On Bogomolny decomposition and some solutions of some Skyrme-like models

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Euler-Lagrange equations of many models in physics are nonlinear partial differential equations of second order

but, in [Bogomolny 1976] Bogomolny derived the equations, called as Bogomolny equations - sometimes called also, as Bogomol’nyi equations (although historically, they were derived earlier in [Belavin, Polyakov, Schwarz, Tyupkin 1975], for another model - SU(2) Yang-Mills theory):
1. scalar field theory - model $\phi^4$ with spontaneous symmetry breaking

$$E = \int_{-\infty}^{\infty} \left( \frac{1}{2} \left( \frac{d\phi}{dx} \right)^2 + \frac{\lambda}{2} (\phi^2 - \gamma^2)^2 \right) dx,$$

$$\phi(x) \in \mathbb{R}, \quad \lim_{x \to \pm \infty} \phi(x) = \pm \gamma$$

Euler-Lagrange equations for this model

$$\frac{d^2 \phi}{dx^2} = 2\lambda \phi (\phi^2 - \gamma^2)$$

On Bogomolny decomposition and some solutions of some Skyrme-like...
we may avoid solving of them, namely we write the formula for $E$ in (1), as follows

$$E = \int_{-\infty}^{\infty} \left( \frac{1}{2} \left( \frac{d\phi}{dx} + \sqrt{\lambda}(\phi^2 - \gamma^2) \right)^2 - \sqrt{\lambda} \frac{d\phi}{dx}(\phi^2 - \gamma^2) \right) dx,$$

(total derivative of $\sqrt{\lambda}(\phi^3 - \gamma^2 \phi)$)

(3)
we integrate the underbraced term in (3)

\[ E = \int_{-\infty}^{\infty} \frac{1}{2} \left( \frac{d\phi}{dx} + \sqrt{\lambda}(\phi^2 - \gamma^2) \right)^2 dx + \frac{2\sqrt{\lambda}}{3} \gamma^2 |Q|, \quad (4) \]

\[ Q = \phi(\infty) - \phi(-\infty), \]

where \( Q \) - topological charge.
now we require reaching the minimum by the functional (4), so the first term must vanish

\[
\frac{d\phi}{dx} = \sqrt{\lambda}(\gamma^2 - \phi^2)
\]  

(5)

The very-known solution of (5), so called “kink”

\[
\phi(x) = \gamma \tanh (\gamma \sqrt{\lambda}(x - x_0))
\]  

(6)

So, the following inequality (Bogomolny bound) is satisfied

\[
E \geq E_{min} = \frac{2\sqrt{\lambda}}{3} \gamma^2 |Q|
\]  

(7)

where \( E_{min} \) - the minimum of the functional (4).
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Baby Skyrme model - an introduction I

I Skyrme model - very interesting model, possesses solitonic solutions, useful for describing phenomena in world of baryons; good description of low-energy physics of strong interactions, [Makhankov et al. 1989].

II baby Skyrme model - an analogical model (on plane) to the Skyrme model in three-dimensional space.

III the target space of Skyrme model is $SU(2)$, [Skyrme 1961], [Skyrme 1962], [Skyrme 1971] $\Rightarrow$ the target space of baby Skyrme model is $S^2$.

IV in these both models: Skyrme and baby Skyrme, static field configurations can be classified topologically by their winding numbers.
anallogically to the Skyrme model, the baby Skyrme model includes:

1. the quadratic term i.e. the term of nonlinear $O(3)$ sigma model,
2. the quartic term - an analogue of the Skyrme term and necessary in order to avoid the consequences of Derrick-Hobart theorem and
3. the potential - its presence in the case of static field configurations with finite energy, in baby Skyrme model, is necessary. However, the form of this potential - not restricted.
VI the lagrangian of baby Skyrme model, [Adam etal. 2009]:

$$\mathcal{L} = \partial_\mu \mathbf{\tilde{S}} \cdot \partial^\mu \mathbf{\tilde{S}} - \beta (\partial^\mu \mathbf{\tilde{S}} \times \partial^\nu \mathbf{\tilde{S}})^2 - V(\mathbf{\tilde{S}}),$$ \hspace{1cm} (8)

where $|\mathbf{\tilde{S}}|^2 = 1$.

VII we consider the energy functional for restricted baby Skyrme model in (2+0) dimensions (the static $\sigma$ term is absent), of the following form, [Adam etal. 2010]

$$H = \frac{1}{2} \int d^2 x \mathcal{H} = \frac{1}{2} \int d^2 x \left( \frac{\beta}{4} (\epsilon_{ij} \partial_i \mathbf{\tilde{S}} \times \partial_j \mathbf{\tilde{S}})^2 + \gamma^2 V(\mathbf{\tilde{S}}) \right),$$ \hspace{1cm} (9)

where we assume nothing about the form of the potential $V$ (of course, $V \in \mathcal{C}$).
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Our goals I

we want to derive Bogomolny equations for ungauged baby Skyrme model in dimensions: (2+0) and (1+1) and for gauged baby Skyrme model in (2+0) dimensions

in contrary to [Adam etal. 2010] (where only special form of potential was investigated), [Speight 2010] (where special class of potentials was investigated) and [Adam etal. 2012]: we derive Bogomolny equations (we call them as Bogomolny decomposition), by applying so called, concept of strong necessary conditions (firstly presented in [Sokalski 1979] and developed in [Sokalski etal. 2001], [Sokalski etal.II 2001], [Sokalski etal. 2002]), for ungauged and gauged versions of restricted baby Skyrme model.
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Derivation of Bogomolny decomposition for baby Skyrme models

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Summary

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The concept of strong necessary conditions I

from the extremum principle, applied to the functional

$$\Phi[u] = \int_{E^2} F(u, u_x, u_t) \, dx dt,$$  \hspace{1cm} (10)

follow the Euler-Lagrange equations

$$F_{,u} - \frac{d}{dx} F_{,u_x} - \frac{d}{dt} F_{,u_t} = 0,$$ \hspace{1cm} (11)
instead of (11) we consider strong necessary conditions, [Sokalski 1979], [Sokalski etal. 2001], [Sokalski etal.II 2001], [Sokalski etal. 2002]

\[
\begin{align*}
F_{,u} &= 0, \\
F_{,u,t} &= 0, \\
F_{,u,x} &= 0,
\end{align*}
\]

(12) \hspace{1cm} (13) \hspace{1cm} (14)

where \( F_{,u} \equiv \frac{\partial F}{\partial u} \), etc.

all solutions of the system of the equations (12) - (14) satisfy the Euler-Lagrange equation (11)

BUT
The concept of strong necessary conditions III

- these solutions, if they exist, are very often trivial.
- a cure:
  A we make gauge transformation of the functional (10)

\[
\Phi \rightarrow \Phi + \ln v, \tag{15}
\]

where \( \ln v \) is such functional that its local variation with respect to \( u(x, t) \) vanishes: \( \delta \ln v \equiv 0 \implies \) E.-L. equations are invariant with respect to the gauge transformation (15).

B non-invariance of the strong necessary conditions (12) - (14) with respect to the gauge transformation (15) \( \implies \) some non-trivial solutions are possible

- now we apply the strong necessary conditions (12) - (14) to the gauged functional: \( \tilde{\Phi} = \Phi + \ln v \)
we obtain so called *dual equations*. 
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After making stereographic projection

\[ \vec{S} = \left[ \frac{\omega + \omega^*}{1 + \omega \omega^*}, \frac{-i(\omega - \omega^*)}{1 + \omega \omega^*}, \frac{1 - \omega \omega^*}{1 + \omega \omega^*} \right], \]  

(16)

where \( \omega = \omega(x, y) \in \mathbb{C} \) and \( x, y \in \mathbb{R} \), the density of the energy functional (9) has the form

\[ H = -4\beta \frac{(\omega, x\omega^* - \omega, y\omega^*)^2}{(1 + \omega \omega^*)^4} + V(\omega, \omega^*) \]  

(17)
Ungauged restricted baby Skyrme model II

now, we make gauge transformation, [Ł. T. S. 2012]:

\[ \mathcal{H} \rightarrow \tilde{\mathcal{H}} = -4\beta \frac{(\omega, x\omega^*_y - \omega, y\omega^*_x)^2}{(1 + \omega\omega^*)^4} + V(\omega, \omega^*) + \sum_{k=1}^{3} l_k, \quad (18) \]

where \( l_k \) are the densities of the invariants:
\( l_1 = G_1(\omega, \omega^*)(\omega, x\omega^*_y - \omega, y\omega^*_x) \) is the density of topological invariant, \( l_2 = D_x G_2(\omega, \omega^*) \), \( l_3 = D_y G_3(\omega, \omega^*) \), \( D_x \equiv \frac{d}{dx}, D_y \equiv \frac{d}{dy} \)
\( \omega = \omega(x, y), \omega^* = \omega^*(x, y) \in \mathcal{C}^2 \) and \( G_k = G_k(\omega, \omega^*) \in \mathcal{C}^2 \), \( (k = 1, 2, 3) \), are some functions, which are to be determinated.
If we apply the concept of strong necessary conditions to (18), the dual equations are, as follows

\[
\tilde{H}_{, \omega} = 16\beta \frac{(\omega_{, x} \omega_{, y} - \omega_{, y} \omega_{, x})^2 \omega^*}{(1 + \omega \omega^*)^5} + V_{, \omega}(\omega, \omega^*) + \\
G_{1, \omega}(\omega, \omega^*)(\omega_{, x} \omega_{, y} - \omega_{, y} \omega_{, x}) + D_x G_{2, \omega}(\omega, \omega^*) + \\
D_y G_{3, \omega}(\omega, \omega^*) = 0, \tag{19}
\]

\[
\tilde{H}_{, \omega^*} = 16\beta \frac{(\omega_{, x} \omega_{, y} - \omega_{, y} \omega_{, x})^2 \omega^*}{(1 + \omega \omega^*)^5} + V_{, \omega^*}(\omega, \omega^*) + \\
G_{1, \omega^*}(\omega, \omega^*)(\omega_{, x} \omega_{, y} - \omega_{, y} \omega_{, x}) + D_x G_{2, \omega^*}(\omega, \omega^*) + \\
D_y G_{3, \omega^*}(\omega, \omega^*) = 0, \tag{20}
\]
Ungauged restricted baby Skyrme model IV

\[ \tilde{H}_{\omega,x} = -8\beta \frac{(\omega,x\omega^*_y - \omega,y\omega^*_x)\omega^*_y}{(1 + \omega\omega^*)^4} + G_1(\omega,\omega^*)\omega^*_y + G_2,\omega = 0, \]  
(21)

\[ \tilde{H}_{\omega,y} = 8\beta \frac{(\omega,x\omega^*_y - \omega,y\omega^*_x)\omega^*_x}{(1 + \omega\omega^*)^4} - G_1(\omega,\omega^*)\omega^*_x + G_3,\omega = 0, \]  
(22)

\[ \tilde{H}_{\omega,x} = 8\beta \frac{(\omega,x\omega^*_y - \omega,y\omega^*_x)\omega^*_y}{(1 + \omega\omega^*)^4} - G_1(\omega,\omega^*)\omega^*_y + G_2,\omega^* = 0, \]  
(23)
\[ \tilde{H}, \omega, \omega^* = -8\beta (\omega, x\omega^*_y - \omega, y\omega^*_x)\omega, x \left(1 + \omega\omega^* \right)^4 + G_1(\omega, \omega^*)\omega, x + G_3, \omega^* = 0. \] (24)

Now, we need to make the equations (19) - (24) self-consistent \( \Rightarrow \) the necessity of the reduction of the number of independent equations by an appropriate choice of the functions \( G_k, (k = 1, 2, 3) \).

usually, such ansatzes exist only for some special \( V(\omega, \omega^*) \) \( \Rightarrow \) in most cases of \( V(\omega, \omega^*) \) for many nonlinear field models, the reduction of the system of corresponding dual equations, to Bogomolny equations, is impossible.
two operations (they were applied firstly in [Sokalski et al. 2002] for the cases of hyperbolic and elliptic systems of nonlinear PDE’s).

At first, integrating the equations (19) - (20) with respect to \( \omega \) and to \( \omega^* \), correspondingly. We get:

\[
-4\beta \frac{(\omega, y \omega_x^* - \omega, y \omega_y^*)^2}{(1 + \omega \omega^*)^4} + V(\omega, \omega^*) + G_1(\omega, \omega^*)(\omega, x \omega_y^* - \omega, y \omega_x^*) + \\
D_x G_2(\omega, \omega^*) + D_y G_3(\omega, \omega^*) = F(\omega, x, \omega, y, \omega_x^*, \omega_y^*),
\]

where \( F \) is some function, which will be determined later.
making the equations (21) - (24) self-consistent: proper choice of the functions $G_k, k = 1, 2, 3$:

proper multiplying of the equations (21) - (24) by $\omega, x, \omega, y, \omega^*, x, \omega^*, y$, correspondingly, and adding by sides obtained equations, we get

$$-8\beta \frac{(\omega, x\omega^*_y - \omega, y\omega^*_x)^2}{(1 + \omega\omega^*)^4} + G_1(\omega, \omega^*)(\omega, x\omega^*_y - \omega, y\omega^*_x) + D_x G_2(\omega, \omega^*) = 0,$$

(26)
Ungauged restricted baby Skyrme model VII

\[-8\beta \frac{(\omega, x\omega^*, y - \omega, y\omega^*)^2}{(1 + \omega\omega^*)^4} + G_1(\omega, \omega^*) (\omega, x\omega^*, y - \omega, y\omega^*) + D_y G_3(\omega, \omega^*) = 0.\]  

(27)

Hence:

\[D_x G_2(\omega, \omega^*) = D_y G_3(\omega, \omega^*).\]  

(28)
IV now: multiplying again the equations (21) - (24) by \( \omega, x, \omega, y, \omega^*, x, \omega^*, y \) and add by sides, but such, that we get

\[
D_y G_2(\omega, \omega^*) = 0, \quad D_x G_3(\omega, \omega^*) = 0.
\]

(29)

the relations (26), (27) and (29) - so called divergent representation (the divergent representation was derived firstly in [Sokalski etal. 2002] for hyperbolic system of two coupled nonlinear partial differential equations).

V Hence, and from (28)

\[
G_2(\omega, \omega^*) = \text{const}, \quad G_3(\omega, \omega^*) = \text{const}.
\]

(30)
Hence, after inserting (30) into (26), (27) and simplifying, we get:

\[ \omega, x \omega^*, y - \omega, y \omega^*, x = \frac{1}{8\beta} G_1(\omega, \omega^*)(1 + \omega \omega^*)^4. \]  

(31)

The same result follows from (21)-(24) and all solutions of (31) satisfy the equations (21) - (24)

When the equation (25) is satisfied by the solutions of (31) ? We insert (30) and (31), into the equation (25)

\[ V(\omega, \omega^*) + \frac{1}{16\beta} G_1^2(\omega, \omega^*)(1 + \omega \omega^*)^4 = F(\omega, x, \omega, y, \omega^*, x, \omega^*, y). \]  

(32)
Now, in order to determining function $F$, we compare (32) with Hamilton-Jacobi equation, [Rund 1966], [Sokalski etal. 2002]:

$$\tilde{H} = 0,$$

(33)

where, of course $\tilde{H}$ in general, for $\omega = \omega(x^\mu), \omega^* = \omega^*(x^\mu)$, ($\mu = 0, 1, 2, 3$ and $x^0 = t$):

$$\tilde{H} = \Pi_\omega \omega, t + \Pi_{\omega^*} \omega^*, t - \tilde{L},$$

(34)

where $\Pi_\omega = \tilde{\mathcal{L}}_{\omega, t}, \Pi_{\omega^*} = \tilde{\mathcal{L}}_{\omega^*, t}$ - canonical momenta and $\tilde{L}$ - Lagrange density, gauge-transformed on the invariants $I_k, (k = 1, 2, 3)$
Obviously, in our case: \( \tilde{\mathcal{H}} = -\tilde{\mathcal{L}} \). Hence, by inserting into this equation, the relations (30) and (31), and taking into account (33), we get that \( F = 0 \). So, we get

\[
V(\omega, \omega^*) = -\frac{1}{16\beta} G_1^2(\omega, \omega^*)(1 + \omega\omega^*)^4. \tag{35}
\]

Then, of course,

\[
G_1 = \frac{4i\sqrt{\beta}}{(1 + \omega\omega^*)^2} \sqrt{V(\omega, \omega^*)}. \tag{36}
\]
We insert (36) in (31) and we obtain Bogomolny decomposition for the given potential $V(w, w^*)$

$$\omega_x \omega_y^* - \omega_y \omega_x^* = \frac{i}{2\sqrt{\beta}} \sqrt{V(\omega, \omega^*)}(1 + \omega^*)^2. \quad (37)$$

Then, the equation (37) is Bogomolny decomposition (Bogomolny equation) for restricted baby Skyrme model in (2+0) dimensions, for arbitrary potential.

We find an exact solution of Bogomolny decomposition (37) for $V = (\omega \omega^* - \gamma^2)^2$ - “Mexican hat” potential, i.e. it is the model with spontaneously broken symmetry.
Ungauged restricted baby Skyrme model XIV

Figure: The potential
Ungauged restricted baby Skyrme model XV

We use so called “hedgehog” ansatz

\[ \omega = \frac{\sin(f(r)) \cos(N\theta) + i \sin(f(r)) \sin(N\theta)}{1 + \cos(f(r))} \]

(38)

After inserting it into the Bogomolny decomposition, and solving obtained ODE, we get exact solution for \( f(r) \) and we present here the figures of: the function \( f(r) \), energy density and the components \( S^i, i = 1, 2, 3 \), for \( \gamma = 5, N = 1 \) and \( \gamma = 5, N = 5 \), correspondingly (of course, \( \omega = u + iv, r^2 = x^2 + y^2 \)): 
Ungauged restricted baby Skyrme model XVI

Figure: Function $f(r)$ and energy density
Ungauged restricted baby Skyrme model XVII

Figure: Components of vector $\vec{S}$
Ungauged restricted baby Skyrme model XVIII

Figure: Function $f(r)$ and energy density
Ungauged restricted baby Skyrme model XIX

Figure: Components of vector $\vec{S}$
Gauged restricted baby Skyrme model I

- full gauged baby Skyrme model

\[ \mathcal{L} = D_\mu \vec{S} \cdot D^\mu \vec{S} + \frac{\lambda^2}{4} (D^\mu \vec{S} \times D^\nu \vec{S})^2 + (1 - \vec{n} \cdot \vec{S}) + F_{\mu\nu}^2, \quad (39) \]

- gauged restricted baby Skyrme model with the potential \( V = (1 - \vec{n} \cdot \vec{S}) \), the special case of gauged full baby Skyrme model (8), when the \( O(3) \)-like term is absent, [Ł. T. S. II 2012]

\[ \mathcal{L} = \frac{\lambda^2}{4} (D_\mu \vec{S} \times D_\nu \vec{S})^2 + F_{\mu\nu}^2 + (1 - \vec{n} \cdot \vec{S}), \quad (40) \]

where \( \vec{S} \) is three-component vector, such that \( |\vec{S}|^2 = 1 \) and \( D_\mu \vec{S} = \partial_\mu \vec{S} + A_\mu (\vec{n} \times \vec{S}) \) is covariant derivative of vector field \( \vec{S} \).
we consider gauged restricted baby Skyrme model in (2+0)
dimensions, [Ł. T. S. II 2012]

\[
H = \frac{1}{2} \int d^2x \quad \mathcal{H} = \frac{1}{2} \int d^2x \left( \frac{\lambda^2}{4} (\epsilon_{ij} D_i \vec{S} \times D_j \vec{S})^2 + F_{\mu\nu}^2 + \gamma^2 V(1 - \vec{n} \cdot \vec{S}) \right),
\]

where \( x_1 = x, \ x_2 = y \) and \( i, j = 1, 2 \).

we make the stereographic projection

\[
\vec{S} = \begin{bmatrix}
\frac{\omega + \omega^*}{1 + \omega \omega^*}, & -i(\omega - \omega^*) & \frac{1 - \omega \omega^*}{1 + \omega \omega^*}
\end{bmatrix},
\]

where \( \omega = \omega(x, y) \in \mathbb{C} \) and \( x, y \in \mathbb{R} \).
Gauged restricted baby Skyrme model III

Then (after rescalling, the constants $\lambda_1, \lambda_2$ have been appeared, instead of $\lambda$ and $\gamma$ has been included in $V$):

$$\mathcal{H} = \frac{1}{16\lambda_1} N_1^2 (1 + \omega \omega^*)^4 + \lambda_2 (A_{2,x} - A_{1,y})^2 + V\left(\frac{2\omega\omega^*}{1 + \omega\omega^*}\right),$$

where: $N_1 = \frac{8\lambda_1}{(1 + \omega\omega^*)^4} [i(\omega, x\omega^* - \omega, y\omega^*) - A_1 (\omega, y\omega^* + \omega\omega^*)] + A_2 (\omega, x\omega^* + \omega\omega^*)].$
The Euler-Lagrange equations for this model are, as follows

\[
\frac{d}{dx} [N_1(i\omega^*_y + A_2\omega^*)] + \frac{d}{dy} [N_1(-i\omega^*_x - A_1\omega^*)] + \\
\frac{1}{4\lambda_1} N_1^2 \omega^* (1 + \omega^*)^3 - N_1 (-A_1\omega^*_{,y} + A_2\omega^*_{,x}) - \\
V' \left( \frac{2\omega^*}{1 + \omega^*} \right) \frac{2\omega^*}{(1 + \omega^*)^2} = 0, \text{ c.c.} \tag{44}
\]

\[-2\lambda_2 \frac{d}{dy} (A_2_{,x} - A_1_{,x}) + N_1 \cdot (\omega_{,y}\omega^* + \omega\omega^*_{,y}) = 0\]

\[2\lambda_2 \frac{d}{dx} (A_2_{,x} - A_1_{,x}) - N_1 \cdot (\omega_{,x}\omega^* + \omega\omega^*_{,x}) = 0\]
beside the lagrangian, in order to apply concept of strong necessary conditions, we need also topological invariant (also gauge invariance required), its density, [Schroers 1995], [Yang 2001]:

\[
I_1 = \vec{S} \cdot D_1 \vec{S} \times D_2 \vec{S} + F_{12}(1 - \vec{n} \cdot \vec{S}),
\]

(45)

after making the stereographic projection (42), we have:

\[
I_1 = \frac{1}{(1 + \omega\omega^*)^2} \left[ 2(i(\omega, x\omega^* - \omega, y\omega^*) - A_1(\omega, y\omega^* + \omega\omega^*)_x + A_2(\omega, x\omega^* + \omega\omega^*)_y) \right] + \frac{2\omega\omega^*}{1 + \omega\omega^*}(A_{2, x} - A_{1, y}).
\]

(46)
Gauged restricted baby Skyrme model VI

- it is useful to generalize the above expression such that there by the term $A_{2,x} - A_{1,y}$, some function of the argument $\frac{2 \omega \omega^*}{1 + \omega \omega^*}$ may be placed.

$$I_1 = \lambda_4 \left\{ \frac{1}{(1 + \omega \omega^*)^2} \left[ 2 G_1' \cdot (i(\omega,x \omega^* - \omega,y \omega^*,x) - A_1(\omega,y \omega^* + \omega \omega^*,y) + A_2(\omega,x \omega^* + \omega \omega^*,x)) \right] + G_1 \cdot (A_{2,x} - A_{1,y}) \right\} \right.,$$

where $\lambda_4 = const$, $G_1 = G_1 \left( \frac{2 \omega \omega^*}{1 + \omega \omega^*} \right)$ and $G_1'$ denotes the derivative of the function $G_1$ with respect to its argument: $\frac{2 \omega \omega^*}{1 + \omega \omega^*}$. 
we make the following gauge transformation

\[
\mathcal{H} \rightarrow \tilde{\mathcal{H}} = \frac{1}{16\lambda_1} N_1^2 (1 + \omega \omega^*)^4 + \lambda_2 (A_{2,x} - A_{1,y})^2 + V\left(\frac{2\omega \omega^*}{1 + \omega \omega^*}\right) + \sum_{k=1}^{3} I_k, \tag{48}
\]

where \( N_1 = \frac{8\lambda_1}{(1 + \omega \omega^*)^4} [i(\omega, x \omega^* - y, y \omega^*) - A_1 (\omega, y \omega^* + \omega y^*) + A_2 (\omega, x \omega^* + \omega x^*)] \),

\( I_1 \) is given by (47),

\( I_2 = D_x G_2(\omega, \omega^*) \), \( I_3 = D_y G_3(\omega, \omega^*) \), \( D_x \equiv \frac{d}{dx}, D_y \equiv \frac{d}{dy} \) and \( G_k \in \mathcal{C}^2, (k = 1, 2, 3) \), are some functions, which are to be determined.
After applying the concept of strong necessary conditions to (18), we obtain the dual equations

\[
\omega : - \frac{1}{4\lambda_1} N_1^2 \omega^* (1 + \omega \omega^*)^3 + N_1 (-A_1 \omega_{,y} + A_2 \omega_{,x}) + V' \frac{2\omega^*}{(1 + \omega \omega^*)^2} + \\
\lambda_4 \left\{ G_1' \frac{N_1 \omega^*}{2\lambda_1} + \frac{2G_1' (-A_1 \omega_{,y} + A_2 \omega_{,x})}{(1 + \omega \omega^*)^2} - \\
G_1' \frac{N_1}{2\lambda_1} (1 + \omega \omega^*) \omega^* + G_1' \frac{2\omega^*}{(1 + \omega \omega^*)^2} (A_{2,x} - A_{1,y}) \right\} + \\
D_x G_{2,\omega} + D_y G_{3,\omega} = 0
\]

(49)
The case of \((2+0)\)-dimensions

\[
\omega^* : -\frac{1}{4\lambda_1} N_1^2 \omega (1 + \omega \omega^*)^3 + N_1 (-A_1 \omega_y + A_2 \omega_x) + V' \frac{2\omega}{(1 + \omega \omega^*)^2} + 
\]

\[
\lambda_4 \left\{ G_1'' \frac{N_1 \omega}{2\lambda_1} + \frac{2G_1' (-A_1 \omega^*_y + A_2 \omega^*_x)}{(1 + \omega \omega^*)^2} - 
\right.
\]

\[
G_1' \frac{N_1}{2\lambda_1} (1 + \omega \omega^*) \omega + G_1' \frac{2\omega}{(1 + \omega \omega^*)^2} (A_2, x - A_1, y) \right\} + 
\]

\[
D_x G_{2, \omega^*} + D_y G_{3, \omega^*} = 0
\]
Gauged restricted baby Skyrme model $X$

$$
\omega_x : N_1(i\omega^*_y + A_2\omega^*) + \frac{2\lambda_4 G'_1(i\omega^*_y + A_2\omega^*)}{(1 + \omega\omega^*)^2} + G_{2,\omega} = 0, \quad (51)
$$

$$
\omega_y : N_1(-i\omega^*_x - A_1\omega^*) + \frac{2\lambda_4 G'_1(-i\omega^*_x - A_1\omega^*)}{(1 + \omega\omega^*)^2} + G_{3,\omega} = 0, \quad (52)
$$

$$
\omega^*_x : N_1(-i\omega^*_y + A_2\omega) + \frac{2\lambda_4 G'_1(-i\omega^*_y + A_2\omega)}{(1 + \omega\omega^*)^2} + G_{2,\omega^*} = 0, \quad (53)
$$
Gauged restricted baby Skyrme model XI

\[ \omega_{,y} : N_1 (i\omega, x - A_1 \omega) + \frac{2\lambda_4 G'_1 (i\omega, x - A_1 \omega)}{(1 + \omega \omega^*)^2} + G_{3, \omega^*} = 0, \quad (54) \]

\[ A_1 : N_1 (-\omega, y \omega^* - \omega \omega_{,y}) + \frac{2\lambda_4 G'_1 (-\omega, y \omega^* - \omega \omega_{,y})}{(1 + \omega \omega^*)^2} = 0, \quad (55) \]

\[ A_2 : N_1 (\omega, x \omega^* + \omega \omega_{,x}) + \frac{2\lambda_4 G'_1 (\omega, x \omega^* + \omega \omega_{,x})}{(1 + \omega \omega^*)^2} = 0, \quad (56) \]

\[ A_{1, y} : -2\lambda_2 (A_{2, x} - A_{1, y}) - \lambda_4 G_1 = 0, \quad (57) \]
A_{2,x} : 2\lambda_2 (A_{2,x} - A_{1,y}) + \lambda_4 G_1 = 0, \quad (58)

where \( N_1 = \frac{8\lambda_1}{(1+\omega\omega^*)^4} \left[ i(\omega, x\omega^* - y, y\omega^*_x) - A_1(\omega, y\omega^* + \omega\omega^*_y) + A_2(\omega, x\omega^* + \omega\omega^*_x) \right] \)

and \( V', G'_1, G''_1 \) denote the derivatives of the functions \( V \) and \( G_1 \) with respect to their argument: \( \frac{2\omega\omega^*}{1+\omega\omega^*} \).

making the equations (49) - (58) self-consistent:
we put:

\[ G'_1 = -\frac{N_1}{2\lambda_4} (1 + \omega\omega^*)^2, \quad (59) \]

\[ A_{2,x} - A_{1,y} = -\frac{\lambda_4}{2\lambda_2} G_1 \left( \frac{2\omega\omega^*}{1 + \omega\omega^*} \right), \quad (60) \]

\[ G_2 = \text{const}, \quad G_3 = \text{const}, \quad (61) \]

where \( N_1 = \frac{8\lambda_1}{(1 + \omega\omega^*)^4} [i(\omega_x\omega^*_y - \omega_y\omega^*_x) - A_1(\omega_y\omega^* + \omega\omega^*_x) + A_2(\omega_x\omega^* + \omega\omega^*_x)]. \]
then, the equations (51)-(58) become the tautologies and we have the candidate for Bogomolny decomposition:

\[
4 \lambda_1 \left[i(\omega,x \omega^*,y - \omega,y \omega^*,x) - A_1(\omega,y \omega^* + \omega \omega^*,y) + A_2(\omega,x \omega^* + \omega \omega^*,x)\right] \lambda_4 (1 + \omega \omega^*)^2
\]

\[
- G'_1,
\]

\[
2 \lambda_2 (A_2,x - A_1,y) + \lambda_4 G_1 \left(\frac{2 \omega \omega^*}{1 + \omega \omega^*}\right) = 0.
\]

(62)
When the equations (49)-(50) are satisfied, if (62) hold? We insert (59)-(61) into (49)-(50). We get the system of ordinary differential equations for $V$ and the solution of it is:

$$V = \frac{\lambda_4^2}{4} \left( \frac{1}{\lambda_1} (G'_1)^2 + \frac{1}{\lambda_2} G_1^2 \right). \quad (63)$$
So, we obtain Bogomolny decomposition for gauged restricted baby Skyrme model in (2+0) dimensions

\[
4\lambda_1 \left[ i(\omega, x\omega_y^* - \omega, y\omega_x^*) - A_1(\omega, y\omega^* + \omega\omega_x^*) + A_2(\omega, x\omega^* + \omega\omega_x^*) \right] \frac{\lambda_4 (1 + \omega\omega^*)^2}{\lambda_4 (1 + \omega\omega^*)^2} = -G_1',
\]

\[
A_{2,x} - A_{1,y} = -\frac{\lambda_4}{2\lambda_2} G_1 \left( \frac{2\omega\omega^*}{1 + \omega\omega^*} \right).
\]

for the potential \(V(\frac{2\omega\omega^*}{1 + \omega\omega^*})\), satisfying

\[
V = \frac{\lambda_4^2}{4} \left( \frac{1}{\lambda_1} (G_1')^2 + \frac{1}{\lambda_2} G_1^2 \right),
\]

(64)
where $G_1 = G_1 \left( \frac{2\omega \omega^*}{1 + \omega \omega^*} \right) \in C^2$. 
Outline

1 Motivation
   - Bogomolny equations - an introduction
   - Baby Skyrme model - an introduction
   - Our goals
   - The concept of strong necessary conditions

2 Derivation of Bogomolny decomposition for baby Skyrme models
   - The case of (2+0)-dimensions
     - Ungauged restricted baby Skyrme model
     - An example
     - Gauged restricted baby Skyrme model
   - The case of (1+1)-dimensions

3 Summary

4 Acknowledgements

5 References
The case of (1+1)-dimensions I

- the lagrangian:

\[ \mathcal{L} = -4\beta \frac{(\omega, t\omega_x - \omega, x\omega_t)^2}{(1 + \omega\omega^*)^4} + V(\omega, \omega^*) \]  

(66)

- the gauge transformation of Lagrangian, [Ł. T. S. 2012]:

\[ \mathcal{L} \longrightarrow \tilde{\mathcal{L}} = -4\beta \frac{(\omega, t\omega_x - \omega, x\omega_t)^2}{(1 + \omega\omega^*)^4} + V(\omega, \omega^*) + \sum_{k=1}^{3} l_k, \]  

(67)

where now: \( l_1 = G_1(\omega, \omega^*)(\omega, t\omega_x - \omega, x\omega_t) \), \( l_2 = D_t G_2(\omega, \omega^*) \), \( l_3 = D_x G_3(\omega, \omega^*) \), \( D_t \equiv \frac{d}{dt}, D_x \equiv \frac{d}{dx} \)
The case of (1+1)-dimensions II

\[ \omega = \omega(t, x), \omega^* = \omega^*(t, x) \in C^2 \text{ and } G_k = G_k(\omega, \omega^*) \in C^2, \]

\((k = 1, 2, 3)\), are some functions, which are to be determinated.

Further computations are analogical to the computations in the case of (2+0)-dimensions, i.e. dual equations have very similar form and \( G_2 = const, G_3 = const \) and:

\[
\omega, t \omega^*_x - \omega, x \omega^*_t = \frac{1}{g\beta} G_1(\omega, \omega^*)(1 + \omega \omega^*)^4. \quad (68)
\]

but with one difference: the Hamilton-Jacobi equation has now another form. Namely, let us remind [Rund 1966], [Sokalski et al. 2002]:

\[
\tilde{H} = 0, \quad (69)
\]
The case of (1+1)-dimensions III

where, of course $\tilde{\mathcal{H}}$ in general, for $\omega = \omega(x^\mu), \omega^* = \omega^*(x^\mu), (\mu = 0, 1, 2, 3$ and $x^0 = t)$:

$$\tilde{\mathcal{H}} = \Pi_\omega \omega, t + \Pi_{\omega^*} \omega^*, t - \tilde{\mathcal{L}}, \quad (70)$$

$$\Pi_\omega = \tilde{\mathcal{L}}, \omega, t, \Pi_{\omega^*} = \tilde{\mathcal{L}}, \omega^*, t. \quad (71)$$

Obviously, in the current case:

$$\tilde{\mathcal{H}} = -4\beta \frac{(\omega, t \omega^* x - \omega, x \omega^*, t)^2}{(1 + \omega \omega^*)^4} - V(\omega, \omega^*) = 0. \quad (72)$$
The case of (1+1)-dimensions IV

After taking into account (74) and (72), we have

\[ V(\omega, \omega^*) + \frac{1}{16\beta} G^2_1(\omega, \omega^*)(1 + \omega \omega^*)^4 = 0. \]  (73)

Thus, we have obtained the same relation between the potential, as in the case of (2+0)-dimensions. The Bogomolny decomposition for this case has the form:

\[ \omega, t\omega^*_x - \omega, x\omega^*_t = \frac{i}{2\sqrt{\beta}} \sqrt{V(\omega, \omega^*)(1 + \omega \omega^*)^2}. \]  (74)
Summary I

The Bogomolny decomposition (the system of Bogomolny equations) has been derived, by using the concept of strong necessary conditions in:

(2+0)-dimensions, for:

- ungauged restricted baby Skyrme model, for arbitrary form of the potential, in contrary to [Adam et al. 2010] (where only special form of potential was investigated), [Speight 2010] (where special class of potentials was investigated),
- gauged restricted baby Skyrme model, the “gauging” of the model causes the condition for the potential in contrary to ungauged model.
(1+1)-dimensions for ungauged restricted baby Skyrme model, for arbitrary form of the potential.

- The figures of example exact solution of Bogomolny decomposition and corresponding energy densities, for ungauged restricted baby Skyrme model in (2+0)-dimensions, have been presented.

- further investigation of other Skyrme-like models and the further solutions of found Bogomolny decompositions and their physical features: work in progress.
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