Quantum Dirichlet forms and their recent applications

Adam Skalski

Białowieża

25 VI 2018

Outline

Our aim today is to show how one can encode certain properties of a von Neumann algebra or of a quantum group, using quantum Markov semigroups or/and their associated Dirichlet forms

Haagerup property

A discrete group G has the Haagerup property if it admits a mixing (C_0-) unitary representation which weakly contains the trivial representation.

In other words, we have $\pi : G \to B(\mathsf{H})$ such that

$$\forall_{\xi,\eta\in\mathsf{H}} \;\; \langle \xi, \pi(\cdot)\eta\rangle \in \mathsf{C}_0(G),$$

but for some net of unit vectors $(\xi_i)_{i\in I}$ such that

$$\forall_{g\in G} \;\; \pi(g)(\xi_i) - \xi_i \xrightarrow{i\in I} 0$$

Haagerup property via positive definite functions

$\varphi : G \to \mathbb{C}$ is positive definite if $\varphi(e) = 1$ and for all $n \in \mathbb{N}$, $g_1, \dots, g_n \in G$, $\lambda_1, \dots, \lambda_n \in \mathbb{C}$

$$\sum_{i,j=1}^{n} \varphi(g_i^{-1} g_j)\overline{\lambda_i}\lambda_j \geq 0.$$

A discrete group G has the Haagerup property if it admits a net of positive definite functions ϕ_i which belong to $\mathrm{C}_0(G)$, and yet converge pointwise to 1.

If G is countable, we can actually arrange $I = \mathbb{R}_+$ and $\phi_t = exp(-t\psi)$.

Haagerup property via positive definite functions

Theorem (Schönberg correspondence)

A symmetric function $\psi : G \to \mathbb{R}$ ($\psi(g) = \psi(g^{-1})$, $g \in G$) is conditionally positive definite if and only if for each $t \geq 0$ the function $\varphi_t := \mathrm{e}^{t\psi}$ is positive definite.

Theorem

A countable group G has Haagerup property if and only if it admits a proper symmetric conditionally positive definite function.

Considering the associated Herz-Schur multipliers acting on $\mathrm{VN}(G)$, i.e. maps of the form

$$M_t(\lambda_g) = \phi_t(g)\lambda_g, \quad g \in G,$$

we get the next reformulation.

Haagerup property via von Neumann algebraic semigroup approximation

Theorem

A countable group G has Haagerup property if and only if the von Neumann algebra $\mathrm{VN}(G)$ admits a symmetric quantum Markov semigroup consisting of L^2-compact Herz-Schur multipliers.

We build the semigroup, first constructing ψ – which can be viewed as a 'generating functional for the semigroup'.

Haagerup property for von Neumann algebras

The following is one of several equivalent versions.

Definition (Caspers + AS, Okayasu + Tomatsu)

Let (M, ϕ) be a von Neumann algebra with a faithful normal semifinite weight. We say that (M, ϕ) has the Haagerup property if there exists a net of normal completely positive, ϕ-reducing maps $(\Phi_i)_{i \in \mathcal{I}}$ on M such that the KMS-induced maps T_i on $L^2(M, \varphi)$ are compact and the net $(T_i)_{i \in \mathcal{I}}$ converges to $I_{L^2(M,\varphi)}$ strongly.

Theorem (C+S, O+T)

The property above does not depend on the choice of the weight.

Haagerup property via Markov semigroups

Let (M, φ) be a von Neumann algebra equipped with a faithful normal state.

Definition

A quantum Markov semigroup $\{\Phi_t : t \geq 0\}$ on (M, φ) is immediately L^2-compact if each of the maps $\Phi_t^{(2)}$ with $t > 0$ is compact.

The next result was inspired by the theorem for finite von Neumann algebras due to Jolissaint and Martin.

Theorem (Caspers + AS)

The following are equivalent:

1. (M, φ) has the Haagerup property;
2. there exists an immediately L^2-compact KMS-symmetric Markov semigroup $\{\Phi_t : t \geq 0\}$ on M.

Haagerup property via Dirichlet forms

Theorem (Caspers + AS)

The following are equivalent:

1. (M, φ) has the Haagerup property;
2. $L^2(M, \varphi)$ admits an orthonormal basis $(e_n)_{n \in \mathbb{N}}$ and a non-decreasing sequence of non-negative numbers $(\lambda_n)_{n \in \mathbb{N}}$ such that $\lim_{n \to \infty} \lambda_n = +\infty$ and the prescription

$$Q(\xi) = \sum_{n=1}^{\infty} \lambda_n |\langle e_n, \xi \rangle|^2, \quad \xi \in \text{Dom } Q,$$

where $\text{Dom } Q = \{\xi \in H_\varphi : \sum_{n=1}^{\infty} \lambda_n |\langle e_n, \xi \rangle|^2 < \infty\}$, defines a completely Dirichlet form.

LCQGs

$\mathbb{G}$ – locally compact quantum group à la Kustermans-Vaes

$\mathsf{L}^\infty(\mathbb{G})$ – the von Neumann algebra, together with the *coproduct* (carrying all the information about $\mathbb{G}$)

$$\Delta : \mathsf{L}^\infty(\mathbb{G}) \to \mathsf{L}^\infty(\mathbb{G})\overline{\otimes}\mathsf{L}^\infty(\mathbb{G})$$

and a canonical *right Haar weight* ϕ

$\mathsf{C}_0(\mathbb{G})$ – the corresponding (reduced) C^*-object

$\mathsf{C}_0^u(\mathbb{G})$ – the universal version of $\mathsf{C}_0(\mathbb{G})$,

$\mathsf{L}^2(\mathbb{G})$ – the GNS Hilbert space of the right invariant Haar weight ϕ on $\mathsf{L}^\infty(\mathbb{G})$

$\mathsf{L}^1(\mathbb{G})$ – predual of $\mathsf{L}^\infty(\mathbb{G})$, with a natural Banach algebra structure.

$$\mathsf{C}_0(\mathbb{G}) \subset \mathsf{L}^\infty(\mathbb{G})$$
$$\mathsf{L}^2(\mathbb{G}) \approx \mathsf{L}^2(\mathsf{L}^\infty(\mathbb{G}), \phi)$$

Dual groups

Each LCQG $\mathbb{G}$ admits the dual LCQG $\widehat{\mathbb{G}}$.

$\mathrm{L}^\infty(\widehat{\mathbb{G}})$, $\mathrm{C}_0(\widehat{\mathbb{G}})$ – subalgebras of $B(\mathrm{L}^2(\mathbb{G}))$

In particular for G – locally compact group

$$\mathrm{L}^\infty(\widehat{G}) = \mathrm{VN}(G), \quad \mathrm{C}_0(\widehat{G}) = C_r^*(G), \quad \mathrm{C}_0^u(\widehat{G}) = C^*(G)$$

We sometimes write

$$\mathrm{L}^\infty(\widehat{\mathbb{G}}) = \mathrm{VN}(\mathbb{G}), \quad \mathrm{C}_0(\widehat{\mathbb{G}}) = C_r^*(\mathbb{G}), \quad \mathrm{C}_0^u(\widehat{\mathbb{G}}) = C^*(\mathbb{G})$$

Simplifications in the compact case

Definition

$\mathbb{G}$ is said to be compact if $\mathrm{C}_0(\mathbb{G})$ is unital (so written as $\mathrm{C}(\mathbb{G})$), equivalently, the weight ϕ is a state.

Any compact quantum group can be described purely algebraically via the Hopf *-algebra $\mathrm{Pol}(\mathbb{G}) \subset \mathrm{C}(\mathbb{G})$, with the counit ϵ.

Convolution semigroups of states on compact quantum groups

A family $(\mu_t)_{t\geq 0_+}$ of states on $\mathrm{Pol}(\mathbb{G})$ is called a convolution semigroup of states if

1. $\mu_{t+s} = \mu_t \star \mu_s := (\mu_t \otimes \mu_s) \circ \Delta, \quad t, s \geq 0;$
2. $\mu_t(a) \overset{t\to 0^+}{\longrightarrow} \mu_0(a) := \epsilon(a), \quad a \in \mathrm{Pol}(\mathbb{G}).$

Such convolution semigroups admit generating functionals:

$$\gamma(a) = \lim_{t\to 0^+} \frac{\mu_t(a) - \epsilon(a)}{t}, \quad a \in \mathrm{Pol}(\mathbb{G}).$$

We associate to it a convolution semigroup of operators $(R_{\mu_t})_{t\geq 0_+}$ on $\mathrm{Pol}(\mathbb{G})$:

$$R_{\mu_t} := (\mathrm{id} \otimes \mu_t) \circ \Delta$$

These extend to operators on $\mathrm{L}^\infty(\mathbb{G})$ which form a Markov semigroup. The corresponding Dirichlet forms contain $\mathrm{Pol}(\mathbb{G})$ in the domain and can be characterised/studied in the purely algebraic manner (see Cipriani, Franz, Kula).

Convolution semigroups of states revisited

$\mathbb{G}$ – locally compact quantum group

A family $(\mu_t)_{t\geq 0_+}$ of states on $\mathrm{C}_0^u(\mathbb{G})$ is called a convolution semigroup of states if

1. $\mu_{t+s} = \mu_t \star \mu_s := (\mu_t \otimes \mu_s) \circ \Delta, \quad t, s \geq 0;$
2. $\mu_t(a) \xrightarrow{t \to 0^+} \mu_0(a) := \epsilon(a), \quad a \in \mathrm{C}_0^u(\mathbb{G}).$

We no longer have the 'algebraic domain' such as $\mathrm{Pol}(\mathbb{G})$. Generating functionals are densely defined, but that is all we know a priori (however: very recently we showed that the domain of the generating functional always contains a dense *-subalgebra).

Convolution operators – revisited once again

$\mathrm{C}_0^u(\mathbb{G})$ admits a canonical involutive operator R^u, so called universal unitary antipode (playing the role of the inverse operation).

Theorem

Let $\mu \in S(\mathrm{C}_0^u(\mathbb{G}))$. The associated operator $R_\mu : \mathrm{L}^\infty(\mathbb{G}) \to \mathrm{L}^\infty(\mathbb{G})$ (which can be informally thought of as the map $(\mu \otimes \mathrm{id}) \circ \Delta$) is unital, completely positive, ϕ-preserving. The map R_μ is KMS-symmetric iff $\mu = \mu \circ \mathrm{R}^u$. Its KMS implementation (acting on $\mathrm{L}^2(\mathbb{G})$) is always bounded and belongs to $\mathrm{L}^\infty(\hat{\mathbb{G}})$.

Convolution semigroups on locally compact quantum groups– main result

Theorem

Let $\mathbb{G}$ be a locally compact quantum group. There exist $1-1$ correspondences between:

1. w^*-continuous convolution semigroups $(\mu_t)_{t\geq 0}$ of R^{u}-invariant states of $\mathrm{C}_0^u(\mathbb{G})$;
2. C_0^*-semigroups $(T_t)_{t\geq 0}$ of normal, unital, completely positive maps on $\mathrm{L}^\infty(\mathbb{G})$ that are KMS-symmetric with respect to ϕ and satisfy $\Delta \circ T_t = (T_t \otimes \mathrm{id}) \circ \Delta$ for every $t \geq 0$;
3. completely Dirichlet forms Q on $\mathrm{L}^2(\mathbb{G})$ with respect to ϕ that are invariant under $\mathcal{U}(\mathrm{L}^\infty(\hat{\mathbb{G}})')$ (modulo multiplication of forms by a positive number).

Applications to geometric properties of quantum groups

Theorem

Let $\mathbb{G}$ be a second countable locally compact quantum group. Then $\hat{\mathbb{G}}$ has Property (T) of Kazhdan if and only if every convolution semigroup of R^u-invariant states on $\mathrm{C}_0^u(\mathbb{G})$ has a bounded generator.

Theorem

Let $\mathbb{G}$ be a second countable locally compact quantum group. Then $\hat{\mathbb{G}}$ has the Haagerup property if and only if there exists a convolution semigroup of R^u-invariant states on $\mathrm{C}_0^u(\mathbb{G})$ such that the L^2-implementations of the associated convolution operators, acting on $\mathrm{L}^2(\mathbb{G})$, in fact belong to $\mathrm{C}_0(\hat{\mathbb{G}})$.

Ongoing work

In the compact case every ‘generating functional’, i.e. a hermitian, conditionally positive functional on the algebra $\mathrm{Pol}(\mathbb{G})$, vanishing at 1, indeed generates a convolution semigroup of states.

Suppose we have a dense unital *-subalgebra A in the (unitization) of $C_0^u(\mathbb{G})$, and a functional $\gamma : A \to \mathbb{C}$ with the properties as above – what to assume about A to guarantee that L ‘generates’ a convolution semigroup of states?