Graph C*-algebras with applications to quantum spaces

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Why C*-algebras?

C*-algebras are a subclass of **operator algebras**.

The field of arose from quantum mechanics as a way to take into account the "noncommutative" behaviour seen at the quantum level.

Operator algebras were introduced by Murray and von Neumann in a series of papers by Murray and von Neumann titled *Rings of Operators*.

There they studied what we now call **von Neumann algebras**: self-adjoint subalgebras of bounded operators on Hilbert space, which are closed in the strong operator topology.

Later, Gelfand and Naimark studied such algebras that are instead closed in the operator norm topology: what we now call **C*-algebras**.

Why C*-algebras?

C*-algebras have since proven useful for giving models for many physical and mathematical structures.

C*-algebras allow one to study such structures using both algebraic and analytic tools.

Examples include:

- topological spaces
- topological groups
- dynamical systems
- directed graphs
- groupoids
- tilings
- foliations
- quantum groups
- quantum spaces
- ...and much more!

C*-algebras

A ***-algebra** is a $\mathbb{C}$-algebra A equipped with a map $* : A \to A$ satisfying

- $(a + \lambda b)^* = a^* + \bar{\lambda} b^*$, for every $a, b \in A$, $\lambda \in \mathbb{C}$,
- $(ab)^* = b^* a^*$, for every $a, b \in A$, and
- $(a^*)^* = a$, for every $a \in A$.

A **C*-norm** on a *-algebra A is a norm $\|\cdot\| : A \to \mathbb{R}_{\geq 0}$ satisfying

- $\|ab\| \leq \|a\| \|b\|$, for every $a, b \in A$,
- $\|a^*\| = \|a\|$, for every $a \in A$, and
- (**C*-equality**) $\|a^* a\| = \|a\|^2$, for every $a \in A$.

C*-algebras

Definition: A **C*-algebra** is a *-algebra A which is complete with respect a C*-norm $\|\cdot\| : A \to \mathbb{R}_{\geq 0}$.

First examples:

- The complex numbers $\mathbb{C}$ with $a^* = \bar{a}$ and $\|a\| = |\bar{a}a|^{1/2}$.
- $M_n(\mathbb{C})$ with $(a_{i,j})^*_{i,j=1,\dots,n} = (\overline{a_{j,i}})_{i,j=1,\dots,n)}$ and the operator norm

$$\|a\| = \sup\{\|ax\|_{\mathbb{C}^n} \mid x \in \mathbb{C}^n, \|x\| \leq 1\}.$$

- $\mathscr{B}(H)$ for a Hilbert space H, with $*$ given by the adjoint and the operator norm

$$\|a\| = \sup\{\|ax\|_H \mid x \in H, \|x\| \leq 1\}.$$

C*-algebras: Examples

The previous examples are all unital C*-algebras. But C*-algebras need not be unital.

- Let H be a Hilbert space. Recall that a bounded operator $T : H \to H$ is **compact** if, for every bounded subset $K \subset H$, the subset $\overline{T(K)}$ is compact.

  Equivalently, there are finite rank operators $T_n \in \mathscr{B}(H)$ such that $\lim_{n \to \infty} \|T - T_n\| = 0$.

  Let $\mathscr{K}(H) = \{T \in \mathscr{B}(H) \mid T \text{ is compact}\}$, equipped with the adjoint and operator norm.

  Then $\mathscr{K}(H)$ is a C*-algebra which is unital if and only if $\dim H < \infty$, in which case $\mathscr{K}(H) = \mathscr{B}(H)$.

- More generally, any norm-closed subalgebra of $A \subset \mathscr{B}(H)$ satisfying $A^* = A$ is a C*-algebra, which may or may not be unital.

C*-algebras: Examples

Not all C*-algebras have such a linear algebra flavour:

Let X be a locally compact Hausdorff space.

A continuous function $f : X \to \mathbb{C}$ **vanishes at infinity** if for every $\epsilon > 0$ there exists a compact subset $K \subset X$ such that $|f(x)| < \epsilon$ for every $x \in X \backslash K$.

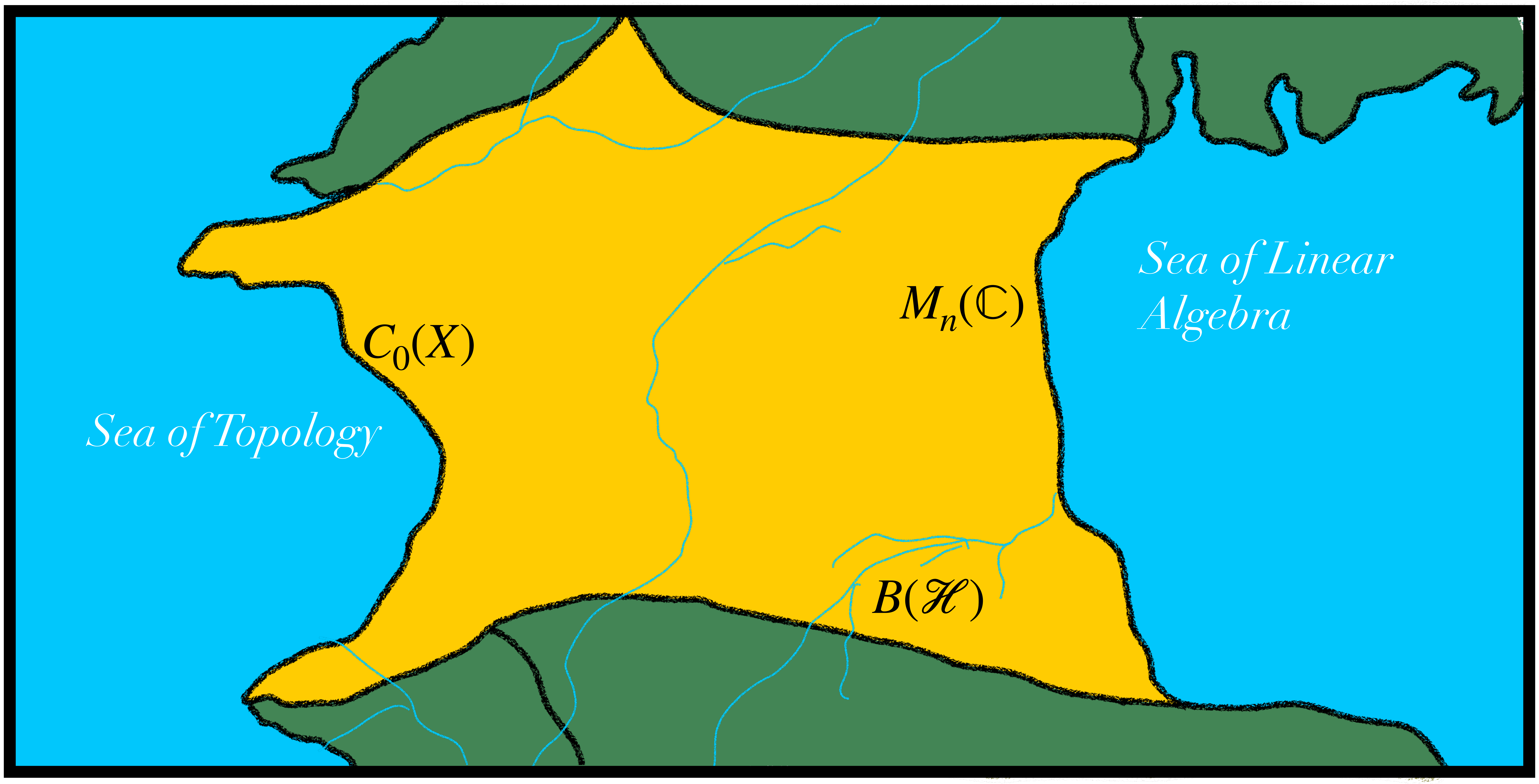
Let $C_0(X) = \{f : X \to \mathbb{C} \mid f \text{ continuous and vanishing at infinity}\}$, and define $f^*(x) = \overline{f(x)}$, $x \in X$ and $\|f(x)\| = \sup_{x \in X} |f(x)|$.

Then $C_0(X)$ is a commutative C*-algebra.

$C_0(X)$ is a unital commutative C*-algebra if and only if X is compact.

In this case we have $C_0(X) = C(X) = \{f : X \to \mathbb{C} \mid f \text{ is continuous}\}$.

C*-landia



Sea of Topology
$C_0(X)$
$M_n(\mathbb{C})$
Sea of Linear Algebra
$B(\mathscr{H})$

*-homomorphisms

If A and B are C*-algebras, then by a *-homomorphism $\varphi : A \to B$ we mean an algebra map which is in addition *-preserving.

Note that we didn't ask for φ to be continuous: A miracle of the C*-identity is that a *-homomorphism is automatically continuous!

By an ideal I in A, unless otherwise stated, we always mean a closed, self-adjoint (that is, $I = I^*$) two-sided ideal.

The kernel of a *-homomorphism $A \to B$ is always an ideal in A.

Unitisation of a non-unital C*-algebra

When A is nonunital, it is often convenient to embed it into a unital C*-algebra.

We say that B is a unitisation of A if A sits in B as an **essential ideal**: If $I \subset B$ is an ideal, then $A \cap I \neq \emptyset$.

Let A be a nonunital C*-algebra. Its **minimal unitisation**, denoted $\tilde{A}$, is defined as follows.

Put

$$\tilde{A} := \{(a, \lambda) \in A \oplus \mathbb{C}\},$$

with multiplication given by $(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda\mu)$. The addition and adjoint are componentwise.

The norm is given by $\|(a, \lambda)\| = \sup\{\|ab + \lambda b\|_A \mid b \in A, \|b\|_A \leq 1\}$.

Then $\tilde{A}$ is a unital C*-algebra and $A \subset \tilde{A}$ is an essential ideal.

Spectrum

Let A be a C*-algebra and $a \in A$. The **spectrum** of a is

$$\mathrm{sp}(a) := \{\lambda \in \mathbb{C} \mid a - \lambda 1 \text{ is not invertible } \}.$$

If A is unital, then $1 = 1_A$ and we mean invertible in A. If A is non-unital, then $1_A = 1_{\tilde{A}}$ and we mean invertible in $\tilde{A}$.

We have that $\varnothing \neq \mathrm{sp}(a) \subset B(0, \|a\|)$.

An element $a \in A$ is **normal** if $a^*a = aa^*$. For normal elements, we have that

$$\sup\{\,|\lambda| \mid \lambda \in \mathrm{sp}(a)\} = \|a\|.$$

In other words, the **norm only depends on spectral data**. This means that for a given C*-algebra A, there is a unique C*-norm on A.

Spectrum

As the name suggests, the spectrum of an element in a C*-algebra generalizes the notion of the spectrum of a matrix.

Thus if $A = M_n(\mathbb{C})$ and $a \in A$, we have that

$$\mathrm{sp}(a) = \{\lambda \in \mathbb{C} \mid \lambda \text{ is an eigenvalue of } a\}.$$

Moving to the west coast of C*-landia, if $A = C(X)$ for some compact Hausdorff space X and $f \in C(X)$, we have

$$\mathrm{sp}(f) = \{f(x) \mid x \in X\} .$$

Characters

Let A be a unital C*-algebra.

A **character** on A is a *-homomorphism $\varphi : A \to \mathbb{C}$.

Since *-homomorphisms are continuous, we can equip the character space $\Omega(A)$ with the weak-* topology.

If X is a compact Hausdorff space, then for every $x \in X$, the point evaluation

$$\mathrm{ev}_x(f) = f(x), f \in C(X)$$

is a *-homomorphism. Conversely, if $\varphi : C(X) \to \mathbb{C}$ is a *-homomorphism, one can show that there is $x \in X$ such that $\varphi(f) = f(x)$. Thus characters are in 1-1 correspondence with points of X.

In fact, X and $\Omega(C(X))$ are homeomorphic.

Conversely, when A is unital and commutative, $\Omega(A)$ is a compact Hausdorff space and $A \cong C(\Omega(A))$.

The Gelfand Theorem

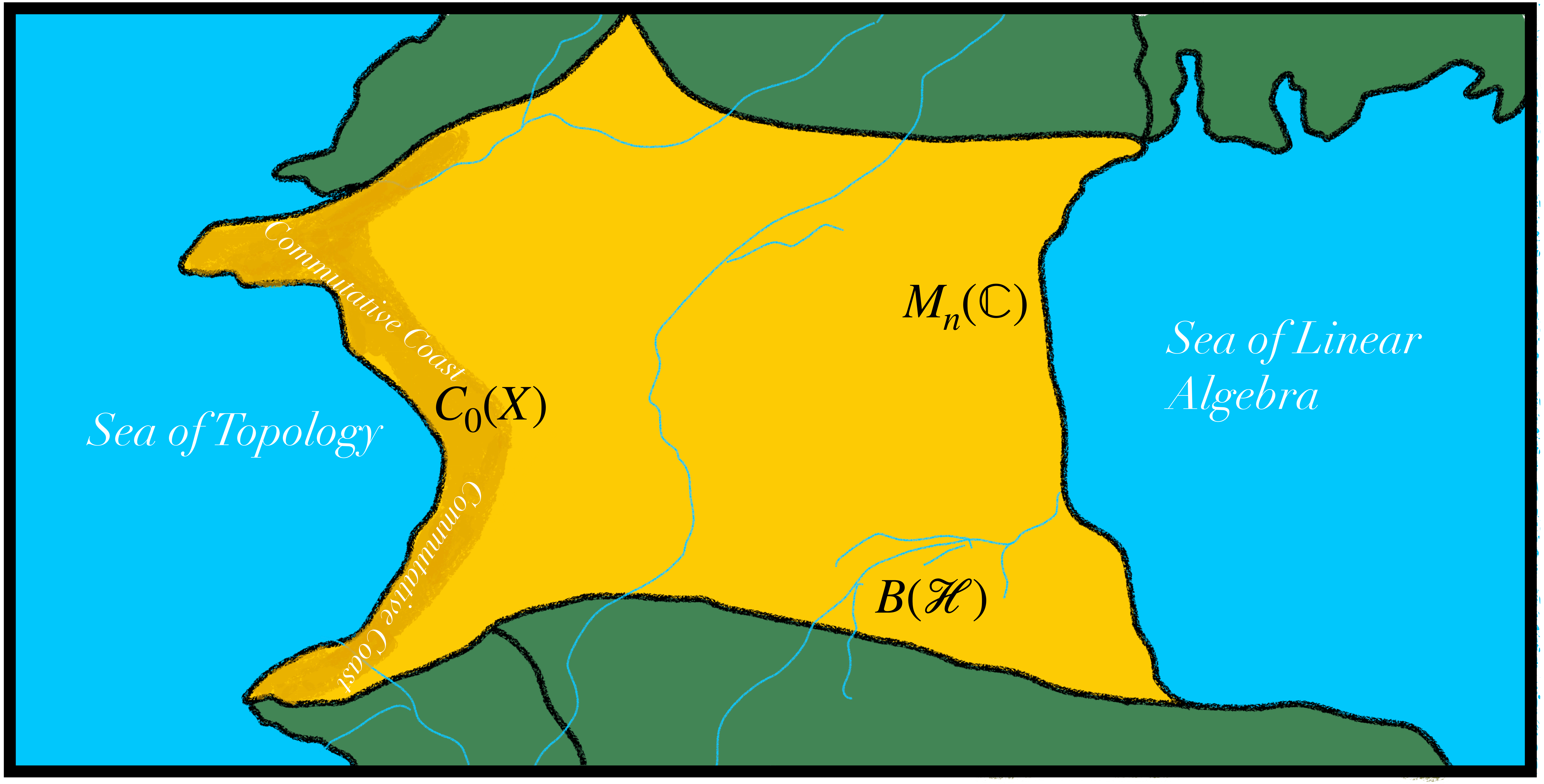
Theorem (Gelfand–Naimark). *For any commutative C*-algebra* A *there exists a locally compact Hausdorff space* X *such that* $A \cong C_0(X)$. A *is unital if and only if* X *is compact, in which case* $A \cong C(X)$.

If X, Y *are compact Hausdorff spaces, then any continuous map* $\varphi : Y \to X$ *induces a *-homomorphism* $C(X) \to C(Y)$ *given by* $f \mapsto f \circ \varphi$*, and vice-versa.*

Thus we have a duality between the category of compact Hausdorff spaces with continuous maps and the category of unital commutative C*-algebras with *-homomorphisms.

This is why we often refer to the study of C*-algebras as "**noncommutative topology**" .

C*-landia



Commutative Coast
$M_n(\mathbb{C})$
Sea of Linear Algebra
$C_0(X)$
Sea of Topology
Commutative Coast
$B(\mathscr{H})$

Functional calculus

The Gelfand–Naimark theorem not only gives us a complete description of commutative C*-algebras, it also gives us one of our most indispensable tools: the **continuous functional calculus**.

An element $a \in A$ is **normal** if $a^*a = aa^*$.

Let A be a unital C*-algebra and $a \in A$ be a normal element. Then the C*-subalgebra $C^*(a, 1_A) \subset A$ is commutative. We have that $\mathrm{sp}(a) \cong \Omega(C^*(a, 1_A))$.

Thus $C^*(a, 1_A) \cong C(\mathrm{sp}(a))$, and this *-isomorphism sends $a \in C^*(a, 1_A)$ to the identity map on $\mathrm{sp}(a)$.

Let $f \in C(\mathrm{sp}(a))$. Then we may define an element $f(a) \in C^*(a, 1_A) \subset A$ by considering the image of f under the *-isomorphism which sends the identity on $C(\mathrm{sp}(a))$ to $a \in C^*(a, 1_A)$.

Functional calculus

Let A be a unital C*-algebra and let $a \in A$ be normal.

Then we know that, for any $f \in C(\mathrm{sp}(\mathrm{a}))$, there is an element $f(a) \in C^*(a, 1_A)$.

The element $f(a)$ is again normal and satisfies

$$\mathrm{sp}(f(a)) = f(\mathrm{sp}(a)) = \{f(\lambda) \mid \lambda \in \mathrm{sp}(a)\}.$$

Furthermore, if $g \in C(\mathrm{sp}(f(a)))$ then we have that

$$g(f(a)) = g \circ f(a).$$

Special elements

Let A be a C*-algebra. An element $a \in A$ is **self-adjoint** if $a = a^*$. Evidently, any self-adjoint element is normal. Every $a \in A$ is of the form $a = b + ic$ for $b, c \in A_{sa}$:

$$a = (a + a^*)/2 + i\,[(a - a^*)/(2i)].$$

If a is self-adjoint, then $\mathrm{sp}(a) \subset [-\|a\|, \|a\|] \subset \mathbb{R}$.

We say that a is **positive** if a is self-adjoint and $\mathrm{sp}(a) \subset \mathbb{R}_{\geq 0}$.

An element $p \in A$ is a **projection** if $p = p^* = p^2$. Every projection is positive.

If A is unital, then an element $u \in A$ is **unitary** if $u^*u = uu^* = 1_A$.

Note that for $M_n(\mathbb{C})$, all these terms mean what they should!

Special elements

Notation:

$$A_{\mathrm{sa}} = \{a \in A \mid a^* = a\} \qquad \mathscr{P}(A) = \{p \in A \mid p = p^2 = p^*\}$$

$$A_+ = \{a \in A \mid a \geq 0\} \qquad \mathscr{U}(A) = \{u \in A \mid u^*u = uu^* = 1\}$$

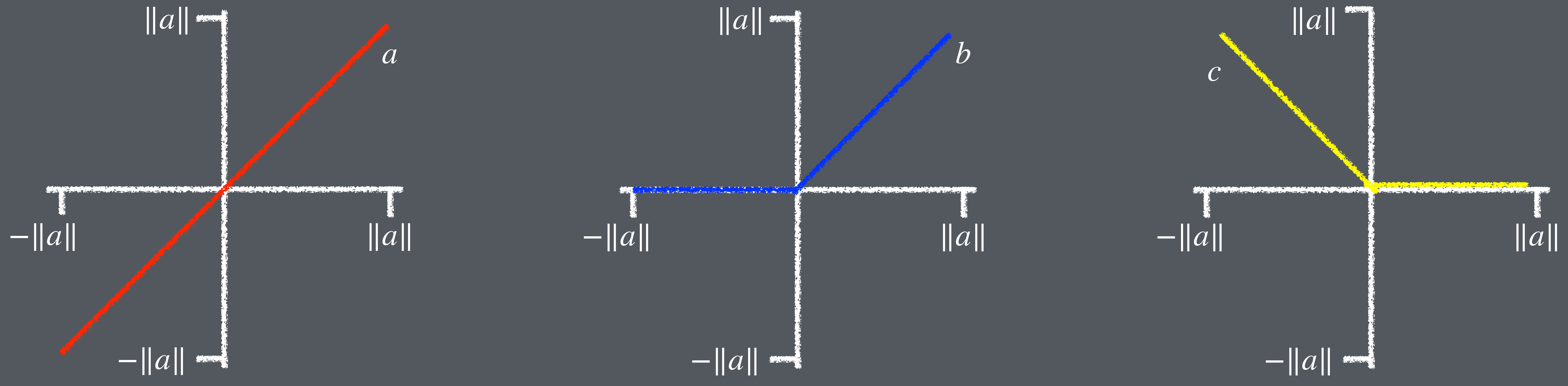
Let $a, b \in A_{sa}$.

We write $a \leq b$ if $b - a \in A_+$.

This makes $(A_{\mathrm{sa}}, \leq)$ into a partially ordered set.

Fun with the functional calculus

Let $a \in A_{\text{sa}}$. Then there exists $b, c \in A_+$ such that $cb = bc = 0$ and $a = b - c$.



$$f_a(t) = t$$

$$f_b(t) = \begin{cases} t, & t \geq 0 \\ 0, & t \leq 0 \end{cases}$$

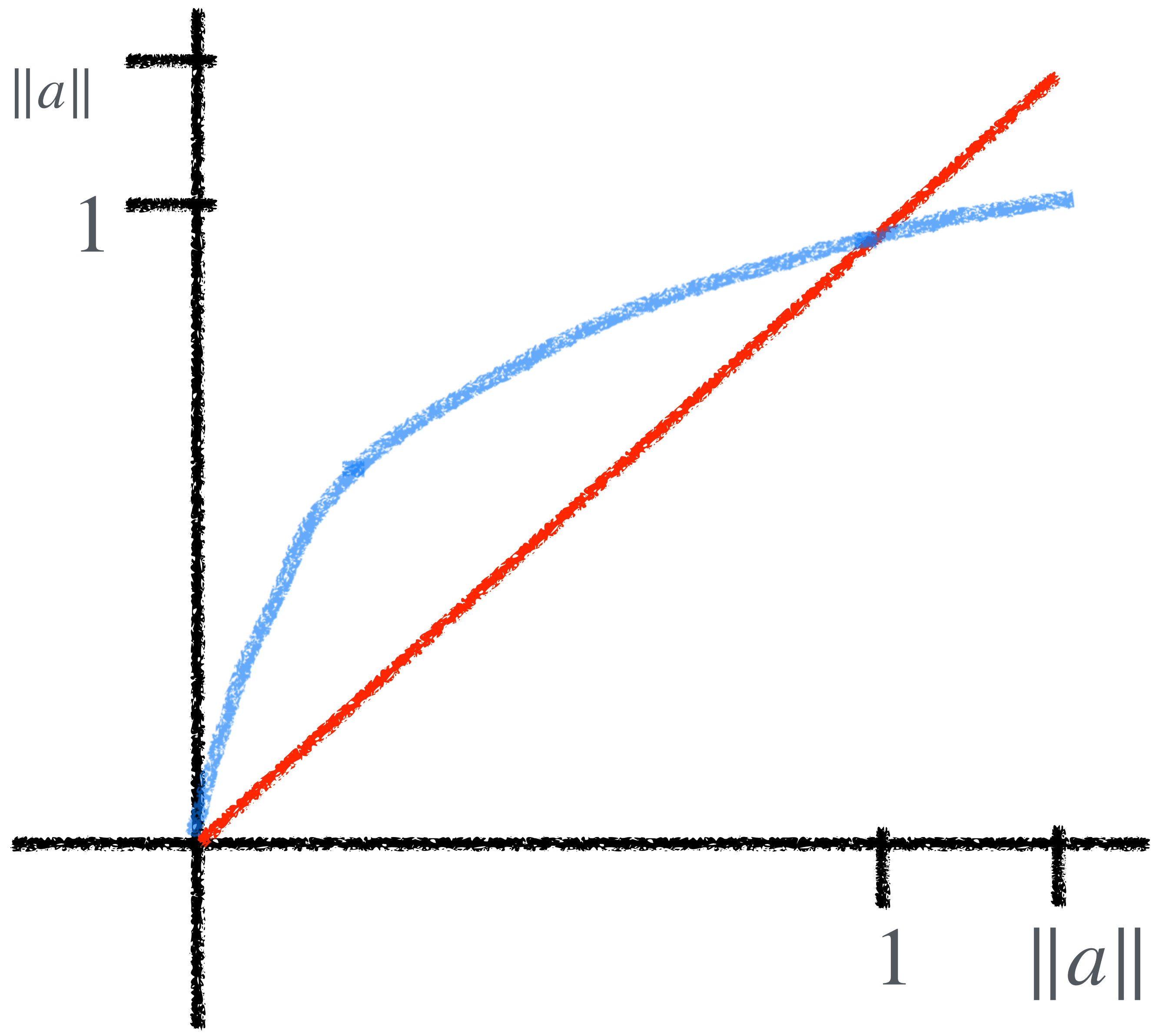
$$f_c(t) = \begin{cases} 0, & t \geq 0 \\ t, & t \leq 0 \end{cases}$$

$$b = f_b(a)$$

$$c = f_c(a)$$

Characterizing positive elements

Let $a \in A_+$. Then there exists a unique $b \in A_+$ such that $b^2 = a$. Since b is unique, we denote it by $\sqrt{a}$.



Note that any element of the form a^*a is self-adjoint.

In fact, one can show that a^*a is always positive.

Thus we have that $A_+ = \{a^*a \mid a \in A\}$.

Positive elements and the order structure of the self-adjoint elements, are an important part of the structure of a C*-algebra.

Positive elements and the order structure will allow us to define **hereditary C*-subalgebras**, important sub-objects, especially when we are dealing with simple C*-algebras which do not have ideals.

They also allow us to put an order on projections, which will play a role in the **K-theory** of a C*-algebra.

Hereditary C*-subalgebras

Let A be a C*-algebra. The order structure on A_+ also allows us to define so-called **hereditary subalgebras**.

A C*-subalgebra B of A is hereditary if, whenever $a \in B \cap A_+$ and $b \in A_+$ such that $b \leq a$, then $b \in B$.

Ideals are always hereditary.

If $a \in A_+$, then the hereditary C*-subalgebra generated by a is $\overline{aAa}$.

If $p \in \mathscr{P}(A) \subset A_+$ is a projection, then $\overline{pAp} = pAp$ is a unital hereditary C*-subalgebra (with unit p). Such a subalgebra is called a **corner** of A.

A hereditary C*-subalgebra $B \subset A$ often inherits many important structural properties from A, for example, simplicity.

Representations

A **representation** of a C*-algebras A is a pair (π, H) consisting of a Hilbert space H and a *-homomorphism $\pi : A \to B(H)$. A representation is **faithful** if π is injective.

A **state** on A is linear functional $\varphi : A \to \mathbb{C}$ satisfying $\varphi(A_+) \in \mathbb{R}_{\geq 0}$ and $\|\varphi\| = 1$.

Every state $\varphi : A \to \mathbb{C}$ gives rise to a representation via the **GNS (Gelfand—Naimark—Segal) construction:**

Define an inner product on A by

$$\langle a, b \rangle = \varphi(a^*b), \qquad a, b \in A.$$

Let $N_\varphi := \{a \in A \mid \varphi(a^*a) = 0\}$. Then the completion of A/N_φ with respect to this inner product is a Hilbert space H_φ.

The GNS theorem

For every $a \in A$, left multiplication by a on A/N_φ defines a bounded linear operator T_a. Extending this to H_φ gives us $T_a \in B(H_\varphi)$, and the map a $\pi_\varphi : A \to B(H_\varphi),\ a \mapsto T_a$ defines a representation of A on H_φ.

A C*-algebra A always have a large supply of states.

In fact, given any $a \in A \backslash \{0\}$, there is a state $\varphi : A \to \mathbb{C}$ satisfying $\varphi(a) \neq 0$.

It follows that taking the representation given by the direct sum of all states produces a faithful representation of A.

Theorem (Gelfand–Naimark–Segal). *Every C*-algebra A is *-isomorphic to a closed self-adjoint subalgebra of $B(\mathscr{H})$ for some Hilbert space $\mathscr{H}$.*

Directed graphs

Representations give us a way of constructing C*-algebras from other mathematical structures, such as directed graphs and certain "quantum" spaces, which will be our focus.

Then we can use the tools of C*-algebras to study such objects and the underlying structure can also give us information about the C*-algebra.

First, we turn to graphs:

Definition: A **directed graph** is $E = (E^0, E^1, r, s)$ consists of

- a countable set E^0, called the **vertices** of E,
- a countable set E^1, called the **edges** of E,
- a **range** map $r : E^1 \to E^0$, and
- a **source** map $s : E^1 \to E^0$.

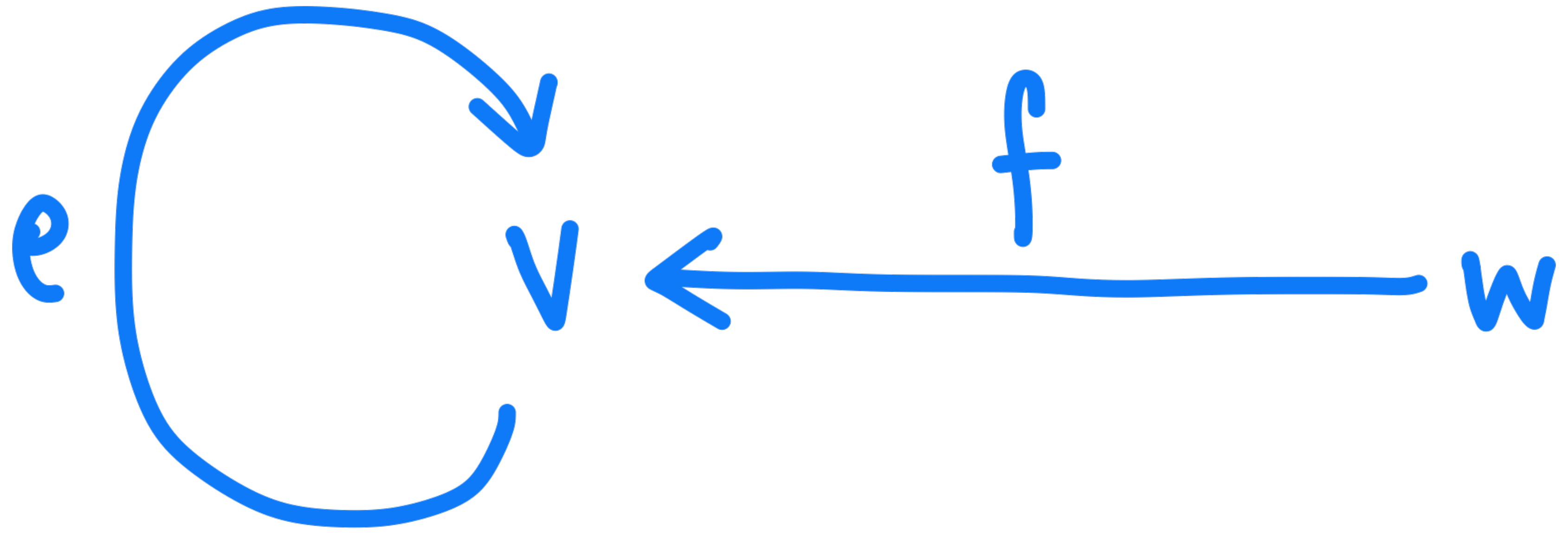
Drawing directed graphs

It is often useful to draw our directed graphs:

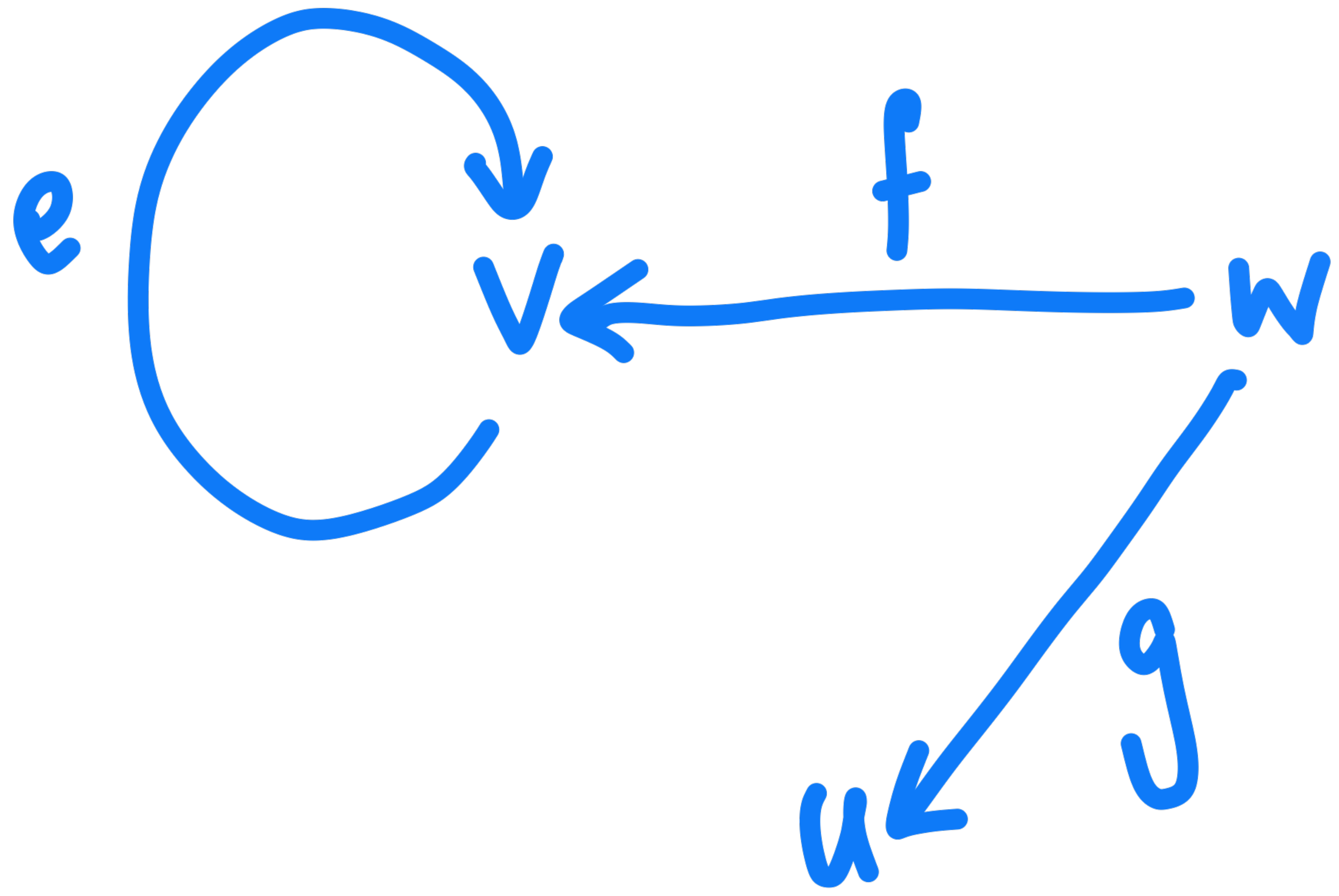
For example, Let $E = (E^0, E^1, r, s)$ where $E^0 = \{v, w\}$, $E^1 = \{e, f\}$,

- $r(e) = s(e) = v$,
- $r(f) = v$, and
- $s(f) = w$

Then we draw



Loops, sources, and sinks



An edge that has the same vertex as its range and source is called a **loop** based at that vertex.

For example, the edge e is a loop based at v.

A vertex with only outgoing edges is called a **source**.

In this graph, the vertex w is a source.

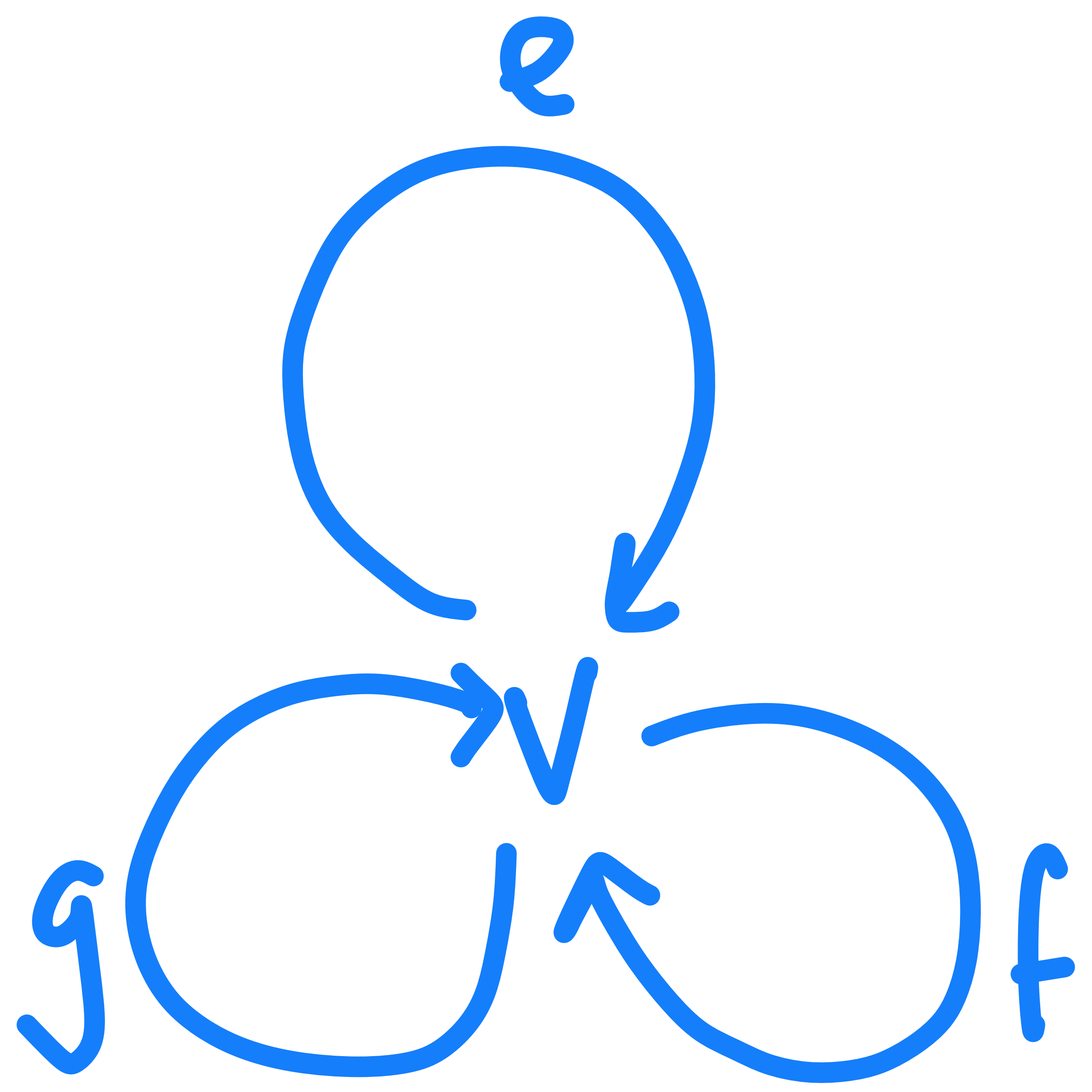
A vertex that only receives edges is called a **sink**.

Here, the vertex u is a sink.

Directed graphs from drawings

Of course, any such drawing also determines a directed graph.

For example,



determines the graph $E = (E^0, E^1, r, s)$ where $E^0 = \{v\}$, $E^1 = \{e, f, g\}$ and $r(e) = s(e) = r(f) = s(f) = r(g) = s(g) = v$.

Operators from graphs

Now we will represent our graphs by operators on Hilbert space, so that we can construct C*-algebras.

First we need the notion of a partial isometry. Let A be a C*-algebra. A partial isometry is an element $s \in S$ satisfying $ss^*s = s$.

The C*-equality implies that the following are equivalent:

- s is a partial isometry,
- $s^*ss^* = s^*$,
- s^*s is a projection,
- ss^* is a projection.

Cuntz–Krieger E-families

For now, we will restrict to row-finite graphs. A graph $E = (E^0, E^1, r, s)$ is **row finite** if every vertex receives at most finitely many edges.

Let $E = (E^0, E^1, r, s)$ be a directed graph. A **Cuntz–Krieger E-family** $\{S, P\}$ on a Hilbert space H consists of pairwise orthogonal projections $\{P_v \mid v \in E^0\}$ on H (so $P_v P_w = \delta_{v,w} P_v$, or equivalently P_v and P_w have orthogonal ranges in H whenever $v \neq w$) and partial isometries $\{S_e \mid e \in E^1\}$ on H satisfying

(CK1) $S_e^* S_e = P_{s(e)}$ for every $e \in E^1$, and

(CK2) $P_v = \displaystyle\sum_{\{e \in E^1 \mid r(e) = v\}} S_e S_e^*$ for every $v \in E^0$ that is not a source.

One can always find Cuntz–Krieger E-families such that $P_v, S_e \neq 0$ for every $v \in E^0$ and $e \in E^1$.

Indeed, for $v \in E^0$, let H_v be a separable infinite-dimensional Hilbert space and set

$$H = \bigoplus_{v \in E^0} H_v.$$

Let P_v to be the projection $H \to H_v$. Since H_v is infinite-dimensional, we can decompose it into direct sum

$$H_v = \bigoplus_{r(e)=v} H_{v,e}$$

where each $H_{v,e}$ is infinite-dimensional. Then let $S_e : H_{s(e)} \to H_{r(e),e}$ to be a unitary which is a partial isometry when viewed as an element in $\mathscr{B}(H)$.

The following are straight forward implications of the CK1 and CK2 and the fact that if $p = \sum_{i=1} p_i$ for projections $p, p_1, \ldots, p_n$ in a C*-algebra, then $p_i p_j = \delta_{ij} p_1$

Proposition. Let E be a row-finite graph. Then any Cuntz–Krieger E-family $\{S, P\}$ satisfies the following:

- the projections $\{S_e S_e^* \mid e \in E^1\}$ are mutually orthogonal;
- if $S_e^* S_f \neq 0$ then $e = f$,
- if $S_e S_f \neq 0$ then $s(e) = r(f)$,
- if $S_e S_f^* \neq 0$ then $s(e) = s(f)$.

Paths

The proposition allows us to define partial isometries associated to **paths** in the graph.

A path in E of length $n \in \mathbb{Z}_{>0}$ is a sequence $\mu = \mu_1\mu_2\ldots\mu_n$ of edges $\mu_i \in E^1$ such that $s(\mu_i) = r(\mu_{i+1})$ for $1 \leq i \leq n-1$.

Since

- the projections $\{S_eS_e^* \mid e \in E^1\}$ are mutually orthogonal;
- if $S_e^*S_f \neq 0$ then $e = f$,
- if $S_eS_f \neq 0$ then $s(e) = r(f)$,
- if $S_eS_f^* \neq 0$ then $s(e) = s(f)$,

we can define $S_\mu := S_{\mu_1}S_{\mu_2}\cdots S_{\mu_n}$. Then $S_\mu^*S_\mu = S_{\mu_n}^*S_{\mu^n} = P_{s(\mu_n)}$, so we set $s(\mu) := s(\mu_n)$.

Similarly, we set $r(\mu) := r(\mu_1)$.

From CK $\boldsymbol{E}$-families to C*-algebras

For a Cuntz–Krieger E-family $\{S, P\}$ on H, we define $C^*(\{S, P\})$ to be the C*-algebra generated by $\{P_v \mid v \in E^0\} \cup \{S_e \mid e \in E^1\}$ in $\mathscr{B}(H)$.

Let $E^n := \{\text{paths of length } n\}$ and let $E^* := \cup_{n \in \mathbb{Z}_{\geq 0}} E^n$ denote the set of all finite length paths. Then

$$C^*(\{S, P\}) = \overline{\text{span}}\{S_\mu S_\nu^* \mid \mu, \nu \in E^*, s(\mu) = s(\nu)\} .$$

Next time, we'll see more about how the combinatorics of the graph can tell us about properties of its associated C*-algebra(s).