots dx_N}(\Delta_{\mathbb{R}^1}^{(N)}, \mathbb{C})$ as a differential operator,

$$H_0 = -\frac{\hbar^2}{2m} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} \,. \tag{94}$$

Then H_0 is defined at least on a minimal domain consisting of smooth functions $\Psi(\Sigma_{j=1}^N \delta_{x_j})$ that vanish together with their first derivatives when adjacent particle coordinates come together; *i.e.*, that satisfy the boundary conditions,

$$\lim_{x_j \to x_{j+1}} \Psi(\Sigma_{j=1}^N \delta_{x_j}) = 0 \qquad (j = 1, \ldots, N-1)\,,$$

$$\lim_{x_j \to x_{j+1}} \frac{\partial \Psi(\Sigma_{j=1}^N \delta_{x_j})}{\partial x_j} = \lim_{x_j \to x_{j+1}} \frac{\partial \Psi(\Sigma_{j=1}^N \delta_{x_j})}{\partial x_{j+1}} = 0 \quad (j = 1, \ldots, N-1)\,. \tag{95}$$

But to fully specify the unbounded operator H_0 as a self-adjoint operator in Hilbert space, it is necessary to widen its domain of definition by means of less restrictive boundary conditions — and the different possible choices then lead to physically inequivalent (bosonic, fermionic, or even intermediate) particle systems.

In particular, one way to relax Eqs. (95) is to allow $\Psi(\Sigma_{j=1}^N \delta_{x_j})$ and $\partial \Psi(\Sigma_{j=1}^N \delta_{x_j})/\partial x_j$ to take on arbitrary values in the limit as $x_j \to x_{j+1}$ (for each $j = 1, \ldots, N-1$), but to require that in this limit, $\partial\Psi/\partial x_j = \partial\Psi/\partial x_{j+1}$. Then a vector $\Psi(\Sigma_{j=1}^N \delta_{x_j})$ in the domain of H_0 defines a corresponding *totally symmetric* wave function on the space of coordinates $(x_1, \ldots, x_N)$ that is in the domain of the usual self-adjoint Hamiltonian operator; and we see that we have a bosonic system. Alternatively, requiring $\Psi(\Sigma_{j=1}^N \delta_{x_j})$ to be zero in the limit as $x_j \to x_{j+1}$ (for each $j = 1, \ldots, N-1$), with $\partial\Psi/\partial x_j = -\partial\Psi/\partial x_{j+1}$ in this limit, means that a choice of $\Psi(\Sigma_{j=1}^N \delta_{x_j})$ in the domain of H_0 determines a *totally antisymmetric* wave function on the space of coordinates $(x_1, \ldots, x_N)$ that is in the domain of the usual Hamiltonian operator; and then we have a fermionic system.

Further discussion of unbounded self-adjoint operators and inequivalent self-adjoint extensions of symmetric operators is beyond the scope of these lecture notes; but see, for instance, the discussion of von Neumann's theory of deficiency indices in the instructive book by Reed and Simon.[72]

But $d = 1$ is a very special case. More generally, for $d \geq 2$ and $N \geq 2$, there are diffeomorphisms trivial at infinity that *do* implement the exchange of any pair of particle coordinates. Then the (a) and the (s) representations of Eq. (64) are unitarily *inequivalent* (we omit the details of the proof).[18] We shall see below how such inequivalent representations are obtained from one-dimensional unitary representations of the symmetric group S_N by inducing, a technique that generalizes to predict more exotic possibilities for the quantum statistics of particles.

To sum up, we first wrote down the Bose and Fermi N-particle representations in the Fock space of canonical nonrelativistic field theory, where they are given by Eqs. (64). Now we have explicitly realized them in the form of Eq. (68), over the configuration spaces $\Delta^{(N)}_{\mathbb{R}^d} \cong \Gamma^{(N)}_{\mathbb{R}^d}$, which we obtained as $\mathcal{K}$-orbits in $\mathcal{S}'(\mathbb{R}^d)$. For distinct N, the measures on $\mathcal{S}'(\mathbb{R}^d)$ corresponding to these representations are supported by mutually disjoint orbits, and are thus mutually singular; it follows that the representations for distinct N are mutually inequivalent. For fixed $N \geq 2$, and $d = 1$, the (s) and (a) representations of (64) are, however, unitarily equivalent; while for $d \geq 2$, they are inequivalent as a consequence of the corresponding inequivalent cocycles.

Other finite-dimensional orbits

The N-particle orbits are not the only finite-dimensional orbits in $\mathcal{S}'(\mathbb{R}^d)$ under the action of $\mathcal{K}(\mathbb{R}^d)$. We may, for instance, construct orbits containing functionals with terms that are derivatives of Dirac δ-functions, or multiple derivatives of them.[73] Suppose for specificity that $d \geq 2$, and consider the functional $-\lambda \cdot \nabla\delta_{\mathbf{x}} \in \mathcal{S}'(\mathbb{R}^d)$ that is defined on $f \in \mathcal{S}(\mathbb{R}^d)$ by the formula

$$\langle -\lambda \cdot \nabla\delta_{\mathbf{x}}, f\rangle = \lambda \cdot (\nabla f)(\mathbf{x})\,; \qquad (96)$$

where $\mathbf{x} \in \mathbb{R}^d$ is fixed, and where $\lambda \neq 0$ is a d-component vector. It is straightforward to determine the action of $\phi \in \mathcal{K}(\mathbb{R}^d)$ on $-\lambda \cdot \nabla\delta_{\mathbf{x}}$ using the definition $\langle \phi\,[-\lambda \cdot \nabla\delta_{\mathbf{x}}]\,, f\,\rangle = \langle -\lambda \cdot \nabla\delta_{\mathbf{x}}, f \circ \phi\rangle$. Writing $-\lambda' \cdot \nabla\delta_{\mathbf{x}'} = \phi\,[-\lambda \cdot \nabla\delta_{\mathbf{x}}\,]$, we obtain

$$\mathbf{x}' = \phi(\mathbf{x}),$$

$$[\lambda']^j = \frac{\partial\phi^j}{\partial x^k}(\mathbf{x})\,\lambda^k \quad (j = 1, \ldots, d)\,. \qquad (97)$$

We see from Eq. (97) that the set of pairs $\{(\mathbf{x}, \lambda),\, \lambda \neq 0\}$ labels a single orbit under $\mathcal{K}(\mathbb{R}^d)$, which we shall label $\Delta^{\nabla,(1)} \subset \mathcal{S}'(\mathbb{R}^d)$. [When $d = 1$, diffeomorphisms cannot act so as to change the sign of λ, and we have in that case two distinct orbits: $\{(x, \lambda),\, \lambda < 0\}$ and $\{(x, \lambda),\, \lambda > 0\}$.] Analogously with the N-particle orbits, we now also have orbits whose elements are sums of N derivatives of δ-distributions:

$$\Delta^{\nabla,(N)} = \sum_{j=1}^{N} -\lambda_j \cdot \nabla\delta_{\mathbf{x}_j} \quad (\lambda_j \neq 0, \forall j)\,. \qquad (98)$$

Comparing Eq. (97) with Eq. (22), we also see that it is natural here to interpret λ as a (non-zero) *tangent vector* to the manifold $\mathbb{R}^d$ at $\mathbf{x}$, and λ' as a tangent vector to $\mathbb{R}^d$ at $\mathbf{x}'$; then $\lambda' = [Ad(\phi^{-1})]\lambda$. We have, in accord with our conventions, a right action of the diffeomorphism group on the configuration space — diffeomorphisms act as usual on the manifold, and they lift by means of the usual derivative map to act on tangent vectors to the manifold.

The Lebesgue measure $d\mathbf{x}d\lambda$ defined on the orbit $\Delta^{\nabla,(1)}$ is quasiinvariant under the action (97) of diffeomorphisms. Thus we have a unitary representation of the semidirect product group $\mathcal{S}(\mathbb{R}^d) \times \mathcal{K}(\mathbb{R}^d)$, in the Hilbert space $\mathcal{H} = L^2_{d\mathbf{x}d\lambda}(\Delta^{\nabla,(1)}, \mathbb{C})$ given by

$$[U(f)\Psi](\mathbf{x},\lambda) = \exp[i\lambda \cdot (\nabla f)(\mathbf{x})]\,\Psi(\mathbf{x},\lambda)\,,$$
$$[V(\phi)\Psi](\mathbf{x},\lambda) = \Psi(\phi(\mathbf{x}), [Ad(\phi^{-1})]\lambda)\mathcal{J}_\phi(\mathbf{x})\,. \qquad (99)$$

In Eq. (99), we have chosen the cocycle that is identically one. Note that the Jacobian of ϕ occurs here without the square root sign — one factor of $\sqrt{\mathcal{J}_\phi(\mathbf{x})}$ results from the action of ϕ on $\mathbf{x}$, while another factor of $\sqrt{\mathcal{J}_\phi(\mathbf{x})}$ results from its action in λ.

The corresponding representation of the current algebra (43) is easily obtained from Eqs. (65); but we notice immediately that $\rho(f)$ for $f(\mathbf{x}) \geq 0$ is not going to be positive definite in this representation — a possibility already anticipated in the discussion following those equations. Therefore $\rho(f)$ cannot describe the mass density. Let us instead construct charge density and electric current density operators from Eqs. (99), using $\rho(f)\Psi = (q/i)\,\partial_a\, U(af)\Psi|_{a=0}$ and $J(\mathbf{g})\Psi = (q\hbar/mi)\,\partial_b\, V(\phi^{\mathbf{g}}_b)\Psi|_{b=0}$.

We obtain

$$[\rho(f)\Psi](\mathbf{x},\lambda) = q\,[\lambda \cdot (\nabla f)(\mathbf{x})]\,\Psi(\mathbf{x},\lambda)\,,$$

$$[J(\mathbf{g})\Psi](\mathbf{x},\lambda) = \frac{q\hbar}{m}\frac{1}{2i}\{\mathbf{g}(\mathbf{x})\cdot\nabla\Psi(\mathbf{x},\lambda) + \nabla\cdot[\mathbf{g}(\mathbf{x})\Psi(\mathbf{x},\lambda)]\}$$

$$+\frac{q\hbar}{m}\frac{1}{2i}(\partial_j g^k)(\mathbf{x})\{\lambda^j\frac{\partial}{\partial\lambda^k}\Psi(\mathbf{x},\lambda) + \frac{\partial}{\partial\lambda^k}[\lambda^j\Psi(\mathbf{x},\lambda)]\}, \qquad (100)$$

where as usual summation over the repeated indices j and k is assumed. In order to interpret these equations, let us compare them with a unitary representation of $\mathcal{S}(\mathbb{R}^d)\times\mathcal{K}(\mathbb{R}^d)$ describing two particles having equal and opposite charge. In such a representation, the charge density operators $\rho(f)$ will likewise fail to be positive definite when $f(\mathbf{x})\geq 0$. We use as the configuration space the $\mathcal{K}$-orbit $\Delta^{(2)}_{(1,-1)} = \{\delta_{\mathbf{x}_1}-\delta_{\mathbf{x}_2}\,|\,\mathbf{x}_1\neq\mathbf{x}_2\}\subset\mathcal{S}'(\mathbb{R}^d)$; and with methods that are by now familiar, we obtain:

$$[U(f)\Phi](\mathbf{x}_1,\mathbf{x}_2) = \exp\{i[f(\mathbf{x}_1)-f(\mathbf{x}_2)]\}\,\Phi(\mathbf{x}_1,\mathbf{x}_2),$$

$$[V(\phi)\Phi](\mathbf{x}_1,\mathbf{x}_2) = \Phi(\phi(\mathbf{x}_1),\phi(\mathbf{x}_2))\sqrt{\mathcal{J}_\phi(\mathbf{x}_1)\mathcal{J}_\phi(\mathbf{x}_2)}, \qquad (101)$$

where $\Phi\in L^2_{d\mathbf{x}_1 d\mathbf{x}_2}(\Delta^{(N)}_{(1,-1)},\mathbb{C})$. Suppose now that Φ is such that it vanishes except for small particle separations. For $\mathbf{x}_1$ sufficiently close to $\mathbf{x}_2$ (so that f changes slowly from $\mathbf{x}_1$ to $\mathbf{x}_2$), we have from the first of Eqs. (101)

$$[\rho(f)\Phi](\mathbf{x}_1,\mathbf{x}_2) = q[f(\mathbf{x}_1)-f(\mathbf{x}_2)]\Phi(\mathbf{x}_1,\mathbf{x}_2)$$

$$\approx q[\lambda\cdot(\nabla f)(\mathbf{x})]\Phi(\mathbf{x}_1,\mathbf{x}_2), \qquad (102)$$

where $\lambda = (\mathbf{x}_1-\mathbf{x}_2)$ and $\mathbf{x} = (1/2)(\mathbf{x}_1+\mathbf{x}_2)$. Then we have in this approximation the first of Eqs. (100), with $\Psi(\mathbf{x},\lambda) = \Psi(\mathbf{x}_1,\mathbf{x}_2)$. From the second of Eqs. (101) we write

$$[V(\phi)\Psi](\mathbf{x},\lambda) =$$

$$=\Psi(\tfrac{1}{2}[\phi(\mathbf{x}_1)+\phi(\mathbf{x}_2)],\,\phi(\mathbf{x}_1)-\phi(\mathbf{x}_2))\sqrt{\mathcal{J}_\phi(\mathbf{x}_1)\mathcal{J}_\phi(\mathbf{x}_2)}, \qquad (103)$$

which approximately equals the second of Eqs. (99) when $\mathbf{x}_1$ and $\mathbf{x}_2$ are near each other. So for any particular choice of the group element $(f,\phi)\in\mathcal{S}(\mathbb{R}^d)\times\mathcal{K}(\mathbb{R}^d)$, the unitary representation given by Eqs. (99) is an approximation to the representation given by Eqs. (101) when acting on wave functions describing tightly-bound particles of equal and opposite charge. This justifies the interpretation of Eqs. (99) as describing a *quantum dipole*. The dipole moment $q\lambda$ is not fixed but variable; it ranges over $\mathbb{R}^d-\{0\}$ (when $d\geq 2$). The wave function $\Psi(\mathbf{x},\lambda)$ is a probability amplitude for finding a neutral particle at $\mathbf{x}$ with dipole moment $q\lambda$.

To describe a dipole particle with fixed net charge $q_0 q$, we construct a CUR of $\mathcal{S}(\mathbb{R}^d)\times\mathcal{K}(\mathbb{R}^d)$ on the orbit $\Delta^{\nabla,(1)}_{q_0} = \{q_0\delta_{\mathbf{x}}+\lambda\cdot\nabla\delta_{\mathbf{x}}\,|\,\mathbf{x}\in\mathbb{R}^d,\,\lambda\in\mathbb{R}^d-0\}\subset\mathcal{S}'(\mathbb{R}^d)$. To describe N identical neutral point dipoles in $\mathbb{R}^d$, we naturally use the orbit $\Delta^{\nabla,(N)}$ as the quantum configuration space. Then the orbit $\Delta^{\nabla,(N)}_{q_0}$ is defined analogously, and describes N identical charged point dipoles. The more complicated orbit $\Delta^{(N)}_{q_0}+\Delta^{\nabla,(N)}_{q_1} := \{\Sigma_{j=1}^{N} q_0\,\delta_{\mathbf{x}_j} + \Sigma_{j=N+1}^{2N}[q_1\,\delta_{\mathbf{x}_j}+\lambda_j\cdot\nabla\delta_{\mathbf{x}_j}]\}$ describes $2N$ point particles in $\mathbb{R}^d$; of these, N are ordinary particles with charge $q_0 q$ (and no dipole moment), while the remaining N are indistinguishable point dipoles having net charge $q_1 q$.

Similarly, quadrupole and higher multipole point particles correspond to interesting classes of orbits in $\mathcal{S}'(\mathbb{R}^d)$ under the diffeomorphism group, and corresponding representations of the local current algebra. The structure of these orbits is actually quite nontrivial, and the resulting particles can be understood as interesting, tightly-bound composite systems of three, four, or more charged components.[73]

The quantum statistics of dipole, quadrupole, and higher multipole point particles can be described by cocycles on the corresponding orbits.

In short, a great variety of distinct quantum systems in $\mathbb{R}^d$ are described by inequivalent CURs of the semidirect product group $\mathcal{S}(\mathbb{R}^d)\times\mathcal{K}(\mathbb{R}^d)$. Many of these are classified by quasiinvariant measures concentrated on finite-dimensional configuration spaces, that in the case of irreducible representations can be realized as single $\mathcal{K}$-orbits in $\mathcal{S}'(\mathbb{R}^d)$. Additional possibilities, particularly those associated with the statistics of indistinguishable particles, are classified by unitary 1-cocycles on these orbits.

But not all irreducible CURs of $\mathcal{S}(\mathbb{R}^d)\times\mathcal{K}(\mathbb{R}^d)$ correspond to quasiinvariant measures on $\mathcal{S}'(\mathbb{R}^d)$ concentrated on a single orbit. As noted earlier, an infinite gas of indistinguishable point particles is described using the configuration space $\Gamma^{(\infty)}_{\mathbb{R}^d}$, which is embedded in $\mathcal{S}'(\mathbb{R}^d)$ by Eq. (74). The resulting set $\Delta^{(\infty)}_{\mathbb{R}^d}$ is not a single orbit under $\mathcal{K}(\mathbb{R}^d)$, but an *uncountable union* of orbits. Different techniques must then be used to construct quasiinvariant measures (see below), leading to still more inequivalent CURs of the semidirect product group that are essential for statistical physics.

Likewise, an infinite gas of indistinguishable point dipoles may be described by the configuration space $\Delta^{(\nabla,\infty)}_{\mathbb{R}^d}\subset\mathcal{S}'(\mathbb{R}^d)$, defined by

$$\Delta^{(\nabla,\infty)}_{\mathbb{R}^d} = \{\Sigma_{j=1}^{\infty}\lambda_j\cdot\nabla\delta_{\mathbf{x}_j}\,|\,\lambda_j\neq 0;\ \mathbf{x}_j\neq\mathbf{x}_k \text{ for } j\neq k\}, \qquad (104)$$

where $\{\mathbf{x}_j\}\subset\mathbb{R}^d$ is locally finite. Here, too, the configuration space is an uncountable union of orbits under the action of the diffeomorphism group.

4. Induced Representations

In Sec. 3.3, we associated representations describing the different possible statistics of a system of N identical point particles in $\mathbb{R}^d$ (for $d \geq 2$) with distinct 1-cocycles on the N-particle orbits $\Delta_{\mathbb{R}^d}^{(N)} \subset \mathcal{S}'(\mathbb{R}^d)$. The $\Delta_{\mathbb{R}^d}^{(N)}$ are naturally identified with the configuration spaces $\Gamma_{\mathbb{R}^d}^{(N)}$, whose elements are N-point subsets of $\mathbb{R}^d$. Next we want to explain how these representations of the diffeomorphism group occur as *induced representations* obtained from the unitary representations of the symmetric group S_N.

4.1. *Mackey's Theory*

Inducing is a method developed by George Mackey,[67,68,69] that has its origin in the study of finite-dimensional Lie groups. It allows one to construct CURs of a second-countable, locally compact group G from CURs of a closed subgroup $H \subset G$. Its generalization to include the infinite-dimensional groups discussed here remains to some degree incomplete. Nevertheless, extension of this method to infinite-dimensional groups provides further insight into the representations we have constructed. In this subsection we outline Mackey's method, and in Sec. 4.2 we describe its generalization to N-particle representations of the diffeomorphism group $\mathcal{K}(\mathbb{R}^d)$. This leads to an understanding of the different statistics that are possible for a system of N indistinguishable quantum particles.

In general let H be a closed subgroup of a group G, and let $\Delta = H\backslash G$ be the quotient space whose elements are the distinct right cosets Hg, for $g \in G$. Then $g \mapsto Hg$ defines a natural projection from G onto Δ. Moreover, G acts on Δ by right multiplication: for $g_1 \in G$, $g_1 : Hg \to Hgg_1$. Since Δ is to play the role of configuration space, we have defined it here so that the action of G on Δ is a right action, consistent with our convention for diffeomorphism groups as well as with Mackey's lecture notes.[69] We shall denote an element of Δ by γ, and write the group action as $(g, \gamma) \to g\gamma$, but recalling that $g\gamma$ is here a right action.

The group H is itself a coset, and thus it is also a *particular* element $\gamma_0 \in \Delta$. With regard to the group action on Δ, the elements of H are precisely those that leave the element γ_0 fixed in Δ. Therefore we refer to H as the *stability subgroup* of G for the point γ_0. For any $\gamma \in \Delta$, define the stability subgroup $G_\gamma = \{ g \in G \,|\, g\gamma = \gamma \}$; in particular, $H = G_{\gamma_0}$. Note that G acts *transitively* on Δ. This means that any element of Δ can be reached by applying some element of G to the fixed element γ_0; and thus Δ itself, if it is a subset of some larger space, is a single G-orbit

in that space. For $\gamma \in \Delta$, let g be any element of G such that $\gamma = g\gamma_0$; we observe that $G_\gamma = g^{-1}Hg$.

If G is a second-countable, locally compact Lie group, it is equipped with a Borel σ-algebra of measurable sets, and with unique left- and right-invariant measures called *Haar measures* defined on this σ-algebra. Then Δ also becomes a measurable space (equipped with the Borel σ-algebra induced in $H\backslash G$ from the Borel structure in G). An important technical point is the existence of a *Borel section* in G; *i.e.*, a measurable subset $\hat{\Delta}_0 \subset G$ that intersects each right coset of H in precisely one point.

It can also be shown that there is a unique class of measures on Δ that are quasiinvariant under the group action. Let ν be any such measure on Δ, and let ν_g denote the transformed measure satisfying $\nu_g(B) = \nu(gB)$ for any Borel set B in Δ. Then, just as in the discussion leading up to Eq. (70), we have the Radon–Nikodym derivative $\alpha_g = d\nu_g/d\nu$ defined as a measurable function on Δ, and satisfying the cocycle equation $\alpha_{g_1 g_2}(\gamma) = \alpha_{g_2}(g_1\gamma)\,\alpha_{g_1}(\gamma)$.

Suppose now that $T(h)$ is a CUR of the subgroup H, acting in an inner product space $\mathcal{W}$. Here $\mathcal{W}$ may be the one-dimensional space $\mathbb{C}$, the finite-dimensional vector space $\mathbb{C}^n$, or an infinite-dimensional Hilbert space. Let $\widetilde{\Psi}$ be a measurable function on G taking values in $\mathcal{W}$, and having the property of *equivariance* under the representation $T(h)$; that is,

$$\widetilde{\Psi}(hg) = T(h)\widetilde{\Psi}(g) \tag{105}$$

almost everywhere (with respect to the Haar measure) in G. Observe that $T(h)$ acts here on the *vector value* of $\widetilde{\Psi}$, transforming it by *left* multiplication of its argument by h. If $\widetilde{\Psi}(g)$ is an equivariant function, we have the desired formula

$$T(h_1)[T(h_2)\widetilde{\Psi}(g)] = T(h_1)\widetilde{\Psi}(h_2 g) = \widetilde{\Psi}(h_1 h_2 g) = T(h_1 h_2)\widetilde{\Psi}(g)\,.$$

We then define a representation of G acting in the space of equivariant functions. This is the representation we shall say is *induced* by T. For $g, g_1 \in G$, let

$$[\widetilde{V}(g_1)\widetilde{\Psi}](g) = \widetilde{\Psi}(gg_1)\sqrt{\alpha_{g_1}(Hg)}\,. \tag{106}$$

That is, $\widetilde{V}(g_1)$ acts by *right* multiplication of the argument of $\widetilde{\Psi}$. It is easy to verify that $\widetilde{V}(g_1)\widetilde{\Psi}$ is likewise an equivariant function; indeed, we have

$$T(h)[\widetilde{V}(g_1)\widetilde{\Psi}](g) = T(h)\widetilde{\Psi}(gg_1)\sqrt{\alpha_{g_1}(Hg)}$$

$$= \widetilde{\Psi}(hgg_1)\sqrt{\alpha_{g_1}(Hg)} = [\widetilde{V}(g_1)\widetilde{\Psi}](hg)$$

as desired, where the last equality uses the fact that $Hhg = Hg$. It is equally straightforward to verify that Eq. (106) respects the group law in G, using the cocycle equation satisfied by α.

Finally, the measure ν allows us to define an inner product on the space of equivariant functions. Given two measurable, equivariant functions $\widetilde{\Phi}, \widetilde{\Psi} : G \to \mathcal{W}$, consider the function $\langle \widetilde{\Phi}(g), \widetilde{\Psi}(g)\rangle_{\mathcal{W}}$ which is defined for each g using the inner product in $\mathcal{W}$. This is a measurable, complex-valued function on G. Furthermore, for any element $h \in H$, we have $\langle \widetilde{\Phi}(hg), \widetilde{\Psi}(hg)\rangle_{\mathcal{W}} = \langle T(h)\widetilde{\Phi}(g), T(h)\widetilde{\Psi}(g)\rangle_{\mathcal{W}}$, using the definition of equivariance. Since $T(h)$ is unitary, this is again $\langle \widetilde{\Phi}(g), \widetilde{\Psi}(g)\rangle_{\mathcal{W}}$. Thus the latter expression is a complex-valued function on G that for any g is constant on the right coset $\gamma = Hg$. That is, it is actually a well-defined complex-valued function on the configuration space Δ. Set

$$\langle \widetilde{\Phi}, \widetilde{\Psi}\rangle = \int_{\Delta} \langle \widetilde{\Phi}(g), \widetilde{\Psi}(g)\rangle_{\mathcal{W}}\, d\nu(\gamma)\,, \tag{107}$$

and let $\widetilde{\mathcal{H}}$ be $\{\widetilde{\Psi} \mid \langle \widetilde{\Psi}, \widetilde{\Psi}\rangle < \infty\}$. Restricting $\widetilde{V}(g_1)$ in Eq. (106) to act in $\widetilde{\mathcal{H}}$, we obtain the *unitary* representation of G that is induced by the representation T of H.

Let us look next at how a unitary 1-cocycle is associated with this induced representation. As in Sec. 3.1, let $\mathcal{H} = L^2_{d\nu}(\Delta, \mathcal{W})$; with the inner product in $\mathcal{H}$ given as in Eq. (67) by $(\Phi, \Psi) = \int_\Delta \langle \Phi(\gamma), \Psi(\gamma)\rangle_{\mathcal{W}}\, d\nu(\gamma)$, for $\Phi, \Psi \in \mathcal{H}$. Note that there is *no* equivariance condition on functions in $\mathcal{H}$. Choose a *fixed Borel section* $\hat{\Delta}_0 \subset G$. For convenience, we shall select $\hat{\Delta}_0$ so that its intersection with H itself is the identity element. Define the unitary operator $Q : \widetilde{\mathcal{H}} \to \mathcal{H}$ by $Q\widetilde{\Psi} = \Psi$, where $\Psi(\gamma)$ is defined to be equal to $\widetilde{\Psi}(g)$ with g selected as the *particular* element belonging to $\hat{\Delta}_0$ for which $Hg = \gamma$.

Now let us write $V(g_1) = Q\widetilde{V}(g_1)Q^{-1}$, in order to define the unitary operator $V(g_1)$ acting in $\mathcal{H}$. For $g \in \hat{\Delta}_0$, the product gg_1 may or may not be an element of $\hat{\Delta}_0$; but there exists a unique element $h \in H$ such that $hgg_1 \in \hat{\Delta}_0$. Evidently h is determined by g and g_1, and in turn $g \in \hat{\Delta}_0$ is specified uniquely by the coset $Hg = \gamma$. Thus we may write our element h as $h_{g_1}(\gamma)$. Then we have for $\Psi \in \mathcal{H}$, and for $g \in \hat{\Delta}_0$ with $Hg = \gamma$, the action

$$[V(g_1)\Psi](\gamma) = [Q\widetilde{V}(g_1)Q^{-1}\Psi](\gamma) = [\widetilde{V}(g_1)Q^{-1}\Psi](g)$$

$$= [Q^{-1}\Psi](gg_1)\sqrt{\alpha_{g_1}(\gamma)}$$

using Eq. (106). From the definition of Q,

$$[Q^{-1}\Psi](h_{g_1}(\gamma)gg_1) = \Psi(Hgg_1) = \Psi(g_1\gamma)\,;$$

and from Eq. (105),

$$[Q^{-1}\Psi](h_{g_1}(\gamma)gg_1) = T(h_{g_1}(\gamma))[Q^{-1}\Psi](gg_1)\,.$$

Thus

$$[Q^{-1}\Psi](gg_1) = T^*(h_{g_1}(\gamma))\,[Q^{-1}\Psi](h_{g_1}(\gamma)gg_1) = T^*(h_{g_1}(\gamma))\Psi(g_1\gamma)\,.$$

Finally, we obtain

$$[V(g_1)\Psi](\gamma) = T^*(h_{g_1}(\gamma))\Psi(g_1\gamma)\sqrt{\alpha_{g_1}(\gamma)}\,. \tag{108}$$

Notice that if $g_1\gamma = \gamma$, $h_{g_1}(\gamma)gg_1 = g$ (where $g \in \hat{\Delta}_0$ and $Hg = \gamma$), so that $h_{g_1}(\gamma) = gg_1^{-1}g^{-1}$. If $g_1\gamma_0 = \gamma_0$, then $h_{g_1}(\gamma_0) = g_1^{-1}$, since $\hat{\Delta}_0$ intersects H at the identity element. In this case $T^*(h_{g_1}(\gamma)) = T(g_1)$.

To sum up, we have written the induced representation in a form parallel to that of Eq. (68). We have the configuration space $\Delta = H\backslash G$ of right cosets. We have the real cocycle $\alpha_g = d\nu_g/d\nu$, where ν is a measure on Δ quasiinvariant under the right group action. And we have the unitary 1-cocycle $\chi_{g_1}(\gamma) = T^*(h_{g_1}(\gamma))$ acting on the representation-space $\mathcal{W}$ of T.

It is straightforward to check that Eq. (71) holds for $\chi_{g_1}(\gamma)$; *i.e.*, that

$$T^*(h_{g_1g_2}(\gamma)) = T^*(h_{g_1}(\gamma))\,T^*(h_{g_2}(g_1\gamma))\,. \tag{109}$$

Indeed, let g (again) be the element of $\hat{\Delta}_0$ for which $Hg = \gamma$, so that $h_{g_1}(\gamma)gg_1 \in \hat{\Delta}_0$. Let g' be the element of $\hat{\Delta}_0$ for which $Hg' = g_1\gamma$, so that $h_{g_2}(g_1\gamma)g'g_2 \in \hat{\Delta}_0$. Since $Hgg_1 = g_1\gamma$, we have $g' = h_{g_1}(\gamma)gg_1$. Therefore $h_{g_2}(g_1\gamma)h_{g_1}(\gamma)gg_1g_2 \in \hat{\Delta}_0$, and we have the equation

$$h_{g_1g_2}(\gamma) = h_{g_2}(g_1\gamma)\,h_{g_1}(\gamma)\,. \tag{110}$$

Eq. (109) immediately follows, since T is a representation of H.

4.2. *Some Induced Diffeomorphism Group Representations*

Now the groups of diffeomorphisms that we are discussing do not belong to the category of second-countable, locally compact groups. In particular, they have no Haar measures. Thus extension of the concept of induced representations to diffeomorphism groups must rely on measures constructed by other means — *e.g.*, the cylinder set measures on N-particle orbits $\Gamma^{(N)}_{\mathbb{R}^d}$ described in Sec. 3.3. Here we describe some results for these orbits, with $N > 1$ and $d > 1$. In doing so we clarify the role of *label* and *value* permutations, which was left rather obscure in earlier work.[70,74]

Consider the action of (let us say) the diffeomorphism group $\mathcal{K}(\mathbf{R}^d)$ on $\Gamma^{(N)}_{\mathbb{R}^d}$. For a fixed element $\gamma_0 \in \Gamma^{(N)}_{\mathbb{R}^d}$, the stability subgroup $\mathcal{K}_{\gamma_0}(\mathbf{R}^d)$ is $\{\eta \in \mathcal{K}(\mathbf{R}^d) \,|\, \eta\gamma_0 = \gamma_0\}$. We have that $\mathcal{K}_{\gamma_0}(\mathbf{R}^d)$ is a closed subgroup of $\mathcal{K}(\mathbf{R}^d)$. While both groups are infinite-dimensional, the codimension of $\mathcal{K}_{\gamma_0}(\mathbf{R}^d)$ in $\mathcal{K}(\mathbf{R}^d)$ is finite; and the configuration space $\Gamma^{(N)}_{\mathbb{R}^d}$ is identified with the quotient space $\mathcal{K}_{\gamma_0}(\mathbf{R}^d)\backslash\mathcal{K}(\mathbf{R}^d)$ of right cosets.

Let $\gamma_0 = \{\mathbf{y}_1, \ldots, \mathbf{y}_N\} \subset \mathbb{R}^d$ be a fixed N-point subset of $\mathbb{R}^d$, $N > 1$. One way for a diffeomorphism η of $\mathbb{R}^d$ to leave γ_0 fixed is, of course, for it to leave each of the points $\mathbf{y}_j$ individually fixed. But when $d > 1$, $\eta \in \mathcal{K}_{\gamma_0}(\mathbf{R}^d)$ can also implement any *permutation* of the points that belong to γ_0. Thus the stability subgroup $\mathcal{K}_{\gamma_0}(\mathbf{R}^d)$, for $d > 1$, maps naturally onto the symmetric group $S(\gamma_0)$ whose elements are permutations of the set $\{\mathbf{y}_1, \ldots, \mathbf{y}_N\}$. For $\hat{\sigma}_1, \hat{\sigma}_2 \in S(\gamma_0)$, we define $[\hat{\sigma}_1\hat{\sigma}_2](\mathbf{y}_j) = \hat{\sigma}_2(\hat{\sigma}_1(\mathbf{y}_j))$. Given $\eta \in \mathcal{K}_{\gamma_0}(\mathbf{R}^d)$, the corresponding permutation $\hat{\sigma}^{\eta} \in S(\gamma_0)$ is then defined by $\hat{\sigma}^{\eta}(y_j) = \eta(y_j)$, and $\eta \to \hat{\sigma}^{\eta}$ is a group homomorphism from $\mathcal{K}_{\gamma_0}(\mathbf{R}^d)$ onto $S(\gamma_0)$.

Now a permutation $\hat{\sigma} \in S(\gamma_0)$, like a diffeomorphism $\eta \in \mathcal{K}_{\gamma_0}(\mathbf{R}^d)$, acts only on the *values* of the $\mathbf{y}_j$ as elements of $\mathbb{R}^d$; it does not "see" the index j we are using to label the elements of γ_0. To relate such value permutations to index permutations, we here let S_N denote the group of permutations of the set containing the first N counting numbers $\{1, \ldots, N\}$. For $\sigma_1, \sigma_2 \in S_N$, we take the group law in S_N to be defined by $(\sigma_1\sigma_2)[j] = \sigma_2[\sigma_1[j]]$ for $j = 1, \ldots, N$. In writing $\gamma_0 = \{\mathbf{y}_1, \ldots, \mathbf{y}_N\}$, let us index the elements of γ_0 in such a way that $\mathbf{y}_1 < \cdots < \mathbf{y}_N$ with respect to the lexicographical ordering in $\mathbb{R}^d$ we introduced in Sec. 3.3. Then a permutation $\hat{\sigma}$ of the elements of γ_0 acts on the *lexicographically ordered* N-tuple $(\mathbf{y}_1, \ldots, \mathbf{y}_N)$ to give the (non-lexicographically) ordered N-tuple $(\mathbf{y}_{\sigma[1]}, \ldots, \mathbf{y}_{\sigma[N]})$, where $\sigma \in S_N$. For $\hat{\sigma}_1, \hat{\sigma}_2 \in S(\gamma_0)$, we have $[\hat{\sigma}_1\hat{\sigma}_2](\mathbf{y}_j) = \hat{\sigma}_2(\hat{\sigma}_1(\mathbf{y}_j)) = \hat{\sigma}_2(\mathbf{y}_{\sigma_1[j]}) = \mathbf{y}_{\sigma_2[\sigma_1[j]]} = \mathbf{y}_{(\sigma_1\sigma_2)[j]}$, establishing the desired isomorphism $S(\gamma_0) \cong S_N$. We also have the corresponding group homomorphism from $\mathcal{K}_{\gamma_0}(\mathbf{R}^d)$ to S_N, which we denote by $\eta \to \sigma^{\eta}$. Note, however, that the isomorphism between $S(\gamma_0)$ and S_N is not canonical; it depends explicitly on our introduction of the lexicographical ordering of the points in γ_0.

Let $[\mathbb{R}^d]^N$ denote the space of all ordered N-tuples $(\mathbf{x}_1, \ldots \mathbf{x}_N)$ of points in $\mathbb{R}^d$, and let $D \subset [\mathbb{R}^d]^N$ be the set of N-tuples where $\mathbf{x}_j = \mathbf{x}_k$ (for some $j \neq k$); D is called the *diagonal* in $[\mathbb{R}^d]^N$. There is a natural projection

$$p : [\mathbb{R}^d]^N - D \;\to\; \Gamma^{(N)}_{\mathbb{R}^d}$$

given by $p : (\mathbf{x}_1, \ldots \mathbf{x}_N) \to \{\mathbf{x}_1, \ldots \mathbf{x}_N\} = \gamma$. Thus $[\mathbb{R}^d]^N - D$ is a *covering space* of $\Gamma^{(N)}_{\mathbb{R}^d}$. It has $N!$ sheets, corresponding to the distinct permutations of each configuration in $\Gamma^{(N)}_{\mathbb{R}^d}$. More specifically, let us consider as in Sec. 3.3 the *preferred sector* of $[\mathbb{R}^d]^N - D$ defined to be

$$\widetilde{\Delta}_0 = \{\, (\mathbf{x}_1, \ldots, \mathbf{x}_N) \in [\mathbb{R}^d]^N - D \,|\, \mathbf{x}_1 < \ldots < \mathbf{x}_N \}$$

where we again use the lexicographical ordering. The points in the preferred sector are in one-to-one correspondence with the elements of $\Gamma^{(N)}_{\mathbb{R}^d}$, so that $\widetilde{\Delta}_0$ serves as a fundamental domain in the covering space. Again we have an isomorphism $S(\gamma) \cong S_N$, with $\hat{\sigma}(\mathbf{x}_j) = \mathbf{x}_{\sigma[j]}$. Let us think of a point in $\widetilde{\Delta}_0$ (conventionally) as associated with the identity element in the symmetric group $S(\gamma)$. Then each element $\hat{\sigma} \in S(\gamma)$ may be regarded as acting on the preferred sector of the covering space $[\mathbb{R}^d]^N - D$ to generate a distinct *sheet* in that space. The preferred sector $\widetilde{\Delta}_0$ will play the role that the Borel section $\hat{\Delta}_0$ played in the preceding discussion of induced representations; while the covering space $[\mathbb{R}^d]^N - D$ plays the role formerly played by the group G.

Thus far we have described the action of S_N on $\widetilde{\Delta}_0 \subset [\mathbb{R}^d]^N - D$, given by $\sigma : (\mathbf{x}_1, \ldots, \mathbf{x}_N) \to (\mathbf{x}_{\sigma[1]}, \ldots, \mathbf{x}_{\sigma[N]})$. But we have not yet defined the group action of S_N, or of $S(\gamma_0)$, on the full space $[\mathbb{R}^d]^N - D$, which requires defining the action of σ on the other sectors. This is needed to construct a Hilbert space of equivariant wave functions on $[\mathbb{R}^d]^N - D$. Moreover we have the natural lifting of the action of the full diffeomorphism group $\mathcal{K}(\mathbb{R}^d)$ from $\Gamma^{(N)}_{\mathbb{R}^d}$ to all of $[\mathbb{R}^d]^N - D$, given by $\phi : (\mathbf{x}_{\sigma[1]}, \ldots, \mathbf{x}_{\sigma[N]}) \to (\phi(\mathbf{x}_{\sigma[1]}), \ldots, \phi(\mathbf{x}_{\sigma[N]}))$. This defines a right action of $\mathcal{K}(\mathbb{R}^d)$ on $[\mathbb{R}^d]^N - D$. But we have not yet defined any left action of $\mathcal{K}_{\gamma_0}(\mathbb{R}^d)$ on $[\mathbb{R}^d]^N - D$. We now show the correct way to take these steps.

Let $(\mathbf{y}_1, \ldots, \mathbf{y}_N)$ be fixed as before, with $\mathbf{y}_1 < \cdots < \mathbf{y}_N$. Let $(\mathbf{x}_{\sigma[1]}, \ldots, \mathbf{x}_{\sigma[N]})$ denote a general element of $[\mathbb{R}^d]^N - D$, where the indices are such that $\mathbf{x}_1 < \cdots < \mathbf{x}_N$, and $\sigma \in S_N$. For each such N-tuple, select a particular diffeomorphism $\phi^{(\mathbf{x}_{\sigma[1]}, \ldots, \mathbf{x}_{\sigma[N]})} \in \mathcal{K}(\mathbb{R}^d)$ for which $\mathbf{x}_{\sigma[j]} = \phi^{(\mathbf{x}_{\sigma[1]}, \ldots, \mathbf{x}_{\sigma[N]})}(\mathbf{y}_j)$; *i.e.*, such that

$$(\mathbf{x}_{\sigma[1]}, \ldots, \mathbf{x}_{\sigma[N]}) = \phi^{(\mathbf{x}_{\sigma[1]}, \ldots, \mathbf{x}_{\sigma[N]})}(\mathbf{y}_1, \ldots, \mathbf{y}_N)\,. \qquad (111)$$

Now for $\eta \in \mathcal{K}_{\gamma_0}(\mathbf{R}^d)$ we write a left action of η on $[\mathbb{R}^d]^N - D$ as follows, in analogy with the left multiplication by elements of the stability subgroup $H \subset G$ in Mackey's theory:

$$\eta : (\mathbf{x}_{\sigma[1]}, \ldots, \mathbf{x}_{\sigma[N]}) \to [\eta\, \phi^{(\mathbf{x}_{\sigma[1]},\ldots,\mathbf{x}_{\sigma[N]})}](\mathbf{y}_1, \ldots, \mathbf{y}_N) =$$

$$= \phi^{(\mathbf{x}_{\sigma[1]},\ldots,\mathbf{x}_{\sigma[N]})}(\eta\,(\mathbf{y}_1, \ldots, \mathbf{y}_N)) = \phi^{(\mathbf{x}_{\sigma[1]},\ldots,\mathbf{x}_{\sigma[N]})}(\mathbf{y}_{\sigma^\eta[1]}, \ldots, \mathbf{y}_{\sigma^\eta[N]})$$

$$= (\mathbf{x}_{\sigma(\sigma^\eta[1])}, \ldots, \mathbf{x}_{\sigma(\sigma^\eta[N])}) = (\mathbf{x}_{(\sigma^\eta\sigma)[1])}, \ldots, \mathbf{x}_{(\sigma^\eta\sigma)[N])}) . \qquad (112)$$

In short, the action of $\sigma \in S_N$ on $[\mathbb{R}^d]^N - D$ is as a *label* permutation, and the desired left action of $\eta \in \mathcal{K}_{\gamma_0}$ on $[\mathbb{R}^d]^N - D$ is by way of the label permutation σ^η. Notice that the result in Eq. (112) is independent of the particular diffeomorphism $\phi^{(\mathbf{x}_{\sigma[1]},\ldots,\mathbf{x}_{\sigma[N]})}$ selected to obey Eq. (111). The (arbitrary) lexicographical ordering enters only in the choice of the homomorphism $\eta \to \sigma^\eta$; it does not otherwise affect the result.

Let us stress carefully this distinction between value and label permutations. Diffeomorphisms, in their *right* action on $[\mathbb{R}^d]^N - D$, "see" only the *values* of the points. Say that $\eta \in \mathcal{K}_{\gamma_0}(\mathbf{R}^d)$ exchanges the two lowest values of the lexicographically ordered set γ_0: $\eta(\mathbf{y}_1) = \mathbf{y}_2$ and $\eta(\mathbf{y}_2) = \mathbf{y}_1$, so that $\sigma^\eta = (12)$. When η acts in its *right* action on the permuted N-tuple $(\mathbf{y}_{\sigma[1]}, \ldots, \mathbf{y}_{\sigma[N]})$ it exchanges the two *lowest* values of the entries, *not* the first two entries; and when η acts in its right action on some other, general element of $[\mathbb{R}^d]^N - D$, it does not typically implement a permutation at all. But this is not the action with respect to which equivariance is defined. The *left* action of η on $[\mathbb{R}^d]^N - D$, given by Eq. (112), exchanges the *first* two entries of *any* N-tuple of points, even when η does not belong to the stability subgroup of the corresponding configuration. This *label* action defines the equivariance of wave functions on the covering space.

The (continuous) homomorphism from $\mathcal{K}_{\gamma_0}(\mathbf{R}^d)$ onto S_N, that was given by $\eta \to \sigma^\eta$, means that any unitary representation T of S_N also defines a CUR of $\mathcal{K}_{\gamma_0}(\mathbf{R}^d)$. That is, there is a certain class of CURs of the stability subgroup that *factor* through unitary representations of S_N. Suppose then that T is an irreducible unitary representation of S_N acting in an inner product space $\mathcal{W}$. Let $\widetilde{\Psi}$ be a measurable function on $[\mathbb{R}^d]^N - D$ taking values in $\mathcal{W}$, equivariant under the representation T; that is, for $(\mathbf{x}_1, \ldots, \mathbf{x}_N) \in [\mathbb{R}^d]^N - D$ (not necessarily lexicographically ordered),

$$\widetilde{\Psi}((\mathbf{x}_{\sigma[1]}, \ldots, \mathbf{x}_{\sigma[N]})) = T(\sigma)\widetilde{\Psi}(\mathbf{x}_1, \ldots, \mathbf{x}_N) \qquad (113)$$

in $[\mathbb{R}^d]^N - D$. This is the analogue of Eq. (105) in the preceding subsection.

Following this analogy, observe that given any two measurable, equivariant (vector-valued) functions $\widetilde{\Phi}$ and $\widetilde{\Psi}$, their $\mathcal{W}$-inner product $\langle\widetilde{\Phi}(\mathbf{x}_1, \ldots, \mathbf{x}_N), \widetilde{\Psi}(\mathbf{x}_1, \ldots, \mathbf{x}_N)\rangle_{\mathcal{W}}$ is a (scalar-valued) function that depends only on $\gamma = p\,(\mathbf{x}_1, \ldots, \mathbf{x}_N) = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$. Let $\widetilde{\mathcal{H}}$ be the Hilbert space of all such functions $\widetilde{\Psi}$ for which $\langle\widetilde{\Psi}(\mathbf{x}_1, \ldots, \mathbf{x}_N), \widetilde{\Psi}(\mathbf{x}_1, \ldots, \mathbf{x}_N)\rangle_{\mathcal{W}}$ is integrable over $\Gamma^{(N)}_{\mathbb{R}^d}$ with respect to $d\mathbf{x}_1 \cdots d\mathbf{x}_N$. In analogy with Eq. (106), the representation of $\mathcal{K}(\mathbb{R}^d)$ *induced* by T is given by

$$[\widetilde{V}(\phi)\widetilde{\Psi}](\mathbf{x}_1, \ldots, \mathbf{x}_N) = \widetilde{\Psi}(\phi(\mathbf{x}_1), \ldots, \phi(\mathbf{x}_N)) \sqrt{\prod_{j=1}^{N} \mathcal{J}_\phi(\mathbf{x}_j)} \,. \qquad (114)$$

As in the preceding subsection, we may write a representation unitarily equivalent to Eq. (114) in the Hilbert space $\mathcal{H} = L^2_{d\mathbf{x}_1\cdots d\mathbf{x}_N}(\Gamma^{(N)}_{\mathbb{R}^d}, \mathcal{W})$ with no equivariance condition. Again define a unitary operator $Q : \widetilde{\mathcal{H}} \to \mathcal{H}$ by $Q\widetilde{\Psi} = \Psi$, where $\Psi(\{\mathbf{x}_1, \ldots, \mathbf{x}_N\})$ is set equal to $\widetilde{\Psi}(\mathbf{x}_1, \ldots, \mathbf{x}_N)$ with $\mathbf{x}_1 < \cdots < \mathbf{x}_N$; *i.e.*, Ψ takes the value of $\widetilde{\Psi}$ on the fundamental domain $\widetilde{\Delta}_0$. Then write $V(\phi) = Q\widetilde{V}(\phi)Q^{-1}$. For $(\mathbf{x}_1, \ldots, \mathbf{x}_N) \in \widetilde{\Delta}_0$ and $\phi \in \mathcal{K}(\mathbb{R}^d)$, there exists a unique permutation $\sigma \in S_N$ such that the N-tuple $(\phi(\mathbf{x}_{\sigma[1]}), \ldots, \phi(\mathbf{x}_{\sigma[N]})) \in \widetilde{\Delta}_0$. Here σ is determined by $\gamma = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$ and by ϕ, so we may write it as $\sigma_\phi(\gamma)$. We then have, for $\Psi \in \mathcal{H}$,

$$[V(\phi)\Psi](\gamma) = [Q\widetilde{V}(\phi)Q^{-1}\Psi](\gamma) = [\widetilde{V}(\phi)Q^{-1}\Psi](\mathbf{x}_1, \ldots, \mathbf{x}_N)$$

$$= [Q^{-1}\Psi](\phi(\mathbf{x}_1), \ldots, \phi(\mathbf{x}_N)) \sqrt{\prod_{j=1}^{N} \mathcal{J}_\phi(\mathbf{x}_j)}$$

$$= T^*(\sigma_\phi(\gamma))[Q^{-1}\Psi](\phi(\mathbf{x}_{\sigma_\phi(\gamma)[1]}), \ldots, \phi(\mathbf{x}_{\sigma_\phi(\gamma)[N]})) \sqrt{\prod_{j=1}^{N} \mathcal{J}_\phi(\mathbf{x}_j)}$$

$$= T^*(\sigma_\phi(\gamma))\Psi(\phi(\gamma)) \sqrt{\prod_{j=1}^{N} \mathcal{J}_\phi(\mathbf{x}_j)} \,. \qquad (115)$$

We have thus written the induced representation in a form parallel to that of Eq. (68), with the unitary 1-cocycle $\chi_\phi(\gamma) = T^*(\sigma_\phi(\gamma))$ acting on the representation-space $\mathcal{W}$ of T. The cocycle of Eq. (93) is the special case of this cocycle that occurs when T is a 1-dimensional representation of S_N.

5. Bosons, Fermions, Paraparticles, Anyons and Plektons

We have seen that for the configuration space $\Gamma_{\mathbb{R}^d}^{(N)}$ of N-point subsets $\gamma = \{x_1, \ldots, x_N\} \subset \mathbb{R}^d$ $(d > 1)$ — or, equivalently, the configuration space $\Delta_{\mathbb{R}^d}^{(N)}$ of generalized functions of the form $\gamma = \Sigma_{j=1}^N \delta_{x_j}$, with $x_j \neq x_k$ for $j \neq k$ — a unitary representation of S_N provides a CUR of the stability subgroup $\mathcal{K}_{\gamma_0}(\mathbb{R}^d) \subset \mathcal{K}(\mathbb{R}^d)$, and induces a unitary representation of $\mathcal{K}(\mathbb{R}^d)$. The latter is characterized either by an equivariance condition satisfied by wave functions on a covering space of $\Gamma_{\mathbb{R}^d}^{(N)}$ (the "coordinate space" on which wave functions obey a symmetry condition), or by a unitary 1-cocycle for the diffeomorphism group action on $\Gamma_{\mathbb{R}^d}^{(N)}$ or $\Delta_{\mathbb{R}^d}^{(N)}$ (the "configuration space" where the permutations themselves act trivially). In addition, it can be shown that unitarily inequivalent representations of S_N induce inequivalent representations of the diffeomorphism group (which, in general, describe physically inequivalent quantum-mechanical systems).[70]

The identity representation of S_N thus leads to the (bosonic) Hilbert space of square-integrable wave functions on the covering space, symmetric under exchange of particle coordinates. The alternating representation of S_N similarly gives us the (fermionic) space of antisymmetric wave functions. In this manner, the one-dimensional unitary representations of S_N $(N > 1)$ *induce* those representations of $\mathcal{K}(\mathbb{R}^d)$ which are just the N-particle Bose and Fermi representations described earlier by Eqs. (64). But these are not the only representations of S_N. Higher-dimensional representations exist (for $N > 2$) associated with the possible Young tableaux; and these may also be used to construct induced representations. The most elementary example is a 2-dimensional, irreducible unitary representation of S_3. In general we obtain a Hilbert space of *multicomponent* wave functions on the covering space, transforming under coordinate permutations according to a higher-dimensional unitary representation of S_N, and a corresponding induced representation of the group of diffeomorphisms of $\mathbb{R}^d$. These describe *paraparticles* that obey the *parastatistics* of Messiah and Greenberg.[10]

The classification of inequivalent unitary representations of the diffeomorphism group thus yields both the quantum kinematics associated with different spaces of configurations (*e.g.*, N-particle quantum mechanics for distinct values of N), and the different possible quantum statistics (Bose, Fermi, or para-) usually associated with systems of N indistinguishable particles. What we have done so far is not sensitive to whether we work with $\mathcal{K}(\mathbb{R}^d)$ or $Diff^c(\mathbb{R}^d)$. Furthermore, the results are not limited to

$M = \mathbb{R}^d$; they extend to more general spatial manifolds, as long as the diffeomorphism group acts transitively on the manifold.

But the framework we have constructed leads directly to additional induced representations associated with the topology of the physical space M, in certain situations when M is not simply connected; and to representations associated with exotic particle statistics, when the spatial dimension $d = 2$. Let us next see how this can occur.[71]

5.1. *Diffeomorphisms and the Fundamental Group*

For the constructions that follow, we shall need some ideas from elementary topology.[75] Two (continuous, directed) arcs in a smooth, connected manifold M having the same end points are called *homotopic* if one can be continuously deformed into the other. This establishes an *equivalence relation* among arcs; if $\beta : [0, 2\pi] \to M$ is such an arc, we denote its homotopy equivalence class by $[\beta]$. For $y \in M$, a *pointed loop* based at y is an arc beginning and ending at y. Two pointed loops β_1, β_2 based at y may be traversed in succession — β_1 followed by β_2 — and reparamaterized, yielding a third pointed loop $\beta_1\beta_2$ based at y; and a pointed loop β_1 may be traversed in the opposite direction, yielding the inverse loop β_1^{-1}. These operations respect homotopy equivalence, so that the set of homotopy classes of pointed loops in M based at y becomes a group $\pi_1(M)$ — known as the *fundamental group* or *first homotopy group* of the manifold. The homotopy class of pointed loops that can be continuously deformed to a point (*i.e.*, the class of "trivial" loops) corresponds to the identity element in the fundamental group. If all loops in M are trivial, we say that M is *simply connected*. Nontrivial homotopy classes of loops are associated with windings about holes in a non-simply connected manifold.

The *universal covering space* $\widetilde{M}$ of M may be constructed as the space of all homotopy classes of arcs in M originating at a fixed point $y \in M$. If $[\beta]$ is a homotopy class of arcs originating at y and terminating at $x \in M$, then $p([\beta]) = x$ defines the projection $p : \widetilde{M} \to M$. The space $\widetilde{M}$ is simply connected, while the distinct elements of $p^{-1}(y)$ correspond to distinct elements of the fundamental group $\pi_1(M)$. If $\mathcal{O}_y \subset M$ is a sufficiently small, simply-connected neighborhood of the point $y \in M$, then $p^{-1}(\mathcal{O}_y) \subset \widetilde{M}$ is the disjoint union of neighborhoods of the elements of $p^{-1}(y)$ in $\widetilde{M}$; and these neighborhoods may again be placed in correspondence with the elements of $\pi_1(M)$. We may think of each element of $p^{-1}(y)$, and likewise each connected neighborhood within $p^{-1}(\mathcal{O}_y)$, as belonging to a different *sheet* in $\widetilde{M}$. Finally, note that for $[\beta] \in \widetilde{M}$ (where

β is an arc from y to x) and $[\beta_1] \in \pi_1(M)$ (where β is a pointed loop based at y), we may form the homotopy class of the composite arc $[\beta_1\beta]$ from y to x, traversing first the loop β_1 and then the arc β. This defines a left action of $\pi_1(M)$ on $\widetilde{M}$.

Consider for example the manifold $M = \mathbb{R}^3 - \mathcal{Z}$, where $\mathcal{Z}$ is an infinite cylinder (interior together with boundary) of fixed radius about the x^3-axis. Let $\mathrm{y} \in M$ be a fixed base point. To any loop β_1 based at y we may assign a *winding number* $n(\beta_1)$, the net number of times the loop circles the missing cylinder $\mathcal{Z}$ in (let us say) a counterclockwise direction looking down from positive-x^3. Homotopic loops clearly have the same winding number, so that $n(\beta_1)$ depends only on the class $[\beta_1]$; and when two loops are traversed successively, their winding numbers add: $n(\beta_1\beta_2) = n(\beta_1) + n(\beta_2)$. Thus we have in this case $\pi_1(M) \cong \mathbb{Z}$ (the additive group of integers).

Now consider the stability subgroup $Diff^{\,c}_{\mathrm{y}}(M)$ of compactly-supported diffeomorphisms of $M = \mathbb{R}^3 - \mathcal{Z}$ leaving y fixed. Imagine further a fixed radial path β_∞ coming in from ∞, perpendicular to the x^3-axis, and terminating at y. For a diffeomorphism $\eta \in Diff^{\,c}_{\mathrm{y}}(M)$, let $\eta \circ \beta_\infty$ be the path obtained by acting on β_∞ with η. Since η becomes trivial at infinity, $\eta \circ \beta_\infty$ coincides with β_∞ far away from the missing cylinder; but $\eta \circ \beta_\infty$ may wind around the excluded region some number of times — *i.e.*, the (homotopy class of the) composite path $\beta_\infty^{-1}\, \eta \circ \beta_\infty$ (beginning at y, traversing first β_∞^{-1} and then $\eta \circ \beta_\infty$) belongs to the fundamental group. Hence we have the map $\eta \to \beta_\infty^{-1}\, \eta \circ \beta_\infty := \beta_1^\eta$, where β_1^η is a pointed loop based at y. This map defines a group homomorphism $Diff^{\,c}_{\mathrm{y}}(M) \to \pi_1(M)$, which we denote $\eta \to [\beta_1^\eta]$. Thus the left action of $\pi_1(M)$ on $\widetilde{M}$ defined above gives us a *left* action of the stability subgroup $Diff^{\,c}_{\mathrm{y}}(M)$ on $\widetilde{M}$, while an irreducible unitary representation T of $\pi_1(M)$ likewise defines a CUR of $Diff^{\,c}_{\mathrm{y}}(M)$.

Moreover, we have a natural lifting of the right action of the full group $Diff^{\,c}(M)$ from M to $\widetilde{M}$ as follows. As before let β be an arc from the base point y to $\mathrm{x} \in M$, so that $[\beta] \in \widetilde{M}$. The composite path $\beta_\infty \beta$ comes in from a fixed direction at infinity, and terminates at x. For an arbitrary diffeomorphism $\phi \in Diff^{\,c}(M)$, the path $\phi \circ (\beta_\infty \beta)$ comes in from the same fixed direction at infinity, and terminates at $\phi(\mathrm{x})$,; so that $\beta_\infty^{-1}\, \phi \circ (\beta_\infty \beta)$ originates at y and terminates at $\phi(\mathrm{x})$. Thus define $\widetilde{\phi} : \widetilde{M} \to \widetilde{M}$ by

$$\widetilde{\phi}\,([\beta]) = [\,\beta_\infty^{-1}\, \phi \circ (\beta_\infty \beta)\,]\,. \tag{116}$$

Evidently $\widetilde{\phi_1\phi_2} = \widetilde{\phi}_1\widetilde{\phi}_2$, while $p(\widetilde{\phi}\,([\beta])) = \phi\,(p\,([\beta])) = \phi(\mathrm{x})$.

Then there springs into being a new family of induced representations. Let T be an irreducible unitary representation of the fundamental group. Let the Hilbert space $\widetilde{\mathcal{H}}$ consist of wave functions $\widetilde{\Psi}$ on $\widetilde{M}$, equivariant with respect to T and square integrable over the base space M (with respect to local Lebesgue measure). In parallel with our earlier construction, the representation induced by T is given by

$$[\widetilde{V}(\phi)\widetilde{\Psi}]([\alpha]) = \widetilde{\Psi}(\widetilde{\phi}\,[\alpha])\,\sqrt{\mathcal{J}_\phi(\mathrm{x})}\,, \tag{117}$$

where $\mathrm{x} = p\,([\alpha])$. In the present example we fix $\theta \in [0, 2\pi)$, and set $T([\beta_1]) = \exp\,[in(\beta_1)\,\theta]$. Notice here that $\widetilde{M}$ has infinitely many sheets; but integration is defined over M, not $\widetilde{M}$. In this example the fundamental group is Abelian, but the construction we describe also extends to more complicated spaces having non-Abelian fundamental groups.

We remark, however, that the construction of a homomorphism from the stability subgroup $Diff^{\,c}_{\mathrm{y}}(M)$ to the fundamental group $\pi_1(M)$, and the lifting of the action of $Diff^{\,c}(M)$ from M to its universal covering space, are not completely general procedures. We have made important use of the fact that $\mathbb{R}^3 - \mathcal{Z}$ extends to spatial infinity, while the diffeomorphisms under discussion become trivial at infinity. Thus the method applies equally well to the manifold $\mathbb{R}^2 - \{(0,0)\}$ (the plane without the origin), or to the plane without a set of N distinct points; but not to the case of the circle S^1, whose fundamental group is also isomorphic to $\mathbb{Z}$. In the S^1 case, rotation by 2π is just the identity diffeomorphism; thus the stability subgroup of a point y is not the disjoint union of components associated with distinct winding numbers. For the analogous construction when M is compact, one needs to extend the diffeomorphism group.

To complete our discussion of induced representations, we now choose a fundamental domain $\widetilde{M}_0$ in $\widetilde{M}$; *i.e.*, a continuous cross-section to be conventionally identified with the identity element in $\pi_1(M)$. In the example $M = \mathbb{R}^3 - \mathcal{Z}$, where $\widetilde{M}$ is the helical covering consisting of homotopy classes of paths from y to $\mathrm{x} \in M$, we may take $\widetilde{M}_0$ to consist of those classes of paths for which the continuous change in azimuthal angle as a path is traversed is greater than or equal to 0 and less than 2π (so that there are no net windings about $\mathcal{Z}$).

For $[\beta] \in \widetilde{M}_0$ (an arc from y to x) and $\phi \in Diff^{\,c}(M)$, consider $\widetilde{\phi}\,([\beta])$ defined by Eq. (116). In general $\widetilde{\phi}\,([\beta])$ does not belong to $\widetilde{M}_0$, but there exists an element $[\beta_1]$ of the fundamental group — *i.e.*, a homotopy class of loops β_1 based at y — such that $[\beta_1\, \widetilde{\phi}\,([\beta]) \in \widetilde{M}_0$. Since $[\beta_1]$ depends on $[\beta]$ and ϕ, while $[\beta] \in \widetilde{M}$ is uniquely specified by its terminal point x, we may write $\beta_1 = \beta_\phi(\mathbf{x})$. Then let $\mathcal{H}$ be the Hilbert space of square-integrable functions on M taking values in $\mathcal{W}$, and $Q : \widetilde{\mathcal{H}} \to \mathcal{H}$ be given by $Q\widetilde{\Psi} = \Psi$, where $\Psi(\mathbf{x})$ takes the value $\widetilde{\Psi}([\beta])$ for $[\beta] \in \widetilde{M}_0$ and $p([\beta]) = \mathbf{x}$. We have the representation in $\mathcal{H}$ that is unitarily equivalent to $\widetilde{V}$,

$$[V(\phi)\Psi](\mathbf{x}) = T^*(\beta_\phi(\mathbf{x}))\Psi(\phi(\mathbf{x}))\sqrt{\mathcal{J}_\phi(\mathbf{x})}\,. \tag{118}$$

5.2. *The Aharonov–Bohm Effect*

The well-known *Aharonov–Bohm effect* in quantum mechanics occurs when a charged particle circles a region of magnetic flux.[76] Imagine an idealized, tightly-wound solenoid of infinite length within the cylinder $\mathcal{Z}$, so as to produce an approximately uniform magnetic field in the x^3-direction in the interior of the solenoid, but effectively zero magnetic field outside the solenoid. Consider a single charged particle confined to the spatial region outside $\mathcal{Z}$ where the magnetic field strength is zero; *e.g.*, by a high potential barrier. Solving the time-independent Schrödinger equation leads to the conclusion that the spectrum of $\hat{L}_3$, the x^3-component of the orbital (kinetic) angular momentum, is shifted from its usual values by an amount proportional to the magnetic flux through the solenoid.

In the presence of an external magnetic field $\mathbf{B}(\mathbf{x})$, considered not as an operator field but as an ordinary vector field on three-dimensional physical space, the expression for the commutator $[J(\mathbf{g}_1), J(\mathbf{g}_2)]$ in the equal-time, nonrelativistic current algebra describing charged particles is modified from Eq. (43) by the addition of a term proportional to $\rho(\mathbf{B}\cdot[\mathbf{g}_1 \times \mathbf{g}_2])$.[77] However, in the region outside the cylinder $\mathcal{Z}$, we have $\mathbf{B}(\mathbf{x}) \equiv 0$; so that even when the magnetic field behind the barrier is non-zero, the Lie algebra of local currents describing the quantum kinematics is unchanged.

As we saw earlier, angular momentum can be expressed in terms of the local, self-adjoint current density operators in a representation of this algebra. The currents, in turn, derive from a representation of the diffeomorphism group. The different possible shifts in the spectrum of $\hat{L}_3$ may be obtained from the distinct (*i.e.*, unitarily inequivalent) representations of the Lie algebra of currents *induced* by characters of the fundamental group of $M = \mathbb{R}^3 - \mathcal{Z}$. In fact, the local current operator in such a representation describing a single particle takes the familiar form $J'(\mathbf{g})\Psi(\mathbf{x}) = (\hbar/2i)[\mathbf{g}(\mathbf{x})\cdot\nabla + \nabla\cdot\mathbf{g}(\mathbf{x})]\Psi(\mathbf{x})$; but the domain of definition of $J'(\mathbf{g})$ consists of wave functions $\Psi(\mathbf{x})$ on $\mathbb{R}^3 - \mathcal{Z}$ satisfying the boundary condition (in cylindrical coordinates)

$$\Psi(r, 2\pi, z) = e^{-i\Phi}\Psi(r, 0, z)\,, \tag{119}$$

where Φ is the total magnetic flux through the solenoid. Extending each wave function Ψ to a corresponding equivariant wave function $\widetilde{\Psi}$ on the universal covering space $\widetilde{M}$ by setting $\widetilde{\Psi}(r,\phi,z) = \Psi(r,\phi,z)$ on the fundamental domain $0 \le \phi < 2\pi$, and $\widetilde{\Psi}(r,\phi+2\pi,z) = e^{-i\Phi}\widetilde{\Psi}(r,\phi,z)$, we can demonstrate (omitting the details here) that $J'(\mathbf{g})$ derives from a one-particle induced representation of $\mathit{Diff}^c(\mathbb{R}^3 - \mathcal{Z})$.[71]

5.3. *Exotic Statistics in Two Space Dimensions*

The existence of unitary representations of $\mathit{Diff}^c(\mathbb{R}^d)$ that are induced by unitary representations of the fundamental group of N-particle configuration space implies additional possibilities for particle statistics in the case $d = 2$. These include the quantum statistics of particles or excitations in two-dimensional space termed *anyons*. When two identical anyons are exchanged without coincidence along a continuous path in the plane, their relative winding number m (the net number of counterclockwise exchanges) is well-defined; it depends only on the homotopy class of the path implementing the exchange. The N-anyon wave function $\widetilde{\Psi}$ is defined on the universal covering space of the space $\Gamma^{(N)}_{\mathbb{R}^2}$ of N-point configurations $\{\mathbf{x}_1, \ldots, \mathbf{x}_N\} \subset \mathbb{R}^2$. It can then acquire a relative phase $e^{im\theta}$ under such an exchange, where θ is a real fixed parameter between 0 and 2π. When $\theta = 0$ we have bosons, and $\theta = \pi$ corresponds to fermions. The name "anyons" derives from the fact that "any" intermediate value of θ is permitted, so that the exchange statistics of anyons actually interpolates those of bosons and fermions.

Of course the physical world is not two-dimensional, and we do not expect to find fundamental particles that satisfy such statistics. Nevertheless we have here a new tool for the description of such phenomena as surface excitations, or quantum vortices in thin superfluid films.

The idea of such intermediate statistics was first suggested by Leinaas and Myrheim, for whom it was necessary to *assume* the exclusion of the "diagonal" from the configuration space.[78,79,80] The result was confirmed (independently) by my work with Menikoff and Sharp,[70,71] where we derived the quantum theory rigorously from induced representations of local nonrelativistic current algebra and the corresponding diffeomorphism group. Here the exclusion of diagonal is a *consequence* of the representation theory, as developed in these lectures. Our early results included the prediction of shifts in angular momentum and energy spectra for systems satisfying intermediate statistics, and the connection with the topology of configuration space and the physics of charged particles circling regions of magnetic flux (as in the Aharonov–Bohm effect).

Subsequently Wilczek introduced the term "anyon" to describe such particles, and proposed a model for them based on charged-particle/flux-tube composites.[81,82] Jackiw and Redlich pointed out that in such models it is the kinetic angular momentum (not the canonical angular momentum) for which the spectrum shifts away from integer multiples of $\hbar$,[83] which is consistent with my earlier development with Menikoff and Sharp. Wilczek also proposed an association between anyons and fractional spin in two space dimensions. This is very natural, since bosons are associated with integer spin and fermions with half-integer spin; and the latter associations are among the most important rigorous results of axiomatic (relativistic) quantum field theory in $3+1$ dimensions.[22] Some applications of ideas about anyons to surface phenomena and related domains of physics followed rapidly.[84,85]

As in the preceding development, the unitary representations of $\mathit{Diff}^{\,c}(\mathbb{R}^2)$ describing anyons are obtained as induced representations. Here the fundamental group of the configuration space $\Gamma^{(N)}_{\mathbb{R}^2}$ is Artin's *braid group* B_N; and this is the group whose one-dimensional representations describe the anyonic wave function symmetry.

A *braid* $b \in B_N$ may be visualized as a set of woollen strands connecting a row of N fixed posts to another row of N posts, where different crossings of the strands above or below each other distinguish different braids. The product of two braids is formed by operating with them successively, while the identity element e is the braid where strands do not cross. Let b_j denote an elementary crossing of strand j *over* strand $j+1$, for $j = 1, \ldots, N-1$; then the inverse braid b_j^{-1} is the elementary crossing of strand $j+1$ over strand j. The braid group itself may be constructed as the free group generated by the elements b_j and b_j^{-1}, *modulo* the (Yang–Baxter) equivalence relations,

$$b_j\, b_{j+1}\, b_j \;=\; b_{j+1}\, b_j\, b_{j+1} \quad (j = 1, \ldots, N-2)\,. \tag{120}$$

For $N=1$ the braid group is trivial; for $N=2$ it is isomorphic to the additive integers $\mathbb{Z}$; while for $N \geq 3$, it is an infinite, non-Abelian group.

In analogy with our discussion of $\mathbb{R}^3 - \mathcal{Z}$, a braid may also be associated with (the homotopy class of) a set of N nonintersecting paths in $\mathbb{R}^2$, coming in from infinity in a specified direction and terminating at the fixed points $\{\mathbf{y}_1, \ldots, \mathbf{y}_N\} = \gamma_0$. As in earlier discussions, we have here a homomorphism from the stability subgroup $\mathit{Diff}^{\,c}_{\gamma_0}(\mathbb{R}^2)$ onto B_N — in general, a compactly-supported diffeomorphism leaving γ_0 fixed acts to transform the homotopy class of the original set of paths from infinity.[94,95] Thus a unitary representation of B_N defines a CUR of the stability subgroup. The inducing construction proceeds in parallel with the development for S_N.

The one-dimensional unitary representations of B_N are specified by the single parameter $\theta \in [0, 2\pi)$, with each b_j represented as multiplication by $e^{i\theta}$. Only the values $\theta = 0$ and $\theta = \pi$ actually determine representations of S_N; the other values of θ lead to the induced representations of the diffeomorphism group describing anyonic statistics. In contrast, unitary representations of C_N allow distinct relative phases (assigned consistently) when different pairs of particles circle each other. After we had identified the braid group as the relevant group,[87,88] Y. S. Wu argued that *only* the one-dimensional representations of B_N should occur in quantum mechanics.[89] However, the diffeomorphism group approach allowed us also to predict the possibility of quantum systems described by higher-dimensional unitary representations of B_N (particles later termed "plektons").[86]

There is also a natural homomorphism h_N from B_N onto S_N, obtained by disregarding the braiding and attending only to the posts connected by the woollen strands constituting the braid. Mathematically, h_N maps the generator b_j to the exchange permutation $(j\ j+1)$. The *kernel* of h_N is the set of braids b for which $h_N(b)$ is the identity permutation; these are just the braids that return each post to its initial position. The kernel forms a nontrivial subgroup C_N, the *colored braid* group. This fact means that the wave function for *distinguishable* particles in $\mathbb{R}^2$ can also acquire an "anyonic" relative phase, as one particle circles another and returns to its original position.[86]

Many details and much subsequent development has been omitted here, including the relation of these ideas to Chern–Simons quantum field theories, their application in describing the integer and fractional quantum Hall effects, and their role in describing possible mechanisms for superconductivity. The reader is referred to more recent review articles, as well as the books by Wilczek and by Khare.[90,91,92,93]

5.4. *Fields Intertwining Current Algebra Representations*

The N-particle unitary Bose or Fermi representations of the diffeomorphism group, and the corresponding representations of the algebra of vector fields, evidently form distinct *hierarchies* in a certain sense — the Bose representations "belong" together, as do the Fermi representations. Likewise the anyonic representations of $Diff^c(\mathbb{R}^2)$ for any fixed value of θ form a hierarchy. To make precise the sense in which this is so, we regard the creation and annihilation fields as *intertwining* operators between N-particle subspaces (for adjacent values of N), and consider the commutation relations that these fields satisfy with the local currents. In effect, we are combining Eqs. (34) or (37) with Eqs. (44)–(45), and generalizing the resulting system to include anyons and possibly other kinds of quantum configurations (*e.g.*, extended objects).

Let $U_N(f)$ and $V_N(\phi)$ be systems of unitary operators satisfying Eq. (62) in Hilbert spaces $\mathcal{H}_N$, describing systems of N identical particles (or N identical configurations of some other sort) in a manifold M. Let $\rho_N(f)$ and $J_N(\mathrm{g})$ be corresponding systems of self-adjoint operators in $\mathcal{H}_N$, satisfying Eqs. (43). Let $h \in \mathcal{H}_1$, and let $\psi^*(h)$ and $\psi(h)$ be intertwining operators labeled by h. That is, take $\psi^*(h) : \mathcal{H}_N \to \mathcal{H}_{N+1}$ and $\psi(h) : \mathcal{H}_{N+1} \to \mathcal{H}_N$, with $\psi(h)$ annihilating the vacuum state $\Omega_0 \in \mathcal{H}_0$. These assumptions assert that the configuration space on which $\mathcal{H}_1$ is modeled establishes the nature of the configuration that ψ^* is taken to create, and ψ is taken to annihilate; while the state vector h describes the actual state in which the configuration is created or annihilated.

Sharp and I proposed that the necessary and sufficient conditions for the indexed set of representations to form a hierarchy should be

$$U_{N+1}(f)\psi^*(h) = \psi^*(U_{N=1}(f)h)U_N(f)\,,$$

$$V_{N+1}(\phi)\psi^*(h) = \psi^*(V_{N=1}(\phi)h)V_N(\phi)\,, \tag{121}$$

where the corresponding relations for the annihilation field ψ are obtained as the adjoint of these equations.[95] Equations (121) are very natural geometrically. Let us think of ψ^* as creating a particle at $\mathbf{x}$ in M or, more generally, as creating a possibly extended configuration α embedded in M. We think of h as an averaging function, defined on the space of singleton configurations. The first relation in Eqs. (121) then asserts that U and ψ^* both act locally in M. The second relation asserts that the result of first creating a single new configuration and then transforming the state vector by a diffeomorphism of M, is the same as the result of first transforming by the diffeomorphism, and then creating the transformed new object. Here the transformation law for singleton configurations is given, of course, by the action of $V_{N=1}(\phi)$ on $\mathcal{H}_1$.

In anticipating that this general structure occurs not only for point particles (including bosons, fermions, and anyons), but also for extended objects such as filaments or tubes, we expect that the creation and annihilation fields are not necessarily distributions over the physical space, but possibly over a space of spatially extended configurations. But ρ and J remain operator-valued distributions over the physical space.

From Eq. (121), we obtain the brackets between ψ^* and the elements of the current algebra. Defining $\rho(f)$ and $J(\mathrm{g})$ so that $\rho(f)\Psi_N = \rho_N(f)\Psi_N$, $J(\mathrm{g})\Psi_N = J_N(\mathrm{g})\Psi_N$, we have

$$[\rho(f), \psi^*(h)] = \psi^*(\rho_{N=1}(f)h)\,,$$

$$[J(\mathrm{g}), \psi^*(h)] = \psi^*(J_{N=1}(\mathrm{g})h)\,, \tag{122}$$

where again the brackets involving ψ are given by the adjoint of these equations. Explicit calculation confirms that the canonical Bose and Fermi nonrelativistic fields satisfy Eqs. (122).

Let us emphasize that *only commutator brackets* occur here and in Eqs. (43), no anticommutation or q-commutation relations. The point is that if we begin with the indexed family of N-particle Bose or Fermi representations of the current algebra (or, alternatively, the corresponding representations of the semidirect product group), we can *construct* the field operators that fulfill Eqs. (122) or (121). Then it is a *consequence of the construction* — no longer an *a priori* assumption — that the Bose fields obey equal-time commutation relations ($-$) and Fermi fields obey equal-time anticommutation relations ($+$) as given by Eqs. (29), even though we assumed only the commutator brackets between fields and currents. As a further consequence we obtain Eqs. (41) for $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$ in terms of the canonical fields, which until this point have been taken to be the *defining* equations for the local currents in these representations.

Similarly, we construct explicit anyon creation and annihilation fields for particles in two-space, obeying Eqs. (121) and (122).[94,95] The result, omitting many interesting details, is that the anyon fields obey q-commutation relations, where $q = \exp i\theta$ is a complex number of modulus one, the relative phase change associated with a single counterclockwise exchange of two anyons. With $[A, B]_q$ defined by Eq. (14), we obtain

$$[\psi(\mathbf{x},t), \psi(\mathbf{y},t)]_q = [\psi^*(\mathbf{x},t), \psi^*(\mathbf{y},t)]_q = 0\,,$$

$$[\psi(\mathbf{y},t), \psi^*(\mathbf{x},t)]_q = \delta(\mathbf{x}-\mathbf{y})\,. \tag{123}$$

These are, of course, generalizations of Eqs. (29); the latter correspond to the choices $q = 1$ or $q = -1$. When $q \neq \pm 1$, the first two relations of

Eq. (123) should be interpreted for consistency as holding for ordered pairs $(\mathbf{x}, \mathbf{y})$ in a half-space of $\mathbb{R}^2 \times \mathbb{R}^2$, while in the complement of that half-space we have the $(1/q)$-bracket. The choice of half-space has no physical consequences, but establishes an arbitrary boundary between sheets in the universal covering spaces of the N-anyon configuration spaces.

We also have the fact that Eqs. (41) hold, whereby $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$ are expressed in terms of the anyon fields. Beginning with these equations, together with the algebraic identity

$$[AB, C]_- = A[B, C]_q + q\,[A, C]_{1/q} B \tag{124}$$

which relates the ordinary commutator to the q-commutator, one verifies that the brackets of ψ and ψ^* with ρ and $\mathbf{J}$ are in fact in accordance with Eqs. (122).

6. Conclusion

In these lectures, we have reviewed at an introductory level how the continuous unitary representations of an infinite-dimensional group — the group of diffeomorphisms of physical space — and the corresponding self-adjoint representations of a nonrelativistic local current algebra, describe and predict the kinematics of a variety of possible quantum systems. Some of these possibilities were already well understood; others, such as anyon and plekton statistics, emerged as predictions of the representation theory. We have also discussed related topics in quantum field theory and group theory.

But we have only scratched the surface of many interesting mathematical and physical questions. There is far more to say about anyons and braid group statistics. The study of quasiinvariant measures and unitary cocycles on the infinite-dimensional configuration space $\Gamma_{\mathbb{R}^d}$ of statistical mechanics, especially measures built up from Poisson and Gibbs measures, continues to evolve rapidly. There are also many exciting developments and partial results for quantum theory on other infinite-dimensional configuration spaces. These include families of quasiinvariant measures derived from self-similar random processes on the generalized configuration space $\Sigma_{\mathbb{R}^d}$ of countable subsets, and the quantum mechanics of extended objects such as vortex configurations.

Hopefully lectures at future COPROMAPH conferences will address some of these important topics, carrying the development further.

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