Graph C*-algebras with applications to quantum spaces II

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Directed graphs

Recall from last time:

A directed graph $E = (E^0, E^1, r, s)$ consists of a countable set E^0 of vertices, a countable set E^1 of edges, and the range and source maps $r, s : E^1 \to E^0$.

Row finiteness and adjacency matrix

A graph $E = (E^0, E^1, r, s)$ is **row finite** if every vertex receives at most finitely many edges.

This can also be described via the adjacency matrix of E.

The **adjacency matrix** $A_E = (a_{v,w})_{v,w \in E^0} \in M_{E^0 \times E^0}(\mathbb{Z})$ of E is defined by

$$a_{v,w} = |\{e \in E^1 \mid r(e) = v, s(e) = w\}|.$$

A graph is row finite if and only if the sum of each row of A_E is finite.

Cuntz–Krieger relations

Let E be a row-finite graph.

A Cuntz–Krieger E-family $\{S, P\}$ on a Hilbert space H consists of pairwise orthogonal projections $\{P_v \in \mathscr{B}(H) \mid v \in E^0\}$ and partial isometries $\{S_e \in \mathscr{B}(H) \mid e \in E^1\}$ satisfying the Cuntz–Krieger relations:

(CK1) $S_e^* S_e = P_{s(e)}$ for every $e \in E^1$, and

(CK2) $P_v = \sum_{e \in r^{-1}(v)} S_e S_e^*$ for every $v \in E^1$ that is not a source.

Cuntz–Krieger E-families for which all operators are non-zero always exist.

An implication of the CK relations

Let $e, f \in E^1$, and consider $S_e S_f \in \mathscr{B}(H)$.

Two applications of the C*-equality tells us that

$$\|S_e S_f\|^4 = \|(S_f^* S_e^*)(S_e S_f)\|^2 = \|(S_f^* S_e^* S_e S_f)(S_f^* S_e^* S_e S_f)\|.$$

Note that $S_e^* S_e = P_{s(e)}$ and $S_f S_f^* \leq P_{r(f)}$. It follows that $S_e^* S_e S_f S_f^* = 0$ if $s(e) \neq r(f)$.

Thus $\|S_e S_f\|^4 = \|S_f^* S_e^* S_e S_f S_f^* S_e^* S_e S_f\| = 0$ if $s(e) \neq r(f)$, and so $S_e S_f = 0$ whenever $s(e) \neq r(f)$.

On the other hand, if $s(e) = r(f)$, we have $S_e^* S_e S_f S_f^* = S_f S_f^*$, so

$$\|S_e S_f\|^4 = \|S_f^* S_e^* S_e S_f S_f^* S_e^* S_e S_f\| = 1$$

whenever $S_f \neq 0$.

Similar calculations give us the following:

Proposition. *Let E be a row-finite graph. Then any Cuntz–Krieger E-family $\{S, P\}$ satisfies the following:*

- *the projections $\{S_e S_e^* \mid e \in E^1\}$ are mutually orthogonal;*
- *if $S_e^* S_f \neq 0$ then $e = f$,*
- *if $S_e S_f \neq 0$ then $s(e) = r(f)$,*
- *if $S_e S_f^* \neq 0$ then $s(e) = s(f)$.*

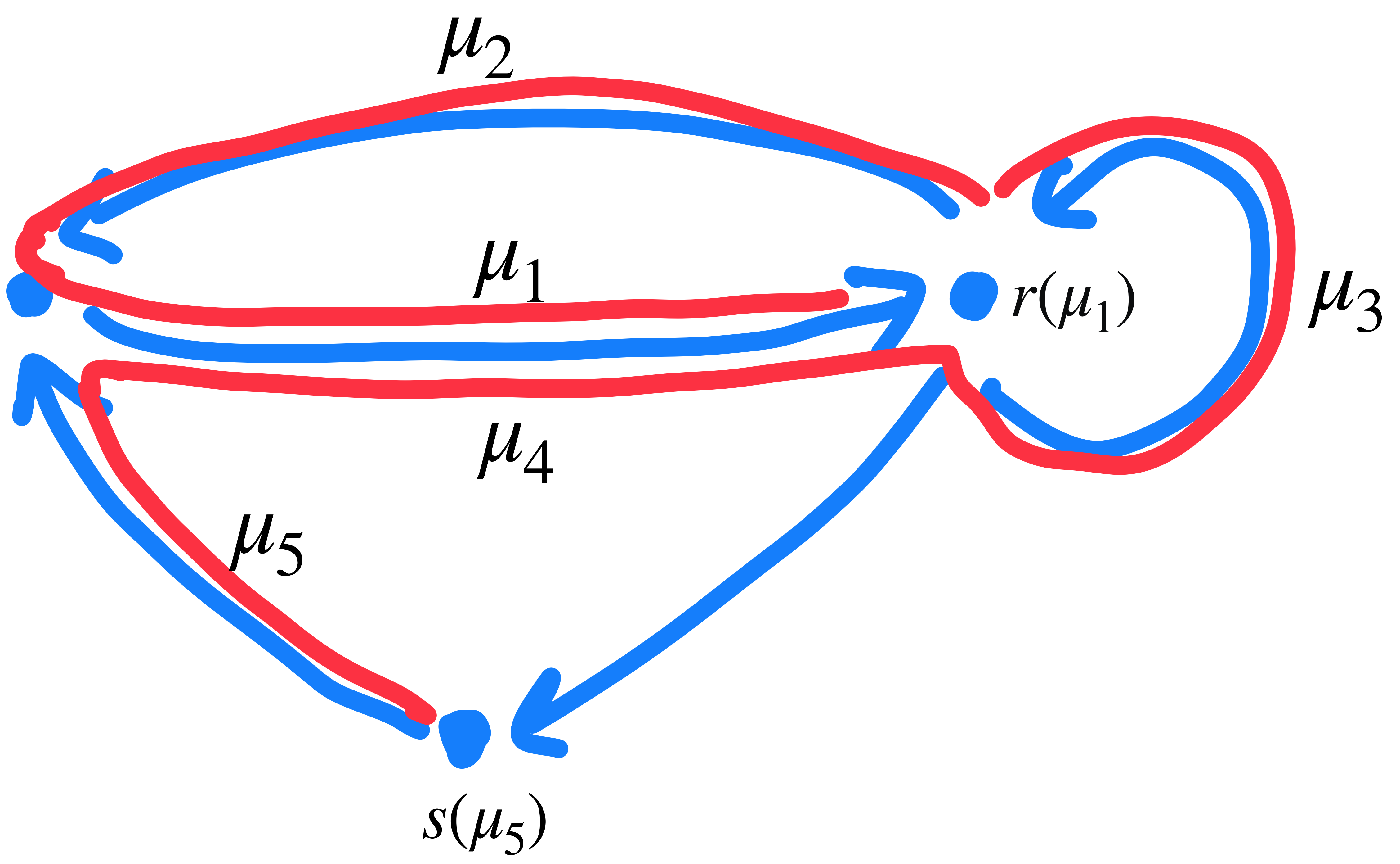
A path in E of length $n \in \mathbb{Z}_{>0}$ is a sequence $\mu = \mu_1 \mu_2 \ldots \mu_n$ of edges $\mu_i \in E^1$ such that $s(\mu_i) = r(\mu_{i+1})$ for $1 \leq i \leq n - 1$. The above allows us to define the operator

$$S_\mu := S_{\mu_1} S_{\mu_2} \cdots S_{\mu_n}.$$

Let $\mu = \mu_1\mu_2\ldots\mu_n$ be a path of length n.

Then

$$
\begin{aligned}
S_\mu^* S_\mu &= S_{\mu_n}^* \cdots S_{\mu_2}^* S_{\mu_1}^* S_{\mu_1} \cdots S_{\mu_2} S_{\mu_n} \\
&= S_{\mu_n}^* \cdots S_{\mu_2}^* P_{s(\mu_1)} S_{\mu_2} \cdots S_{\mu_n} \\
&= S_{\mu_n}^* \cdots S_{\mu_2}^* P_{r(\mu_2)} S_{\mu_2} \cdots S_{\mu_n} \\
&= S_{\mu_n}^* \cdots S_{\mu_2}^* S_{\mu_2} \cdots S_{\mu_n} \\
&\vdots \\
&= S_{\mu_n}^* S_{\mu_n} = P_{s(\mu_n)}.
\end{aligned}
$$



Similarly $S_\mu S_\mu^* \leq S_{\mu_1} S_{\mu_1}^*$. So paths behave like edges.

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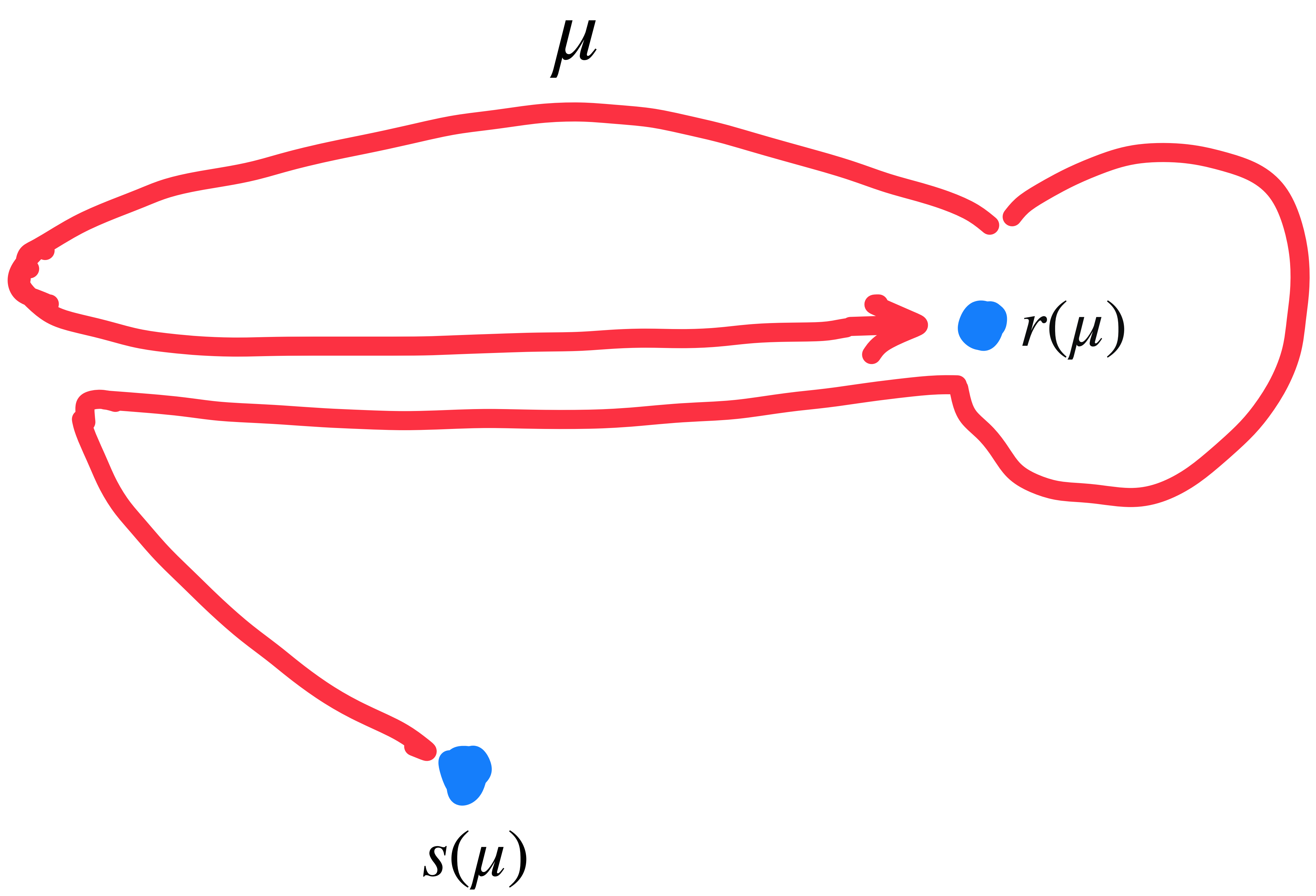

Similarly $S_\mu S_\mu^* \le S_{\mu_1} S_{\mu_1}^*$. So paths behave like edges.

For $n \geq 1$, let E^n denote the paths of length n and let $E^* = \sqcup_{n \geq 0} E^n$ denote the set of all finite paths (we can think of a vertex of a path of length zero)

Proposition. *Let E be a row-finite graph. Then any Cuntz–Krieger E-family $\{S, P\}$ satisfies the following:*

- *the projections $\{S_\mu S_\mu^* \mid \mu \in E^n\}$ are mutually orthogonal;*
- *if $S_\mu^* S_\nu \neq 0$ then $\mu = \nu\mu'$ or $\nu = \nu'\mu$ for some $\mu', \nu' \in E^*$,*
- *if $S_\mu S_\nu \neq 0$ then $\mu\nu \in E^*$,*
- *if $S_\mu S_\nu^* \neq 0$ then $s(\mu) = s(\nu)$.*

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- *if $S_\mu^* S_\nu \neq 0$ then $\mu = \nu\mu'$ or $\nu = \nu'\mu$ for some $\mu', \nu' \in E^*$, and $S_{\nu\mu'}^* S_\nu = S_{\mu'}^*$ or $S_\mu^* S_{\nu'\mu} = S_{\nu'}$*
- *if $S_\mu S_\nu \neq 0$ then $\mu\nu \in E^*$, and $S_\mu S_\nu = S_{\mu\nu}$*
- *if $S_\mu S_\nu^* \neq 0$ then $s(\mu) = s(\nu)$.*

For a Cuntz–Krieger E-family $\{S, P\}$ on H, we define $C^*(\{S, P\})$ to be the C*-algebra generated by $\{P_v \mid v \in E^0\} \cup \{S_e \mid e \in E^1\}$ in $\mathscr{B}(H)$.

Let $E^n := \{\text{paths of length } n\}$ and let $E^* := \cup_{n \in \mathbb{Z}_{\geq 0}} E^n$ denote the set of all finite length paths. Then

$$C^*(\{S, P\}) = \overline{\operatorname{span}}\{S_\mu S_\nu^* \mid \mu, \nu \in E^*, s(\mu) = s(\nu)\}.$$

The graph C*-algebra $C^*(E)$

Let $E = (E^0, E^1, r, s)$ be a directed graph.

Let

$$V_E := \{\lambda_{\mu,\nu} d_{\mu,\nu} \mid \lambda_{\mu,\nu} \in \mathbb{C}, \mu, \nu \in E^*\}.$$

Equip V_E with pointwise addition, multiplication given by

$$d_{\mu_1,\nu_1} d_{\mu_2,\nu_2} = \begin{cases} d_{\mu_1\alpha,\, \nu_2} \text{ if } \exists \alpha \in E^* : \mu_2 = \nu_1\alpha \\ d_{\mu_1,\, \nu_2\beta} \text{ if } \exists \beta \in E^* : \nu_1 = \mu_2\beta \\ 0 \text{ otherwise,} \end{cases}$$

and $* : V_E \to V_E$ by $(d_{\mu,\nu})^* = d_{\nu,\mu}$.

Then V_E is a *-algebra.

The graph C*-algebra $C^*(E)$

Any Cuntz–Krieger E-family $\{S, P\}$ on H gives rise to a *-representation $\pi_{S,P} : V_E \to \mathscr{B}(H)$ by defining

$$\pi_{S,P}(d_{\mu,\nu}) = S_\mu S_\nu^*.$$

Let v be a partial isometry in a C*-algebra A. Then $\|v\|^4 = \|v^*v\|^2 = \|v^*v\| = \|v\|^2$. So $\|v\| \in \{0,1\}$.

In particular, for any Cuntz–Krieger E-family $\{S, P\}$, we have $\|\pi_{S,P}(d_{\mu,\nu})\| \leq 1$.

The graph C*-algebra $C^*(E)$

It follows that

$$\|a\| = \sup\{\|\pi_{S,P}(a)\|_{\{S,P\}} \mid \{S,P\} \text{ a Cuntz–Krieger } E\text{-family}\}$$

is a well-defined C*-norm on V_E. (Note that it is indeed a norm since $\forall \mu, \nu\ \exists$ CK E-family such that $\pi_{S,P}(d_{\mu,\nu}) \neq 0$).

We define the **graph C*-algebra** of E to be $C^*(E) := \overline{V(E)}^{\|\cdot\|}$, the completion of $V(E)$ with respect to a$\|\cdot\|$.

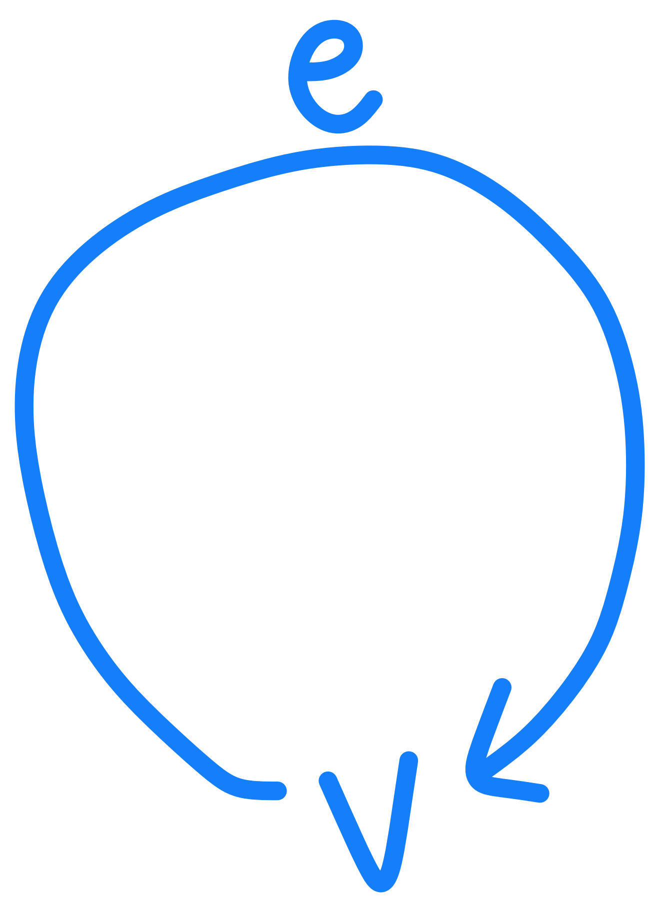
$C^*(E)$ is universal

$C^*(E)$ is the universal C*-algebra for the Cuntz–Krieger relations in the following sense:

Proposition: *Let E be a row-finite directed graph. Suppose A is a C*-algebra generated by a Cuntz-Krieger E-family $\{W, R\}$ with the following property: for every Cuntz–Krieger E-family $\{T, Q\}$ in a C*-algebra B, there is a *-homomorphism $\rho_{T,Q} : A \to B$ such that $\rho_{T,Q}(W_e) = T_e$ for every $e \in E^1$ and $\rho_{T,Q}(R_v) = Q_v$ for every $v \in E^0$.*

Then there is an isomorphism $\varphi : C^*(E) \to A$ satisfying $\varphi(d_{e,s(e)}) = W_e$ and $\varphi(d_{v,v}) = R_v$, for every $e \in E^1, v \in E^0$.

Examples



The Cuntz–Krieger relations for this graph tell us we have one projection p_v and a partial isometry s_e satisfying

$$s_e^* s_e = p_v \text{ and } p_v = s_e s_e^*.$$

In particular, s_e commutes with s_e^* and since $s_e p_v = s_e(s_e^* s_e) = (s_e s_e^*) s_e = p_v s_e$ we see that s_e commutes with p_v.

Moreover, $s_e p_v = s_e s_e^* s_e = s_e$ and $p_v s_e = s_e$, so p_v acts as a unit.

This implies that s_e is a unitary. The Gelfand Theorem says

$$C^*(u,1) \cong C(\mathrm{sp}(u)).$$

Examples

The spectrum of a unitary is always contained in $\mathbb{T} = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$.

The function $f \in C(\mathbb{T})$ given by $f(z) = z$ is a unitary with $\text{sp}(f) = \mathbb{T}$.

For any unitary u we can define the inclusion map $\text{sp}(u) \to \mathbb{T}$.

Then we have an induced map $C(\mathbb{T}) \to C^*(u,1)$ which sends $f \to u$.

Thus by the universal property we have

$$C(\mathbb{T}) = C^*(f,1) \cong C^*(E)$$

Examples

Here we have

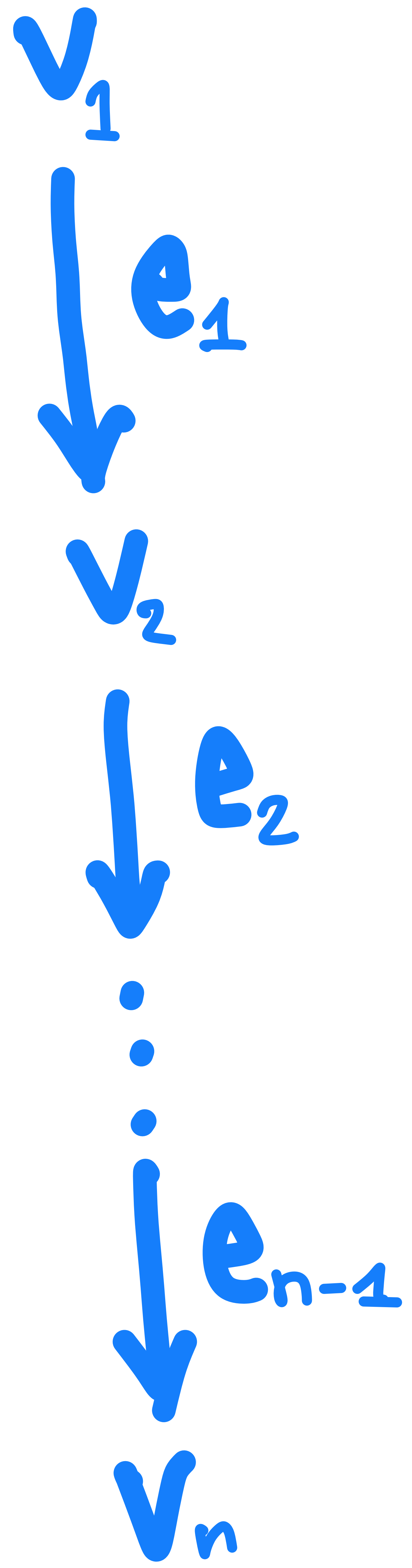
$s_{e_k}^* s_{e_k} = p_{v_k}$ and $s_{e_k} s_{e_k}^* = p_{v_{k+1}}$ for every $1 \leq k \leq n-1$.

Define a map $\varphi : M_n(\mathbb{C}) \to C^*(E)$ by

$\varphi(e_{k+1,k}) = s_{e_k}$ where $e_{k+1,k}$ has 1 in the $(k+1,k)$ entry and zeros elsewhere.

The s_{e_k} satisfy the same relations as the matrix units $e_{k+1,k}$, so the map is a well-defined surjection.

Since $M_n(\mathbb{C})$ is simple, we get $C^*(E) \cong M_n(\mathbb{C})$.



Examples

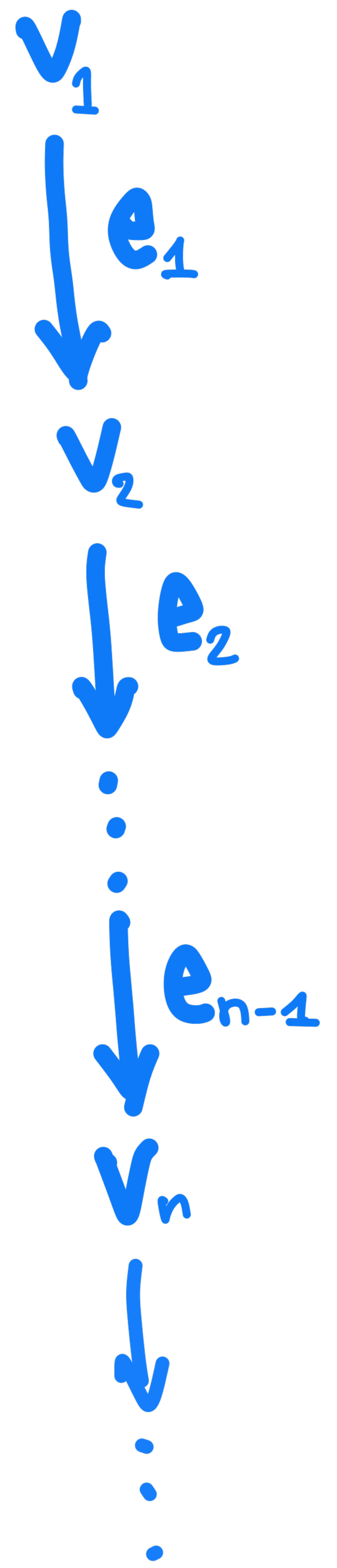
Both the previous examples were unital. In general, we will have a unital C*-algebra whenever there are finitely many vertices. Then

$$1_{C^*(E)} = \sum_{v \in E^0} p_v.$$

Generalising the previous example to infinitely vertices, it is not hard to check that we get a non-unital C*-algebra,

$$C^*(E) \cong \mathscr{K},$$

the C*-algebra of compact operators on separable Hilbert space.



Simplicity

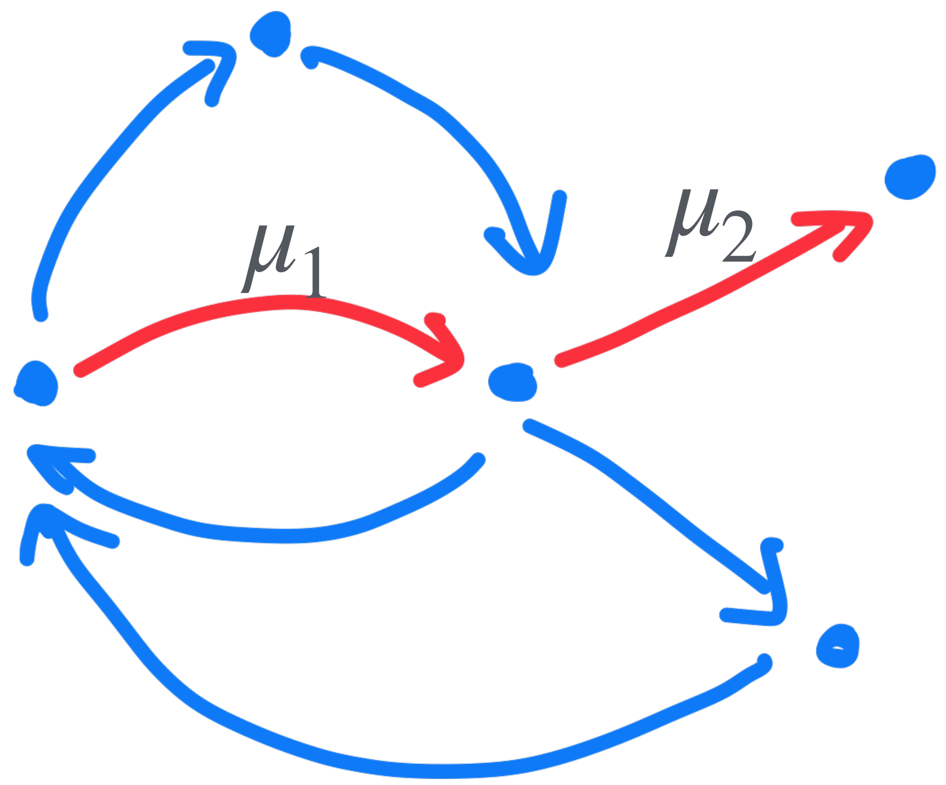
A C*-algebra is **simple** if it has no non-trivial proper ideals (=closed, two sided ideals).

When is a row finite graph C*-algebra simple?

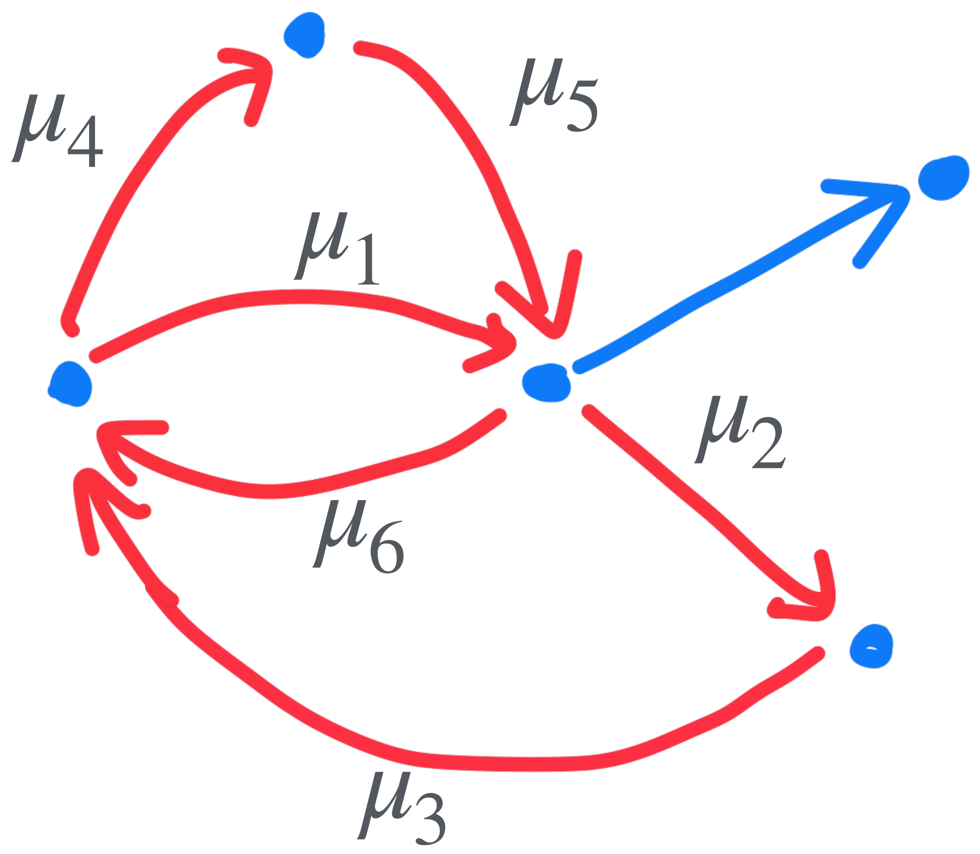
Turns out we determine this directly from the graph.

First, some terminology.

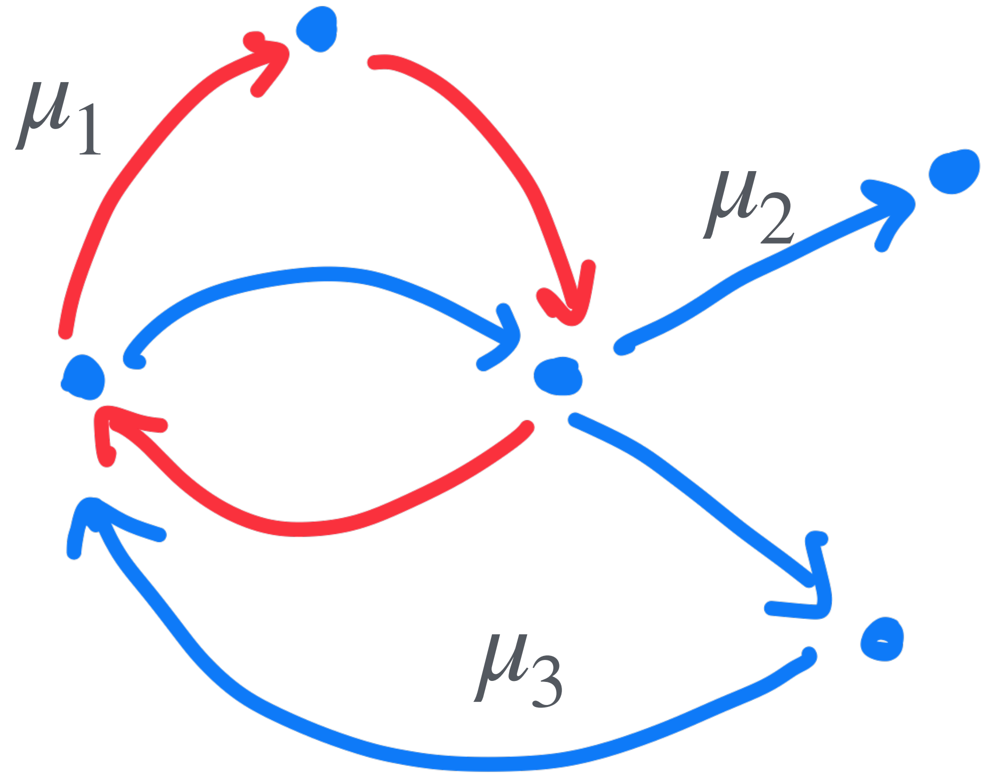
A path $\mu \in E^*$ is a **cycle** if $r(\mu) = s(\mu)$ and $s(\mu_i) \neq s(\mu_j)$ for any $i \neq j$.



not a cycle: $r(\mu) \neq s(\mu)$

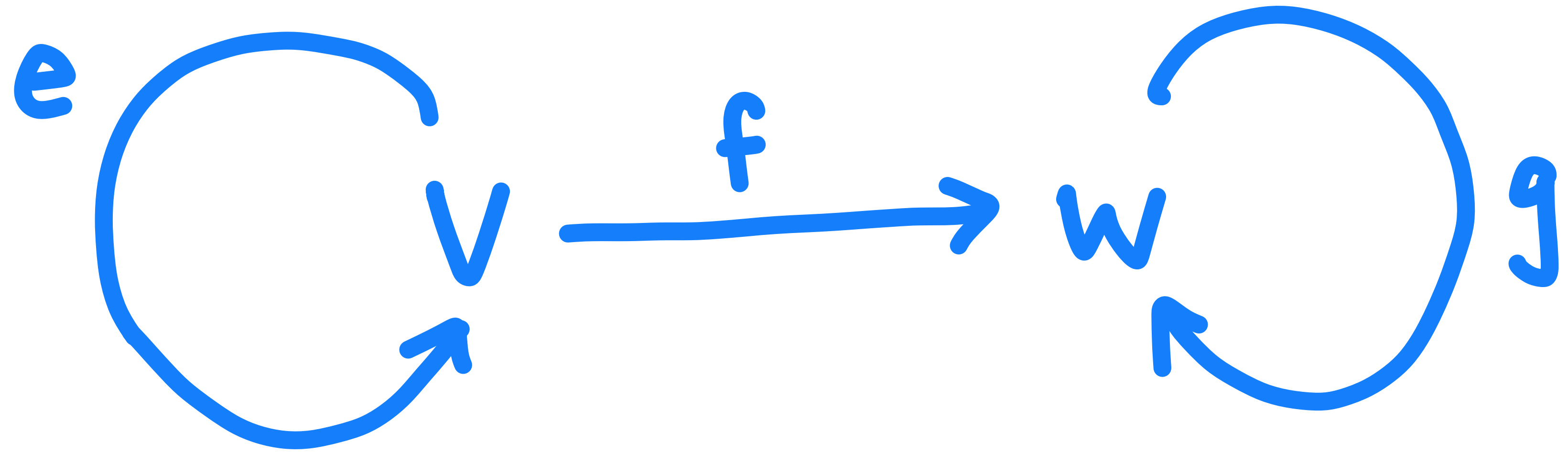


not a cycle: $r(\mu) = s(\mu)$,
but eg. $s(\mu_1) = s(\mu_4)$



cycle

Let μ be a cycle. An entry to μ is an edge $e \in E^1$ such that $r(e) = r(\mu_i)$ for some i, but $e \neq \mu_i$. For example, the cycle e has no entry, but f is an entry to the cycle g.

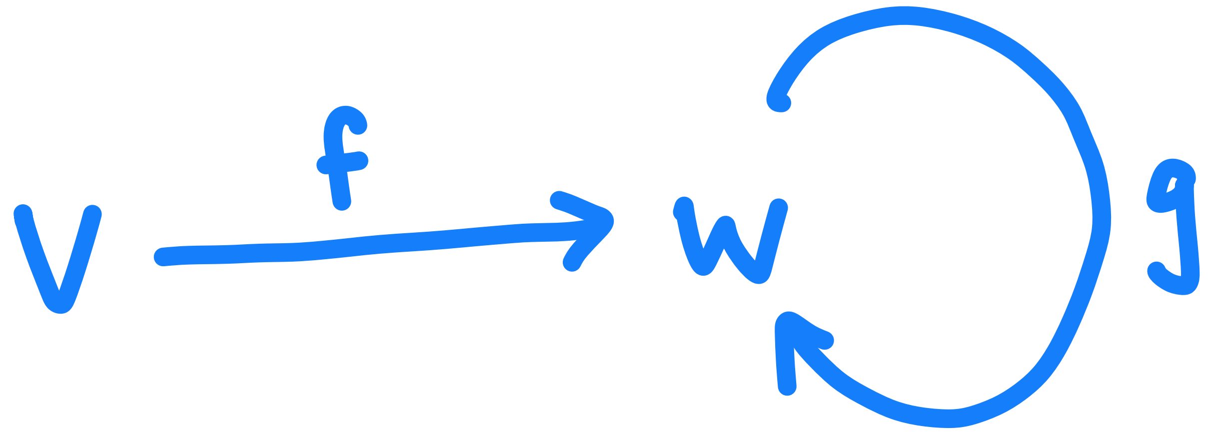


We put a partial order on the vertices as follows: Write $v \leq w$ if there exists $\mu \in E^*$ such that $s(\mu) = v, r(\mu) = w$.

Let $E^\infty = \{\mu = \mu_1\mu_2\cdots \mid s(\mu_i) = s(\mu_{i+1})\}$ and $E^{\leq\infty} = E^\infty \cup \{\mu \in E^* \mid \mu_1 \text{ is a source}\}$.

A graph is cofinal if for every $\mu \in E^{\leq\infty}$ and $v \in E^0$ there exists $w \in \mu$ such that $v \leq w$.

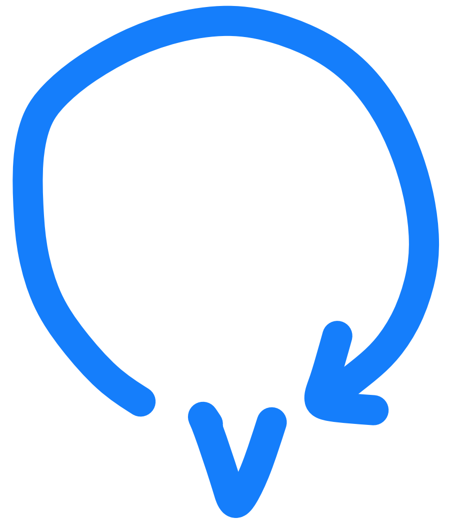
cofinal



not cofinal

Simplicity

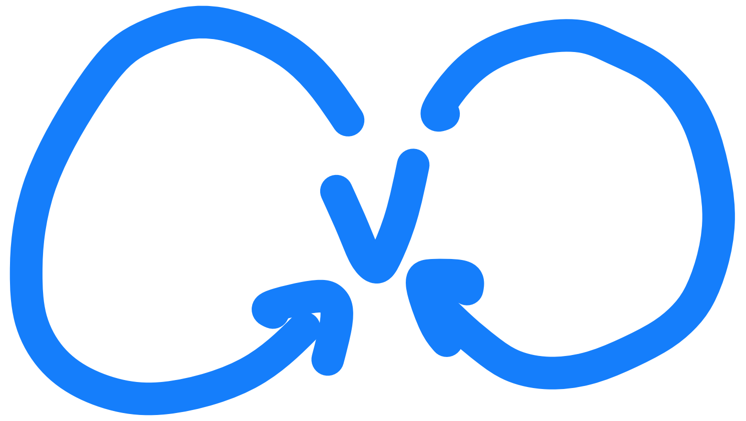
Theorem: *Let E be a row finite graph. Then $C^*(E)$ is simple if and only if every cycle has an entry and E is cofinal.*



$C^*(E) \cong C(\mathbb{T})$ is not simple.

Indeed for any closed subset $X \subset \mathbb{T}$,

$I_X = \{f \in C(\mathbb{T}) \mid f(x) = 0$ for every $x \in X\}$ is an ideal in $C(\mathbb{T})$



$C^*(E)$ is isomorphic to the **Cuntz algebra** $\mathcal{O}_2$. Cuntz showed that for any nonzero $x \in \mathcal{O}_2$ there are $a, b \in \mathcal{O}_2$ such that $axb = 1$.

More generally, we can describe the entire ideal structure of a graph C*-algebra. But I'll return to this later.

Short introduction to K-theory

Loosely speaking, the K-theory of a compact Hausdorff space is an invariant built from isomorphism classes of vector bundles over that space (in the case of K^0) or a related space.

By the Serre–Swan theorem, vector bundles over X can be replaced by finitely generated projective $C(X)$-modules.

In the spirit of noncommutative topology, for a C*-algebra A, we can construct K_0 from finitely generated projective A-modules.

Any finitely-generated A module is of the form $pA^{\oplus n}$ for a projection $p \in M_n(A)$. Thus we can equivalently construct K_0 from projections in $\cup_{n\in\mathbb{Z}_{>0}} M_n(A)$.

K_0-group

Let A be a C*-algebra and let $p, q \in A$ be projections. We say that p and q are **Murray–von Neumann equivalent**, $p \sim q$, if there exists a partial isometry $v \in A$ such that $v^*v = p$ and $vv^* = q$.

Let $M_\infty(A) = \bigcup_{n\in\mathbb{N}} M_n(A)$, where we have identified $M_n(A) \subset M_{n+1}(A)$ by mapping $a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$.

Denote $\mathscr{P}_\infty(A) := \{p \in M_\infty(A) \mid p = p^2 = p^*\}$, and extend Murray–Neumann equivalences to $\mathscr{P}_\infty(A)$. Define addition by

$$[a] + [b] = \left[\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right].$$

$K_0(A)$ is then defined to be the Grothendieck group of the resulting abelian monoid,

$$K_0(A) = \{[p] - [q] \mid p, q \in \mathscr{P}_\infty(A)\}.$$

K_1-group

$K_1(A) := K_0(SA)$, where $SA = \{f \in C([0,1], A) \mid f(0) = f(1) = 0\}$.

We can also realize K_1 via unitaries: Let $\mathcal{U}(M_\infty(A)) = \cup_{n\in\mathbb{Z}_{>0}} \mathcal{U}(M_n(A))$ where we identify the unitary $u \in M_n(A)$ with $\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \in M_{n+1}(A)$. (Replace A with $\tilde{A}$ if A is nonunital).

Then $\mathcal{U}(M_\infty(A))/\sim_h$ becomes an abelian monoid with respect to

$$[a] + [b] = \left[\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right],$$

and $K_1(A)$ is the Grothendieck group of this monoid.

One could then define $K_2(A) = K_1(SA) = K_0(S(SA))$, but it turns out that $K_2(A) \cong K_0(A)$.

K-theory of a graph C*-algebra

The adjacency matrix of a graph E allows us to calculate the K-theory of the corresponding graph C*-algebra $C^*(E)$

Recall that the adjacency matrix $A_E = (a_{v,w})_{v,w \in E^0} \in M_{E^0 \times E^0}(\mathbb{Z})$ of E is defined by

$$a_{v,w} = |\{e \in E^1 \mid r(e) = v, s(e) = w\}|.$$

In the following, we consider $1 - A_E^t$ as a map $\mathbb{Z}^{E^0} \to \mathbb{Z}^{E^0}$:

Theorem. *Let E be a row-finite graph with no sources, and let A_E be its adjacency matrix. Then*

$$K_0(C^*(E)) \cong \operatorname{coker}(1 - A_E^t) \text{ and } K_1(C^*(E)) \cong \ker(1 - A_E^t).$$

Theorem. *Suppose that* $A := C^*(E)$ *and* $B := C^*(F)$ *are simple graph C*-algebras. Then*

$$A \cong B \text{ if and only if } (K_0(A), K_1(A)) \cong (K_0(B), K_1(B)).$$

In fact, this theorem can be greatly generalized. There is an invariant of arbitrary graph C*-algebras which consists of the K-theory of the graph, its ideals, and various compatibility maps. Any two unital graph C*-algebra are isomorphic if and only if their invariants are isomorphic! (Eilers, Restorff, Ruiz, Sørensen)

Note that both these theorems require one to know in advance that a given C*-algebra is a graph C*-algebra. We'll see that sometimes it is possible to deduce exactly when a given C*-algebra is isomorphic to a graph C*-algebra, and apply this to the C*-algebras of quantum flag manifolds.