Two interacting scalar fields: practical renormalization

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Abstract

The main theme of the paper is the detailed discussion of the renormalization of the quantum field theory comprising two interacting scalar fields. The potential of the model is the fourth-order homogeneous polynomial of the fields, symmetric with respect to the transformation $\phi_i \to -\phi_i$. We determine the Feynman rules for the model and then we present a detailed discussion of the renormalization of the theory at one loop. Next, we derive the one loop renormalization group equations for the running masses and coupling constants. At the level of two loops, we use the *FeynArts* package of *Mathematica* to generate the two loops Feynman diagrams and calculate in detail the *setting sun* diagram.

1. Introduction

Renormalization in quantum field theory is an indispensable tool to obtain precise results at higher orders of perturbation theory. The full process of renormalization is a complex task, and it is best learned with examples. We will consider here a theory of two real, interacting scalar fields described by the following Lagrangian density

$$\begin{aligned}\mathcal{L}(\phi_1,\phi_2) = \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 - \frac{1}{2}m_1^2\phi_1^2 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \frac{1}{2}m_2^2\phi_2^2 \\ -\frac{\lambda_1}{4!}\phi_1^4 - \frac{\lambda_2}{4!}\phi_2^4 - \frac{\lambda_3}{4}\phi_1^2\phi_2^2.\end{aligned} \tag{1}$$

The interaction potential density in (1)

$$\mathcal{V}(\phi_1,\phi_2) = \frac{\lambda_1}{4!}\phi_1^4 + \frac{\lambda_2}{4!}\phi_2^4 + \frac{\lambda_3}{4}\phi_1^2\phi_2^2 \tag{2}$$

should be positive definite, so we assume that the coupling constans λ_1, λ_2 and λ_3 fulfill the following conditions

$$\lambda_1 > 0, \quad \lambda_2 > 0, \quad \lambda_1\lambda_2 - 9\lambda_3^2 > 0. \tag{3}$$

From the Lagrangian density (1) one obtains the classical equations of motion for the fields ϕ_1 and ϕ_2

$$(\Box + m_1^2)\phi_1 + \frac{\lambda_1}{3!}\phi_1^3 + \frac{\lambda_3}{2}\phi_1\phi_2^2 = 0, \tag{4a}$$

$$(\Box + m_2^2)\phi_2 + \frac{\lambda_2}{3!}\phi_2^3 + \frac{\lambda_3}{2}\phi_1^2\phi_2 = 0. \tag{4b}$$

In our paper, we concentrate on the renormalization of the model described by the Lagrangian density (1). The contents of the paper are the following: In Section 2, we recall the notion of the Green's function and we establish the Feynman rules of the considered model. In Section 3, we analyze the divergent diagrams for the two and four-point Green's functions. We draw the corresponding Feynman diagrams and derive the analytical expressions corresponding to these diagrams. In Section 4, we discuss the mathematical methods that are used in the calculation of the Feynman diagrams, containing the loop integrals. We discuss the Feynman prescription, Wick rotation and two methods of regularization of the divergent integrals: Pauli-Villars regularization and dimensional regularization. Section 5 is devoted to the discussion of the one loop renormalization of our model. First, we discuss the regularization of the one loop divergent diagrams and then, we explain how the divergencies are removed in the one loop diagrams, through the procedure of renormalization. In Section 6, we derive the one loop renormalization group equations for the masses m_1, m_2 and the coupling constants λ_1, λ_2 and λ_3 in our theory. Section 7 is devoted to the discussion of the two loop renormalization of the model. After Conclusions, we include several appendices, where some additional details of the calculations are presented. We also give the listings of the *Mathematica* programs, that were used in the discussion of the two loop renormalization.

2. Feynman rules

Feynman rules for the Feynman diagrams in the momentum space are determined from the Lagrangian density and serve to derive the analytical expressions for the amplitudes of the given processes. The Lagrangian density (1) contains two fields ϕ_1 and ϕ_2 and three interaction vertices, so the Feynman diagrams will contain the propagators for the fields ϕ_1 and ϕ_2 and the vertices corresponding to the coupling constants λ_1, λ_2 and λ_3. The graphical representations of the basic elements of the Feynman diagrams stemming from the Lagrangian density are given in Fig. 1.

The Green's functions $G^{(n)}(x_1, x_2, \ldots, x_n)$ or n point functions are the quantities from which the S-matrix elements are calculated. They are the vacuum expectation values of the time ordered product of fields and are equal to

$$G^{(n)}_{i_1 \cdots i_n}(x_1, x_2, \ldots, x_n) = \langle 0 | T(\phi_{i_1}(x_1), \phi_{i_2}(x_2), \ldots, \phi_{i_n}(x_n)) | 0 \rangle \, . \tag{5}$$

Here T denotes the time ordered product of the fields.

The Green's function in the momentum space $G^{(n)}_{i_1 \cdots i_n}(p_1, p_2, \ldots, p_n)$ is the Fourier transform of the Green's function $G^{(n)}_{i_1\cdots i_n}(x_1, x_2, \ldots, x_n)$

$$(2\pi)^4\delta(p_1 + \cdots + p_n)G^{(n)}_{i_1\cdots i_n}(p_1, p_2, \ldots, p_n)$$
$$= \int \prod_{k=1}^{n} \mathrm{d}^4 x_k x^{-ip_k x_k} G^{(n)}_{i_1\cdots i_n}(x_1, x_2, \ldots, x_n). \quad (6)$$

The amputated Green's function is obtained from the momentum space Green's function by

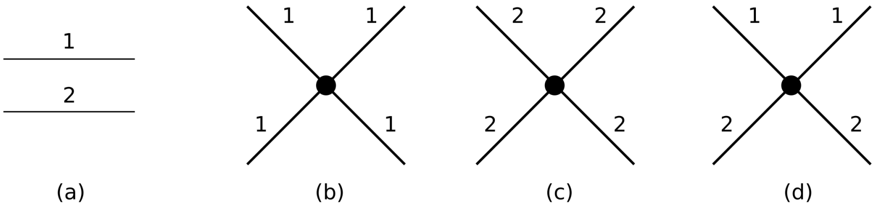


Figure 1: Elements of the Feynman diagrams, (a) propagators of particles 1 and 2, (b) vertex with the coupling constant λ_1, (c) vertex with the coupling constant λ_2, (d) vertex with the coupling constant λ_3.

removing the propagators $\Delta_k(p_k)$ of external fields

$$
{}^{\mathrm{amp}}G^{(n)}_{i_1\cdots i_n}(p_1,p_2,\ldots,p_n)=\prod_{k=1}^{n}\left[\frac{1}{i\Delta_k(p_k)}\right]G^{(n)}_{i_1\cdots i_n}(p_1,p_2,\ldots,p_n). \tag{7}
$$

The S-matrix elements are obtained from the Lehmann-Symanzik-Zimmermann reduction formula

$$
\begin{aligned}
\langle q_1,\ldots,q_n|S|p_1,p_2\rangle &= \left[i\int \mathrm{d}^4x_1 e^{-ip_1x_1}(\Box+m_1^2)\right]\left[i\int \mathrm{d}^4x_2 e^{-ip_2x_2}(\Box+m_2^2)\right]\\
&\times\left[i\int \mathrm{d}^4z_1 e^{iq_1z_1}(\Box+M_{i_1}^2)\right]\cdots\left[i\int \mathrm{d}^4z_n e^{iq_nz_n}(\Box+M_{i_n}^2)\right]\\
&\times\ \langle 0|T(\phi_1(x_1),\phi_2(x_2),\phi_{i_1}(z_1),\ldots,\phi_{i_n}(z_n))|0\rangle
\end{aligned} \tag{8}
$$

and for the Green's function the following perturbative expansion holds

$$
\begin{aligned}
G^{(n)}_{i_1\cdots i_n}(x_1,x_2,\ldots,x_n) &= \sum_{l=0}^{\infty}\frac{(-i)^l}{l!}\int_{-\infty}^{\infty}\mathrm{d}^4z_1\cdots\mathrm{d}^4z_l\\
&\times\ \langle 0|T(\phi^{\mathrm{I}}_{i_1}(x_1),\ldots,\phi^{\mathrm{I}}_{i_n}(x_n),\mathcal{V}(\phi^{\mathrm{I}}_1(z_1),\phi^{\mathrm{I}}_2(z_1)),\ldots,\mathcal{V}(\phi^{\mathrm{I}}_1(z_l),\phi^{\mathrm{I}}_2(z_l))|0\rangle_c\,.
\end{aligned} \tag{9}
$$

The superscript I means that the fields are taken in the interaction picture and the subscript c denotes the connected part.

The Green's functions contain the vacuum expectation value of time ordered product of the fields. In order to calculate this vacuum expectation value the time ordered product is expanded in terms of the normal products of the fields. The Feynman diagrams are the graphical representation of terms of this expansion.

The propagators are calculated from the $G^{(2)}_{i_1 i_2}$ Green's functions

$$\langle 0|T(\phi^{\rm I}_{i_1}(x)\phi^{\rm I}_{i_2}(y))|0\rangle = i\Delta_{i_1}(x-y)\delta_{i_1 i_2} = \int \frac{{\rm d}^4 p}{(2\pi)^4} e^{ip(x-y)} \frac{i}{p^2 - m^2_{i_1} + i\epsilon}\delta_{i_1 i_2}.$$

The Green's function $G^{(2)}_{12}$ vanishes, because the Lagrangian density (1) contains only even powers of fields and the symmetry $\phi_i \to -\phi_i$ holds.

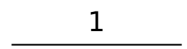
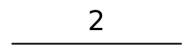
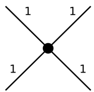
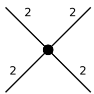
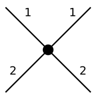
The vertices are obtained from the $G^{(4)}$ Green's functions and the non vanishing functions are $G^{(4)}_{1111}(x_1, x_2, x_3, x_4)$, $G^{(4)}_{2222}(x_1, x_2, x_3, x_4)$ and $G^{(4)}_{1122}(x_1, x_2, x_3, x_4)$. The vertices of the Feynman diagrams obtained from these functions with the analytic correspondence are given in Table 1.

The Green's functions $G^{(4)}_{1111}(x_1, x_2, x_3, x_4)$ and $G^{(4)}_{1122}(x_1, x_2, x_3, x_4)$ at the lowest order including interactions are obtained from the equations

$$\begin{aligned} &G^{(4)}_{1111}(x_1, x_2, x_3, x_4) \\ &\quad = \int_{-\infty}^{\infty} {\rm d}^4 z\, \langle 0|T(\phi^{\rm I}_1(x_1), \phi^{\rm I}_1(x_2), \phi^{\rm I}_1(x_3), \phi^{\rm I}_1(x_4), \mathcal{V}(\phi^{\rm I}_1(z), \phi^{\rm I}_2(z)))|0\rangle_c\,, \end{aligned} \tag{10}$$

$$\begin{aligned} &G^{(4)}_{1122}(x_1, x_2, x_3, x_4) \\ &\quad = \int_{-\infty}^{\infty} {\rm d}^4 z\, \langle 0|T(\phi^{\rm I}_1(x_1), \phi^{\rm I}_1(x_2), \phi^{\rm I}_2(x_3), \phi^{\rm I}_2(x_4), \mathcal{V}(\phi^{\rm I}_1(z), \phi^{\rm I}_2(z)))|0\rangle_c\,. \end{aligned} \tag{11}$$

Table 1: Analytical expressions corresponding to the elements of the Feynman diagrams.

Propagators		$\longrightarrow$	$\dfrac{i}{p^2 - m_1^2 + i\epsilon}$
		$\longrightarrow$	$\dfrac{i}{p^2 - m_2^2 + i\epsilon}$
Vertices		$\longrightarrow$	$-i\lambda_1$
		$\longrightarrow$	$-i\lambda_2,$
		$\longrightarrow$	$-i\lambda_3$

and $G^{(4)}_{2222}(x_1, x_2, x_3, x_4)$ is obtained from $G^{(4)}_{1111}(x_1, x_2, x_3, x_4)$ by the change of indices $1 \leftrightarrow 2$.

The integrand of Eq. (10) contains the term $\mathcal{V}(\phi^{\mathrm{I}}_1(z), \phi^{\mathrm{I}}_2(z)))$ given in Eq. (2), which contains three terms. The connected diagrams are obtained only from the term $(\lambda_1 \phi_1^4)/4!$, so only this term has to be included in the calculation. A similar situation occurs in Eq. (11) which corresponds to the term $(\lambda_3 \phi_1^2 \phi_2^2)/4$ in Eq. (2).

The terms in the Lagrangian density (1) corresponding to the vertices contain the factors $1/4!$ and $1/4$. In Appendix A, we present Wick's theorem, which is used in Appendices B and C to show that these factors are canceled in Feynman diagrams and the Feynman rules are such as those given in Table 1. [1]

To conclude this section let us list the rules for the construction of the analytical expression of the transition amplitude in the momentum representation from the Feynman diagram:

- The amplitude contains an overall factor of i.
- Each vertex produces the factor $-i\lambda_j$: $j = 1$ for four particles of type 1, $j = 2$ for four particles of type 2, $j = 3$ for two particles of type 1 and two particles of type 2.
- For each internal line of a particle of the type j with four-momentum k there corresponds the propagator $\dfrac{i}{k^2 - m_j^2 + i\epsilon}$.
- Four momentum is conserved at each vertex.

[1] In case of diagrams with loops the situation becomes more complicated and it is necessary to introduce the symmetry factors.

- Integrate over each internal four momentum within a loop (not fixed by the four momentum conservation) by applying the integral $\int \frac{\mathrm{d}^4 k}{(2\pi)^4}$.

3. Two and four point divergent diagrams

In this section we will discuss the lowest order divergent diagrams and the analytical expressions corresponding to them.

3.1. Diagrams for the propagators

In Fig. 2 there are the first order corrections to the propagators of particles 1 and 2. We will consider in detail only the diagrams for the propagator of particle 1, because the results for particle 2 are obtained from those of particle 1 by a simple change of indices.

The analytical expression in the coordinate space obtained from Eq. (9) that generates the diagrams in Fig. 2 (a) and (b) is

$$(-i) \int_{-\infty}^{\infty} \mathrm{d}^4 z_1 \, \langle 0 | T(\phi_1^{\mathrm{I}}(x_1), \phi_1^{\mathrm{I}}(x_2), \mathcal{V}(\phi_1^{\mathrm{I}}(z_1), \phi_2^{\mathrm{I}}(z_1))) | 0 \rangle_c \, . \tag{12}$$

After inserting the explicit form of the potential $\mathcal{V}(\phi_1^{\mathrm{I}}(z_1), \phi_2^{\mathrm{I}}(z_1))$ from Eq. (2) we obtain separately for diagrams in Fig. 2 (a) and (b)

$$(-i)\frac{\lambda_1}{4!}\int_{-\infty}^{\infty} \mathrm{d}^4 z_1 \, \langle 0|T(\phi_1^{\mathrm{I}}(x_1), \phi_1^{\mathrm{I}}(x_2), \left(\phi_1^{\mathrm{I}}(z_1)\right)^4)|0\rangle_c \, , \tag{13a}$$

$$(-i)\frac{\lambda_3}{4}\int_{-\infty}^{\infty} \mathrm{d}^4 z_1 \, \langle 0|T(\phi_1^{\mathrm{I}}(x_1), \phi_1^{\mathrm{I}}(x_2), \left(\phi_1^{\mathrm{I}}(z_1)\right)^2, \left(\phi_2^{\mathrm{I}}(z_1)\right)^2)|0\rangle_c \, . \tag{13b}$$

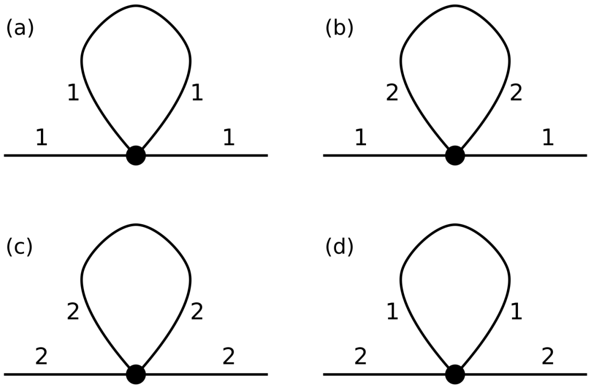


Figure 2: Divergent two point Feynman diagrams; (a) propagator of particle 1 with a loop containing particle 1, (b) propagator of particle 1 with a loop containing particle 2, (c) propagator of particle 2 with a loop containing particle 2, (d) propagator of particle 2 with a loop containing particle 1. The full propagator requires the summation of different diagrams.

From these equations we obtain in Appendix D that the *symmetry factors*[2] for the diagrams in Fig. 2 are equal 1/2.

In the momentum space we obtain the following expressions for the diagrams in Fig. 2

$$(i)(-i\lambda_1)\frac{1}{2}\int \frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{i}{k^2 - m_1^2 + i\epsilon} = \frac{\lambda_1}{2}\int \frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{i}{k^2 - m_1^2 + i\epsilon}, \tag{14a}$$

$$(i)(-i\lambda_3)\frac{1}{2}\int \frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{i}{k^2 - m_2^2 + i\epsilon} = \frac{\lambda_3}{2}\int \frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{i}{k^2 - m_2^2 + i\epsilon}, \tag{14b}$$

$$(i)(-i\lambda_2)\frac{1}{2}\int \frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{i}{k^2 - m_2^2 + i\epsilon} = \frac{\lambda_2}{2}\int \frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{i}{k^2 - m_2^2 + i\epsilon}, \tag{14c}$$

$$(i)(-i\lambda_3)\frac{1}{2}\int \frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{i}{k^2 - m_1^2 + i\epsilon} = \frac{\lambda_3}{2}\int \frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{i}{k^2 - m_1^2 + i\epsilon}. \tag{14d}$$

The 1/2 in Eqs. (14) is the symmetry factor and the integrals are quadratically divergent.

[2] Also known as the *combinatorial factors*.

3.2. Diagrams for the vertices

The analytical expressions in the coordinate space obtained from Eq. (9) that generate the diagrams in Fig. 3 are

$$\begin{aligned}
&\frac{(-i)^2}{2!}\int_{-\infty}^{\infty} \mathrm{d}^4 z_1 \mathrm{d}^4 z_2 \\
&\times \langle 0|T(\phi_1^{\mathrm{I}}(x_1), \phi_1^{\mathrm{I}}(x_2), \phi_1^{\mathrm{I}}(x_3)\phi_1^{\mathrm{I}}(x_4), \mathcal{V}(\phi_1^{\mathrm{I}}(z_1), \phi_2^{\mathrm{I}}(z_1)), \mathcal{V}(\phi_1^{\mathrm{I}}(z_2), \phi_2^{\mathrm{I}}(z_2)))|0\rangle_c \,, \qquad (15a)\\
&\frac{(-i)^2}{2!}\int_{-\infty}^{\infty} \mathrm{d}^4 z_1 \mathrm{d}^4 z_2 \\
&\times \langle 0|T(\phi_2^{\mathrm{I}}(x_1), \phi_2^{\mathrm{I}}(x_2), \phi_2^{\mathrm{I}}(x_3)\phi_2^{\mathrm{I}}(x_4), \mathcal{V}(\phi_1^{\mathrm{I}}(z_1), \phi_2^{\mathrm{I}}(z_1)), \mathcal{V}(\phi_1^{\mathrm{I}}(z_2), \phi_2^{\mathrm{I}}(z_2)))|0\rangle_c \,, \qquad (15b)\\
&\frac{(-i)^2}{2!}\int_{-\infty}^{\infty} \mathrm{d}^4 z_1 \mathrm{d}^4 z_2 \\
&\times \langle 0|T(\phi_1^{\mathrm{I}}(x_1), \phi_1^{\mathrm{I}}(x_2), \phi_2^{\mathrm{I}}(x_3)\phi_2^{\mathrm{I}}(x_4), \mathcal{V}(\phi_1^{\mathrm{I}}(z_1), \phi_2^{\mathrm{I}}(z_1)), \mathcal{V}(\phi_1^{\mathrm{I}}(z_2), \phi_2^{\mathrm{I}}(z_2)))|0\rangle_c \,. \qquad (15c)
\end{aligned}$$

After inserting the explicit form of the potential $\mathcal{V}(\phi_1^{\mathrm{I}}(z_1), \phi_2^{\mathrm{I}}(z_1))$ we obtain separately for

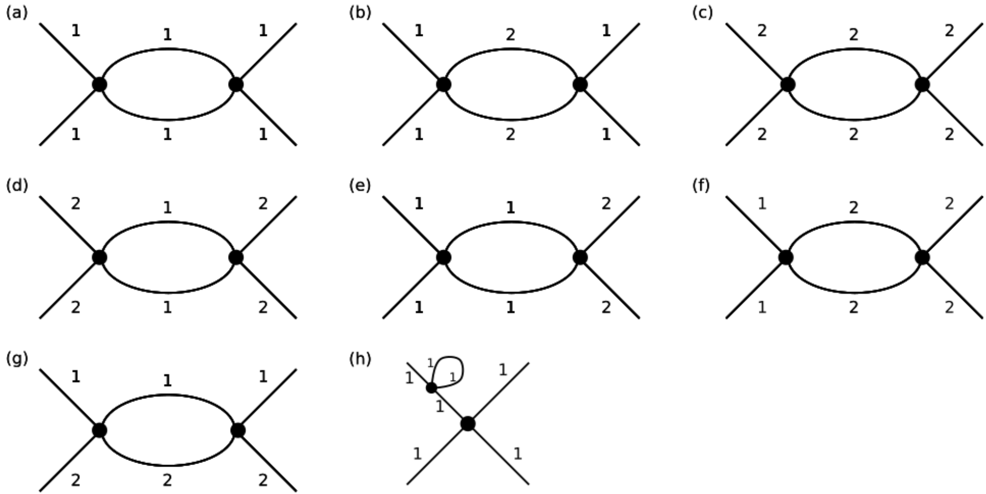


Figure 3: Patterns for the divergent four point Feynman diagrams; (a), (b) correction to the vertex with the coupling constant λ_1; (c), (d) correction to the vertex with the coupling constant λ_2; (e), (f), (g) correction to the vertex with the coupling constant λ_3; (h) this type of diagram is not included in the propagator correction because it is not one particle irreducible diagram (it can be divided into two separate diagrams by cutting only one internal line). To each pattern corresponds more than one Feynman diagram. See Appendix E.

the diagrams in Fig. 3 (a)–(g)

$$(-i)^2\left(\frac{\lambda_1}{4!}\right)^2\frac{1}{2!}\int_{-\infty}^{\infty}\mathrm{d}^4z_1\mathrm{d}^4z_2$$
$$\times\langle 0|T(\phi_1^{\mathrm{I}}(x_1),\phi_1^{\mathrm{I}}(x_2),\phi_1^{\mathrm{I}}(x_3),\phi_1^{\mathrm{I}}(x_4),\left(\phi_1^{\mathrm{I}}(z_1)\right)^4,\left(\phi_1^{\mathrm{I}}(z_2)\right)^4)|0\rangle_c\,, \tag{16a}$$

$$(-i)^2\left(\frac{\lambda_3}{4}\right)^2\frac{1}{2!}\int_{-\infty}^{\infty}\mathrm{d}^4z_1\mathrm{d}^4z_2$$
$$\times\langle 0|T(\phi_1^{\mathrm{I}}(x_1),\phi_1^{\mathrm{I}}(x_2),\phi_1^{\mathrm{I}}(x_3),\phi_1^{\mathrm{I}}(x_4),$$
$$\left(\phi_1^{\mathrm{I}}(z_1)\right)^2,\left(\phi_2^{\mathrm{I}}(z_1)\right)^2,\left(\phi_1^{\mathrm{I}}(z_2)\right)^2\left(\phi_2^{\mathrm{I}}(z_2)\right)^2)|0\rangle_c, \tag{16b}$$

$$(-i)^2\left(\frac{\lambda_2}{4!}\right)^2\frac{1}{2!}\int_{-\infty}^{\infty}\mathrm{d}^4z_1\mathrm{d}^4z_2$$
$$\times\langle 0|T(\phi_2^{\mathrm{I}}(x_1),\phi_2^{\mathrm{I}}(x_2),\phi_2^{\mathrm{I}}(x_3),\phi_2^{\mathrm{I}}(x_4),\left(\phi_2^{\mathrm{I}}(z_1)\right)^4,\left(\phi_2^{\mathrm{I}}(z_2)\right)^4)|0\rangle_c\,, \tag{16c}$$

$$(-i)^2\left(\frac{\lambda_3}{4}\right)^2\frac{1}{2!}\int_{-\infty}^{\infty}\mathrm{d}^4z_1\mathrm{d}^4z_2$$
$$\times\langle 0|T(\phi_2^{\mathrm{I}}(x_1),\phi_2^{\mathrm{I}}(x_2),\phi_2^{\mathrm{I}}(x_3),\phi_2^{\mathrm{I}}(x_4),$$
$$\left(\phi_1^{\mathrm{I}}(z_1)\right)^2,\left(\phi_2^{\mathrm{I}}(z_1)\right)^2,\left(\phi_1^{\mathrm{I}}(z_2)\right)^2\left(\phi_2^{\mathrm{I}}(z_2)\right)^2)|0\rangle_c, \tag{16d}$$

$$(-i)^2\frac{\lambda_1}{4!}\frac{\lambda_3}{4}\frac{2}{2!}\int_{-\infty}^{\infty}\mathrm{d}^4z_1\mathrm{d}^4z_2$$
$$\times\langle 0|T(\phi_1^{\mathrm{I}}(x_1),\phi_1^{\mathrm{I}}(x_2),\phi_2^{\mathrm{I}}(x_3),\phi_2^{\mathrm{I}}(x_4),$$
$$\left(\phi_1^{\mathrm{I}}(z_1)\right)^4,\left(\phi_1^{\mathrm{I}}(z_2)\right)^2,\left(\phi_2^{\mathrm{I}}(z_2)\right)^2)|0\rangle_c, \tag{16e}$$

The three Feynman diagrams in the momentum space corresponding to Eq. (16b) are shown in Fig. 16 and the analytic expressions for these diagrams are

$$-\frac{i\lambda_1^2}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2-m_1^2+i\epsilon}\cdot\frac{i}{(p_1+p_2-k)^2-m_1^2+i\epsilon}, \tag{17a}$$

$$-\frac{i\lambda_1^2}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2-m_1^2+i\epsilon}\cdot\frac{i}{(p_1-p_3-k)^2-m_1^2+i\epsilon}, \tag{17b}$$

$$-\frac{i\lambda_1^2}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2-m_1^2+i\epsilon}\cdot\frac{i}{(p_1-p_4-k)^2-m_1^2+i\epsilon} \tag{17c}$$

and similarly one can obtain the momentum space diagrams for the remaining equations (16). The integrals in Eqs. (17) are logarithmically divergent.

4. Calculation of the loop integrals

The loop integrals contain the integration over momentum. Those integrals contain the products of the particle propagators and integration of such a product is complicated. The Feynman prescription, which we will discuss first, facilitates the integration.

Another convenient scheme that facilitates the integration is the Wick rotation which allows the conversion of the integral from the Minkowski space to the Euclidean space.

As we have seen in Section 3 the loop diagrams are divergent. None of the physical quantities are infinite, so the loop diagrams require a special treatment in the field theory. Ultimately these infinities are removed by the procedure of renormalization, but first these diagrams are made finite by introducing a cut-off into their analytical expressions. The idea of the cut-off is to make the integral convergent by adding to the integral a modification, which depends on an additional parameter with a property that at a certain limit of the parameter, the integral tends to the original divergent integral. In such a way, we can identify a type of divergence, which is then removed. A simple cut-off placed as an upper limit of integration is physically inconvenient, because it breaks the translation invariance of the theory. We will discuss here two methods of removing the divergence of the loop integrals: the *Pauli-Villars regularization* and the *dimensional regularization.* These methods of regularization have desirable properties from a physical point of view.

4.1. Feynman prescription

The Feynman prescription converts the product of two fractions into an integral of one fraction and adds additional integrals to the expression. This facilitates the integration of the product of the propagators

$$\frac{1}{AB} = \int_0^1 \frac{\mathrm{d}\xi}{(A\xi + (1-\xi)B)^2} = \int_0^1 \int_0^1 \frac{\mathrm{d}\xi_1\, \mathrm{d}\xi_2\, \delta(1-\xi_1-\xi_2)}{(A\xi_1 + B\xi_2)^2}. \tag{18}$$

Eq. (18) can be generalized to the product of n terms in the denominator

$$\frac{1}{A_1 \cdots A_n} = (n-1)! \int_0^1 \cdots \int_0^1 \frac{\mathrm{d}\xi_1 \cdots \mathrm{d}\xi_n\, \delta(1-\xi_1-\cdots-\xi_n)}{(A_1\xi_1 + \cdots + A_n\xi_n)^n}. \tag{19}$$

Another generalization is obtained by the differentiation of Eq. (18) with respect to A

$$\frac{1}{A^n B} = \int_0^1 \frac{n!\,\mathrm{d}\xi}{(A\xi + (1-\xi)B)^{n+1}} = \int_0^1 \int_0^1 \frac{n!\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\,\delta(1-\xi_1-\xi_2)}{(A\xi_1 + B\xi_2)^{n+1}}. \tag{20}$$

4.2. Wick rotation

The momentum integrals are performed in the Minkowski space, where the length of a vector $q = (q_0, q_1, q_2, q_3)$ is equal $q^2 = q_0^2 - q_1^2 - q_2^2 - q_3^2$. Let us consider the integral

$$\int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{(q^2 - m^2 + i\epsilon)^n} \tag{21}$$

in the Minkowski space. The integrated function as the function of complex q_0 has poles at $q_0 = \pm(\sqrt{\mathbf{q}^2 + m^2} - i\epsilon)$. This means that the contour integral in the complex q_0 plane over the path shown in Fig. 4 does not include the poles of the integrand, so it vanishes. When the integration radius goes to infinity the integrals over the circular segments tend to zero and it means that the following relation holds

$$\int_{-\infty}^{+\infty} \mathrm{d}q_0 \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^4} \frac{1}{(q^2 - m^2 + i\epsilon)^n} = \int_{-i\infty}^{+i\infty} \mathrm{d}q_0 \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^4} \frac{1}{(q^2 - m^2 + i\epsilon)^n}. \tag{22}$$

The variable q_0 on the right hand side of Eq. (22) is pure imaginary. To convert the integration over real numbers we make the change of variables

$$q_0 \to i q_4, \quad \mathrm{d}q_0 \to i\,\mathrm{d}q_4\,, \quad \mathrm{d}^4 q = \mathrm{d}q_0\,\mathrm{d}^3\mathbf{q} \to i\,\mathrm{d}q_4\,\mathrm{d}^3\mathbf{q} = i\,\mathrm{d}^4 q_\mathrm{E}$$
$$q^2 = q_0^2 - q_1^2 - q_2^2 - q_3^2 \to -q_1^2 - q_2^2 - q_3^2 - q_4^2 = -q_\mathrm{E}^2.$$

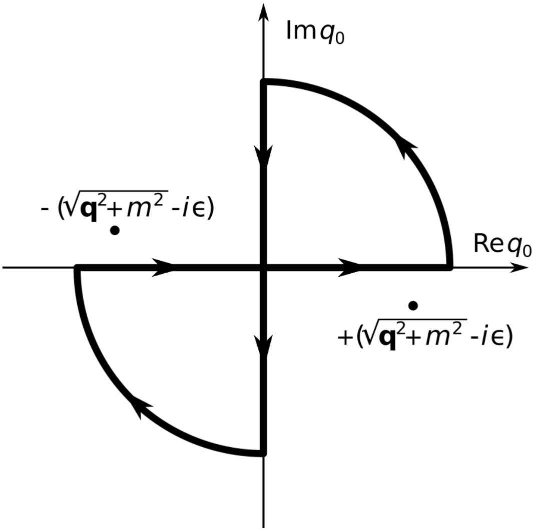


Figure 4: Integration path in the q_0 and the poles for Eq. (21)

Then the Minkowski space integral (21) is converted into the Euclidean space integral and

$$\int \frac{d^4q}{(2\pi)^4} \frac{1}{(q^2 - m^2 + i\epsilon)^n} = i(-1)^n \int \frac{d^4q_E}{(2\pi)^4} \cdot \frac{1}{(q_E^2 + m^2)^n}. \tag{23}$$

The $i\epsilon$ term in the denominator on the right hand side is not necessary, because the function $(q_E^2 + m^2)$ is not singular.

It should be noted that the Wick rotation is valid for any function that has no singularities in the first and third quadrant of the complex q_0 plane.

4.3. Pauli-Villars regularization

The loop diagram contains an integral over the intermediate momentum d^4q of a product of the propagators $\sim \frac{1}{q^2 - m^2}$. The integral with one propagator is quadratically divergent and the integral with two propagators is logarithmically divergent. The Pauli-Villars regularization [1] introduces a cut-off Λ by adding to the propagator terms, which vanish at a limit $\Lambda \to \infty$. In such a way the asymptotic behavior of the propagator is modified and the loop integral becomes finite, but dependent on the cut-off. The prototype of the Pauli-Villars regularization is illustrated by the equation

$$\begin{aligned} \frac{i}{q^2 - m^2 + i\epsilon} \to \left[\frac{i}{q^2 - m^2 + i\epsilon}\right]^{\mathrm{PV}}_{\Lambda} &= \frac{i}{q^2 - m^2 + i\epsilon} - \frac{i}{q^2 - \Lambda + i\epsilon} \\ &= \frac{i(m^2 - \Lambda)}{(q^2 - m^2 + i\epsilon)(q^2 - \Lambda + i\epsilon)}. \end{aligned} \tag{24}$$

Here Λ is the cut-off parameter with the dimension of mass squared. One can see that the modified propagator behaves for large q^2 like $1/q^4$, while the original propagator behaved like $1/q^2$. This can improve the convergence of the loop integrals. The general form of the Pauli-Villars regularization of the propagator is

$$\frac{i}{q^2 - m^2 + i\epsilon} \to \left[\frac{i}{q^2 - m^2 + i\epsilon}\right]^{\text{PV}}_{\Lambda} = \frac{i}{q^2 - m^2 + i\epsilon} - \sum_{j=1}^{n} \frac{iC_j}{q^2 - \Lambda_j + i\epsilon}$$
$$= \frac{iC}{(q^2 - m^2 + i\epsilon)\prod_{j=1}^{n}(q^2 - \Lambda_j + i\epsilon)}. \quad (25)$$

Pauli-Villars regularization is physically equivalent to the introduction of scalar particles obeying the Fermi-Dirac statistics (ghosts) into the theory.

4.4. Dimensional regularization

Dimensional regularization [2, 3] is another method of analyzing the divergent diagrams in quantum field theory. The method consists in calculating the integral in d dimensional spacetime. Suppose we need to calculate the integral

$$\int \frac{\mathrm{d}^4 q_\mathrm{E}}{(2\pi)^4} \frac{1}{(q_\mathrm{E}^2 + m^2)^2}. \quad (26)$$

This integral is 4-dimensional and is logarithmically divergent, because for large q_E the numerator behaves, like $\sim q_\mathrm{E}^3$ and the denominator like $\sim q_\mathrm{E}^4$. In the 3-dimensional space this integral is finite. This suggests to calculate this integral in the space with lower dimensionality d and then, analytically continue the result to $d \to 4$. This will isolate the singularity of the integral.

In the d-dimensional space, after separating the angular part, the integral (26) becomes

$$\int \frac{\mathrm{d}^d q_\mathrm{E}}{(2\pi)^d} \frac{1}{(q_\mathrm{E}^2 + m^2)^2} = \frac{1}{(2\pi)^d} \int \mathrm{d}\Omega_d \int_0^\infty \frac{q_\mathrm{E}^{d-1}\, \mathrm{d}q_\mathrm{E}}{(q_\mathrm{E}^2 + m^2)^2}. \tag{27}$$

For the angular part we have

$$\frac{1}{(2\pi)^d} \int \mathrm{d}\Omega_d = \frac{2}{(2\pi)^d} \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \tag{28}$$

and it is not singular at $d = 4$. Here $\Gamma(z)$ is the Euler gamma function. The remaining integral

$$\int_0^\infty \frac{q_\mathrm{E}^{d-1}\, \mathrm{d}q_\mathrm{E}}{(q_\mathrm{E}^2 + m^2)^2} = \frac{\Gamma(\frac{d}{2})\Gamma(2-\frac{d}{2})}{2\Gamma(2)(m^2)^{2-\frac{d}{2}}} = \frac{(d-2)\pi}{4\sin\left(\frac{d\pi}{2}\right) m^{4-d}} \tag{29}$$

is finite for $d < 4$. The right hand side of Eq. (29) can be calculated for continuous values of the space dimension d. Expanding the right hand side of Eq. (29) around the point $d = 4$ one obtains

$$\frac{\Gamma(\frac{d}{2})\Gamma(2-\frac{d}{2})}{2\Gamma(2) m^{2(2-\frac{d}{2})}} \approx -\frac{1}{d-4} - \frac{1}{2}(1 + \ln m^2) + \cdots \tag{30}$$

and for the integral (27) one gets

$$\int \frac{\mathrm{d}^d q_\mathrm{E}}{(2\pi)^d} \frac{1}{(q_\mathrm{E}^2 + m^2)^2} = -\frac{1}{8\pi^2(d-4)} - \frac{1}{16\pi^2}\left(\gamma + \ln\left(\frac{m^2}{4\pi}\right)\right) + \cdots, \tag{31}$$

where $\gamma = 0.577216\ldots$ is the Euler's constant. From Eq. (31) we see that the integral has a pole at $d = 4$ and the divergent part is equal $-1/(8\pi^2(d-4))$.

Eqs. (32) and (33) below are general formulas, which are helpful in the practical calculation of integrals in the dimensional regularization

$$\int \frac{\mathrm{d}^d q_{\mathrm{E}}}{(2\pi)^d} \frac{1}{(q_{\mathrm{E}}^2 + m^2)^r} = \frac{\Gamma(r - \frac{d}{2})}{(4\pi)^{\frac{d}{2}} \Gamma(r)} \cdot \frac{1}{(m^2)^{r-\frac{d}{2}}} \tag{32}$$

and

$$\Gamma(-n+z) = \frac{(-1)^n}{n!} \Big(\frac{1}{z} + \psi(n+1) + \frac{1}{2} z \Big(\frac{\pi^2}{3} + \psi^2(n+1) - \psi'(n+1) \Big) \Big) + \cdots \tag{33}$$

Here

$$\psi(z) = \frac{\mathrm{d} \ln \Gamma(z)}{\mathrm{d} z}, \quad \psi(n+1) = \sum_{l=1}^{n} \frac{1}{l} - \gamma, \quad \psi'(n+1) = \frac{\pi^2}{6} - \sum_{l=1}^{n} \frac{1}{l^2} \tag{34}$$

and again γ is the Euler constant.

From Eq. (32), by a change of variables, one obtains a more general result

$$\int \frac{\mathrm{d}^d q_{\mathrm{E}}}{(2\pi)^d} \frac{1}{(q_{\mathrm{E}}^2 + m^2 + 2 q_{\mathrm{E}} \cdot p)^r} = \frac{\Gamma(r - \frac{d}{2})}{(4\pi)^{\frac{d}{2}} \Gamma(r)} \frac{1}{(m^2 - p^2)^{r-\frac{d}{2}}} \tag{35}$$

and by differentiation of Eq. (35) with respect of p_μ one can obtain more useful formulas.

4.4.0.1. Dimension of the coupling constant

The action $S = \int \mathcal{L}\, \mathrm{d}^4x$ is dimensionless[3] and it has to be dimensionless also in the d dimensional space. The dimension of the fields depends of the dimension of the space-time and can be determined from the Lagrangian density or from the fields commutation relations. The dimension of the scalar field ϕ in the 4-dimensional space is L^{-1} (L is length). In the d-dimensional space the dimension of the scalar field is $L^{1-\frac{d}{2}}$. In order that the part of the action in d dimensions, obtained from the $\lambda\phi^4$, is dimensionless the coupling constant λ must have the dimension L^{4-d}. To maintain the coupling constant dimensionless it is rescaled by a parameter μ with the dimension of mass, which plays the role of the cut-off. The coupling constant λ then becomes

$$\lambda \to \lambda\mu^{4-d}. \tag{36}$$

In the considered model the three coupling constants λ_1, λ_2 and λ_3 are all rescaled in such a way.

5. Renormalization

The Lagrangian density in quantum field theory is a function of the fields and of the masses and coupling constants. Those masses and coupling constants *do not* correspond to observable quantities, e.g., the observable masses are obtained from the poles of the propagators and are functions of the Lagrangian masses and coupling constants. At the lowest order of

[3] In natural units the dimension of action is position $\times$ momentum which has the same dimension as Planck's constant $\hbar$.

perturbation theory and for the free fields the Lagrangian masses do correspond to the observable masses, but it is not very interesting for most of practical cases. The fact that parameters of the Lagrangian have to be adjusted to physical observables is called *renormalization*. This means that the renormalization has to be performed in any field theory, independent of the presence of infinities.

The infinite expressions first appeared in quantum electrodynamics. This caused an important difficulty, because it ruled out the calculations at higher orders of perturbation theory. It turned out that in quantum electrodynamics the mathematical structure of the infinite expressions was identical to the mathematical structure of the Lagrangian density. This means that the infinite expressions were *renormalizing* the parameters of the Lagrangian and were not modifying the theory in any other way. Such a situation does not occur in all quantum field theories, but only in a small class of renormalizable theories, like quantum electrodynamics, theories with non-abelian gauge symmetry and theories of scalar fields with self interaction of type ϕ^3 or ϕ^4. The theory described by the Lagrangian density in Eq. (1) is renormalizable.

Most of the practical calculations in quantum field theory are done perturbatively with respect to the power of the coupling constants. The exception to this rule is the renormalization, which is discussed with respect to the number of loops present in the Feynman diagrams. We will discuss the renormalization of our model up to two loops.

5.1. One loop regularization

The process of renormalization starts with the identification of the divergent diagrams. The one loop diagrams can appear at any order of the perturbative calculations, but the structure of these diagrams is such that one can extract a simpler one loop divergent sub-diagram and the rest of the diagram is finite. This is the reason that one has to concentrate on the renormalization of those simple sub-diagrams[4]. In our model those simple diagrams at one loop correspond to the two and four point Green's functions. The topological structure of those diagrams is shown in Fig. 5

5.1.1. Regularization of the two point Green's functions

All the diagrams that correspond to the structure in Fig. 5 (a) are shown in Fig. 2 and the mathematical expressions corresponding to those diagrams are given in Eqs. (14). For the diagram in Fig. 2 (a) the mathematical formula in d dimensions obtained from Eq. (14a) becomes

$$
\begin{aligned}
&\frac{\lambda_1}{2}\int\frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{i}{k^2-m_1^2+i\epsilon}\to\frac{\lambda_1\mu^{4-d}}{2}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{i}{k^2-m_1^2+i\epsilon}\\
&\qquad\to\frac{\lambda_1\mu^{4-d}}{2}\int\frac{\mathrm{d}^d k_{\mathrm{E}}}{(2\pi)^d}\frac{1}{k_{\mathrm{E}}^2+m_1^2}=\frac{\lambda_1\mu^{4-d}}{2}\frac{\Gamma(1-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}\Gamma(1)}\frac{1}{(m_1^2)^{1-\frac{d}{2}}}\\
&\qquad\underset{d=4}{\to}\frac{\lambda_1 m_1^2}{16\pi^2}\left(\frac{1}{d-4}+\frac{1}{2}\left(\gamma-1+\ln\left(\frac{m_1^2}{4\pi\mu^2}\right)\right)\right)+\mathcal{O}(d-4). \qquad (37)
\end{aligned}
$$

[4]It can be shown that there is only a finite number of such diagrams.

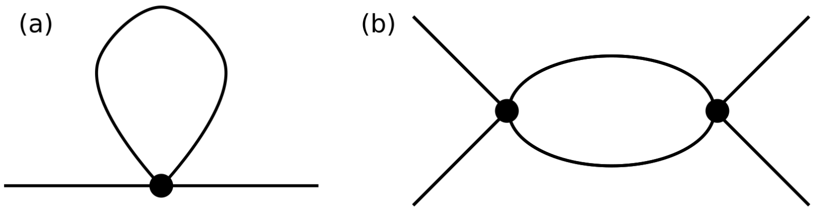


Figure 5: The topological structure of the one loop divergent diagrams for: (a) two point function, and (b) for the four point functions.

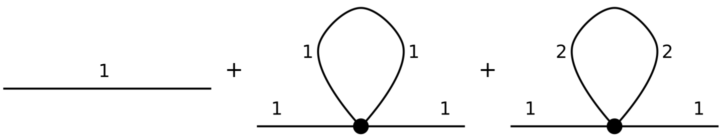


Figure 6: The propagator of particle 1 with one loop corrections.

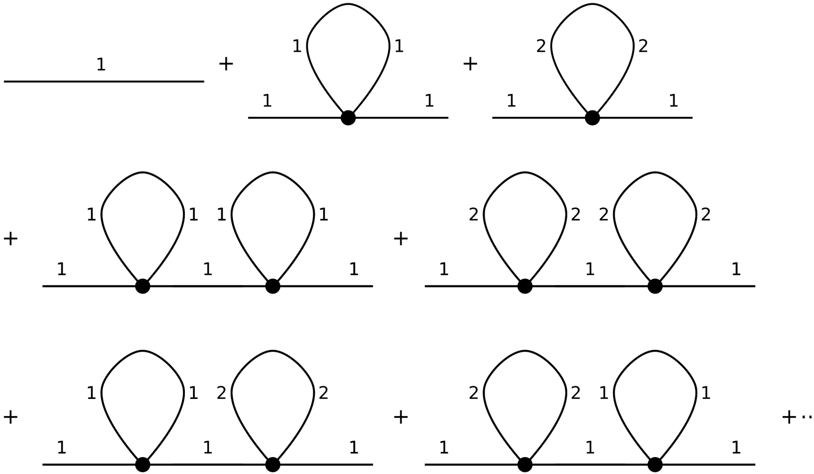


Figure 7: The next iteration for the propagator of particle 1 with one loop corrections.

In the same way we obtain the following result for the diagram in Fig. 2 (b)

$$\frac{\lambda_3}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2-m_2^2+i\epsilon}$$
$$\underset{d=4}{\longrightarrow}\frac{\lambda_3 m_2^2}{16\pi^2}\left(\frac{1}{d-4}+\frac{1}{2}\left(\gamma-1+\ln\left(\frac{m_2^2}{4\pi\mu^2}\right)\right)\right)+\mathcal{O}(d-4). \quad (38)$$

The space-time dimension d is equal to 4 and the integrals in Eqs. (37) and (38) have poles for this value of d and are divergent. The important property of Eqs. (37) and (38) is that the singular part of the integral has been separated from the finite part. The finite part depends on the cut-off parameter μ^2 and the infinite part does not. The μ^2 is a new free parameter of the theory[5], so the dependence of the finite part on it, means that the finite parts of the integrals are not uniquely determined.

Eqs. (37) and (38) are the one loop corrections to the propagator of particle 1, so the full propagator at one loop order will be the sum of the free propagator and the integrals (37) and (38). Graphically this sum is shown in Fig. 6.

The one loop corrections can be iterated and the next iteration is shown in Fig. 7. The iterations can be continued and as a result one obtains the geometric series for the propagator

[5]When we will discuss the renormalization group we will see that the physical predictions of the theory *do not* depend on μ^2.

of particle 1

$$\begin{aligned}
&\frac{i}{p^2 - m_1^2 + i\epsilon} \\
&\qquad + \frac{i}{p^2 - m_1^2 + i\epsilon}(-i\Sigma_1)\frac{i}{p^2 - m_1^2 + i\epsilon} + \frac{i}{p^2 - m_1^2 + i\epsilon}(-i\Sigma_3^1)\frac{i}{p^2 - m_1^2 + i\epsilon} \\
&\qquad + \frac{i}{p^2 - m_1^2 + i\epsilon}(-i\Sigma_1)\frac{i}{p^2 - m_1^2 + i\epsilon}(-i\Sigma_1)\frac{i}{p^2 - m_1^2 + i\epsilon} \\
&\qquad + \frac{i}{p^2 - m_1^2 + i\epsilon}(-i\Sigma_3^1)\frac{i}{p^2 - m_1^2 + i\epsilon}(-i\Sigma_3^1)\frac{i}{p^2 - m_1^2 + i\epsilon} \\
&\qquad + \frac{i}{p^2 - m_1^2 + i\epsilon}(-i\Sigma_1)\frac{i}{p^2 - m_1^2 + i\epsilon}(-i\Sigma_3^1)\frac{i}{p^2 - m_1^2 + i\epsilon} \\
&\qquad + \frac{i}{p^2 - m_1^2 + i\epsilon}(-i\Sigma_3^1)\frac{i}{p^2 - m_1^2 + i\epsilon}(-i\Sigma_1^1)\frac{i}{p^2 - m_1^2 + i\epsilon} + \cdots \\
&\qquad = \frac{i}{p^2 - m_1^2 + i\epsilon} \cdot \frac{1}{1 - \frac{(-i(\Sigma_1 + \Sigma_3^1))i}{p^2 - m_1^2 + i\epsilon}} = \frac{i}{p^2 - m_1^2 - \Sigma_1 - \Sigma_3^1 + i\epsilon}.
\end{aligned} \tag{39}$$

In Eq. (39) we used the shorthand notation for the results of Eqs. (37) and (38)

$$\Sigma_1 = \frac{\lambda_1 m_1^2}{16\pi^2}\Big(\frac{1}{d-4} + \frac{1}{2}\big(\gamma - 1 + \ln\Big(\frac{m_1^2}{4\pi\mu^2}\Big)\big)\Big), \tag{40a}$$

$$\Sigma_2 = \frac{\lambda_2 m_2^2}{16\pi^2}\Big(\frac{1}{d-4} + \frac{1}{2}\big(\gamma - 1 + \ln\Big(\frac{m_2^2}{4\pi\mu^2}\Big)\big)\Big), \tag{40b}$$

$$\Sigma_3^1 = \frac{\lambda_3 m_2^2}{16\pi^2}\Big(\frac{1}{d-4} + \frac{1}{2}\big(\gamma - 1 + \ln\Big(\frac{m_2^2}{4\pi\mu^2}\Big)\big)\Big), \tag{40c}$$

$$\Sigma_3^2 = \frac{\lambda_3 m_1^2}{16\pi^2}\Big(\frac{1}{d-4} + \frac{1}{2}\big(\gamma - 1 + \ln\Big(\frac{m_1^2}{4\pi\mu^2}\Big)\big)\Big). \tag{40d}$$

Σ_2 and Σ_3^2 are necessary for the propagator of particle 2 and they are obtained from the Σ_1 and Σ_3^1 by changing the indices $1 \leftrightarrow 2$.

From Eq. (39) we see that after the one loop corrections for the propagator the value of the mass m_1^2 was shifted

$$m_1^2 \to m_1^2 + \Sigma_1 + \Sigma_3^1. \tag{41}$$

It means that the mass m_1 becomes *renormalized* as the result of the interactions.

The shift of the mass m_2^2 is

$$m_2^2 \to m_2^2 + \Sigma_2 + \Sigma_3^2. \tag{42}$$

5.1.2. Regularization of the four point Green's functions

All the diagrams that correspond to the structure in Fig. 5 (b), with all external particles of type 1, are shown in Fig. 16 and the mathematical expressions corresponding to those diagrams are given in Eqs. (17). For the diagram in Fig. 16 (a) the mathematical formula in d dimensions obtained from Eq. (17a) is obtained by the following steps (here we use the substitution: $p_1 + p_2 = p$)

$$-\frac{i\lambda_1^2}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2-m_1^2+i\epsilon}\cdot\frac{i}{(p-k)^2-m_1^2+i\epsilon} \tag{43a}$$

$$=\frac{i\lambda_1^2}{2}\int_0^1\mathrm{d}z\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{(z(k^2-m_1^2+i\epsilon)+(1-z)((p-k)^2-m_1^2+i\epsilon))^2} \tag{43b}$$

$$=\frac{i\lambda_1^2}{2}\int_0^1\mathrm{d}z\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{(zk^2+(1-z)(p^2+k^2-2pk)-m_1^2+i\epsilon)^2} \tag{43c}$$

$$=\frac{i\lambda_1^2}{2}\int_0^1\mathrm{d}z\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{((k-(1-z)p)^2+z(1-z)p^2-m_1^2+i\epsilon)^2} \tag{43d}$$

$$\to\frac{i\lambda_1^2}{2}\int_0^1\mathrm{d}z\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{(k^2+z(1-z)p^2-m_1^2+i\epsilon)^2} \tag{43e}$$

$$\to-\frac{\lambda_1^2}{2}\int_0^1\mathrm{d}z\int\frac{\mathrm{d}^4k_{\mathrm{E}}}{(2\pi)^4}\frac{1}{(k_{\mathrm{E}}^2-z(1-z)p^2+m_1^2)^2} \tag{43f}$$

$$\to-\frac{\lambda_1^2\mu^{2(4-d)}}{2}\int_0^1\mathrm{d}z\int\frac{\mathrm{d}^dk_{\mathrm{E}}}{(2\pi)^d}\frac{1}{(k_{\mathrm{E}}^2-z(1-z)p^2+m_1^2)^2} \tag{43g}$$

$$=-\frac{\lambda_1^2\mu^{2(4-d)}}{2}\int_0^1\mathrm{d}z\,\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}\Gamma(2)(m_1^2-z(1-z)p^2)^{2-\frac{d}{2}}} \tag{43h}$$

$$\underset{d=4}{\to}\frac{\lambda_1^2\mu^{4-d}}{16\pi^2}\Big(\frac{1}{d-4}+\frac{1}{2}(\gamma+\int_0^1\mathrm{d}z\ln\Big(\frac{|z(1-z)p^2-m_1^2|}{4\pi\mu^2}\Big))\Big). \tag{43i}$$

Let us explain each step in Eqs. (43)

Eq. (43a)→ (43b) Feynman prescription.

Eq. (43b)→ (43c) Transformation of the denominator with propagators.

Eq. (43c)→ (43d) Transformation of the denominator with propagators.

Eq. (43d)→ (43e) Change of the integration variable $k \to (k - (1 - z)p)$.

Eq. (43e)→ (43f) Wick rotation.

Eq. (43f)→ (43g) Transition to the d-dimensional integration.

Eq. (43g)→ (43h) Calculation of the d-dimensional integral, using Eq. (32).

Eq. (43h)→ (43i) Calculation of the limit $d \to 4$. Note, that the factor μ^{4-d} has not been included into the integral, because the dimension of the diagram is μ^{4-d}.

The last integral in Eq. (43i) is elementary and can be calculated, but an explicit integration does not introduce important information.

To continue, we define the functions $F(r_1, r_2, r_3)$ and $F_\Sigma(s, t, u, r_1, r_2)$

$$F(r_1, r_2, r_3) = \int_0^1 \mathrm{d}z \ln\left(\frac{|z(1-z)r_1 - r_2|}{4\pi r_3}\right), \tag{44}$$

$$F_\Sigma(s, t, u, r_1, r_2) = F(s, r_1, r_2) + F(t, r_1, r_2) + F(u, r_1, r_2). \tag{45}$$

And with the help of the function (44), using Eq. (43i), we can write the analytical expressions corresponding to the diagrams in Fig. 16

$$\frac{\lambda_1^2\mu^{4-d}}{16\pi^2}\Big(\frac{1}{d-4}+\frac{1}{2}(\gamma+F(s,m_1^2,\mu^2))\Big), \tag{46a}$$

$$\frac{\lambda_1^2\mu^{4-d}}{16\pi^2}\Big(\frac{1}{d-4}+\frac{1}{2}(\gamma+F(t,m_1^2,\mu^2))\Big), \tag{46b}$$

$$\frac{\lambda_1^2\mu^{4-d}}{16\pi^2}\Big(\frac{1}{d-4}+\frac{1}{2}(\gamma+F(u,m_1^2,\mu^2))\Big). \tag{46c}$$

Here, s, t, and u denote the Mandelstam variables

$$s=(p_1+p_2)^2, \quad t=(p_1-p_3)^2, \quad u=(p_1-p_4)^2.$$

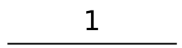
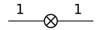
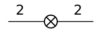
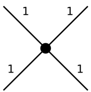
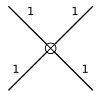
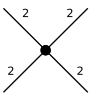
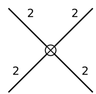
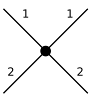
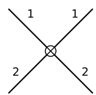
Next we will calculate the diagrams of the type shown in Fig. 3 (b). There are three diagrams of this type which are shown in Fig. 8. The procedure of the derivation of the analytic expressions corresponding to these diagrams is the same as in the previous case and the result is

$$\frac{\lambda_3^2\mu^{4-d}}{16\pi^2}\Big(\frac{1}{d-4}+\frac{1}{2}(\gamma+F(s,m_2^2,\mu^2))\Big), \tag{47a}$$

$$\frac{\lambda_3^2\mu^{4-d}}{16\pi^2}\Big(\frac{1}{d-4}+\frac{1}{2}(\gamma+F(t,m_2^2,\mu^2))\Big), \tag{47b}$$

$$\frac{\lambda_3^2\mu^{4-d}}{16\pi^2}\Big(\frac{1}{d-4}+\frac{1}{2}(\gamma+F(u,m_2^2,\mu^2))\Big). \tag{47c}$$

Table 2: Analytical expressions corresponding to the elements of the Feynman diagrams for the Lagrangian density in Eq. (54).

Propagators		$\longrightarrow \dfrac{i}{p^2 - M_1^2 + i\epsilon}$		$\longrightarrow -i\delta_{M_1}$
		$\longrightarrow \dfrac{i}{p^2 - M_2^2 + i\epsilon}$		$\longrightarrow -i\delta_{M_2}$
Vertices		$\longrightarrow -i\Lambda_1$		$\longrightarrow -i\delta_{\Lambda_1}$
		$\longrightarrow -i\Lambda_2,$		$\longrightarrow -i\delta_{\Lambda_2},$
		$\longrightarrow -i\Lambda_3$		$\longrightarrow -i\delta_{\Lambda_3}.$

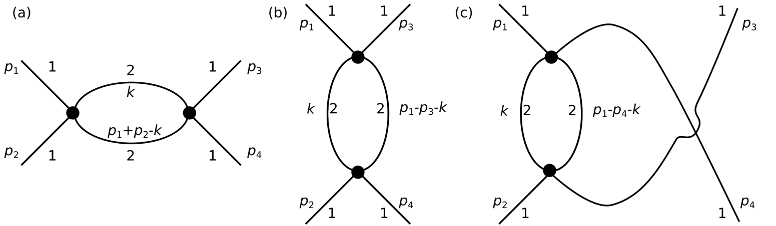


Figure 8: Three Feynman diagrams with four external particles of the type 1 and the vertices of the type 3.

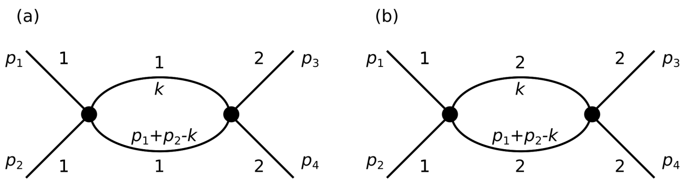


Figure 9: Feynman diagrams corresponding to the patterns in Fig. 3 (e) and (f).

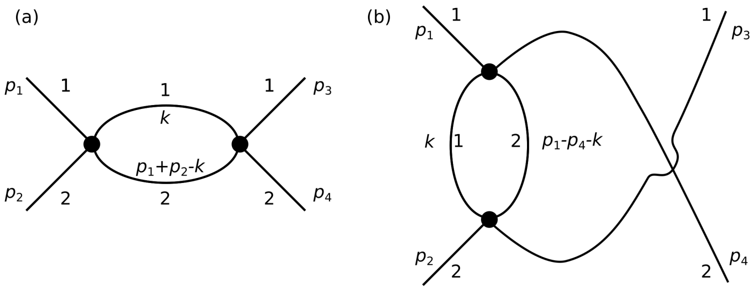


Figure 10: Two Feynman diagrams corresponding to the pattern in Fig. 3 (g).

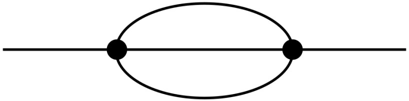


Figure 11: The *setting sun* diagram in the ϕ^4 theory, which is divergent and requires the introduction of a correction, related with the normalization of the scalar field.

The four body vertex with one loop contribution for four external particles of the type 1 is thus equal

$$\lambda_1\mu^{4-d} + \frac{\lambda_1^2\mu^{4-d}}{16\pi^2}\Big(\frac{3}{d-4} + \frac{3}{2}\gamma + \frac{1}{2}\big(F_\Sigma(s,t,u,m_1^2,\mu^2)\big)\Big) \\ + \frac{\lambda_3^2\mu^{4-d}}{16\pi^2}\Big(\frac{3}{d-4} + \frac{3}{2}\gamma + \frac{1}{2}\big(F_\Sigma(s,t,u,m_2^2,\mu^2)\big)\Big). \quad (48)$$

By changing the indices $1 \leftrightarrow 2$ we obtain from Eq. (48) the four body vertex with one loop contribution for four external particles of the type 2

$$\lambda_2\mu^{4-d} + \frac{\lambda_2^2\mu^{4-d}}{16\pi^2}\Big(\frac{3}{d-4} + \frac{3}{2}\gamma + \frac{1}{2}\big(F_\Sigma(s,t,u,m_2^2,\mu^2)\big)\Big) \\ + \frac{\lambda_3^2\mu^{4-d}}{16\pi^2}\Big(\frac{3}{d-4} + \frac{3}{2}\gamma + \frac{1}{2}\big(F_\Sigma(s,t,u,m_1^2,\mu^2)\big)\Big). \quad (49)$$

For the patterns in Fig. 3 (e) and (f) there is only one Feynman diagram, each in the momentum space shown in Fig. 9 and the analytical expressions are:
For the pattern in Fig. 3 (e)

$$-\frac{i\lambda_1\lambda_3}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2-m_1^2+i\epsilon}\cdot\frac{i}{(p_1+p_2-k)^2-m_1^2+i\epsilon} \\ \underset{d=4}{\rightarrow} \frac{\lambda_1\lambda_3\mu^{4-d}}{16\pi^2}\Big(\frac{1}{d-4} + \frac{1}{2}(\gamma + F(s,m_1^2,\mu^2))\Big) \quad (50)$$

and for the pattern in Fig. 3 (f)

$$-\frac{i\lambda_2\lambda_3}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2-m_2^2+i\epsilon}\cdot\frac{i}{(p_1+p_2-k)^2-m_2^2+i\epsilon}$$
$$\underset{d=4}{\rightarrow}\frac{\lambda_2\lambda_3\mu^{4-d}}{16\pi^2}\Big(\frac{1}{d-4}+\frac{1}{2}(\gamma+F(s,m_2^2,\mu^2))\Big). \tag{51}$$

The last diagram that we consider is with pattern in Fig. 3 (g) for which we have two diagrams in momentum space shown in Fig. 10 and the analytic expressions corresponding to these diagrams are

$$-i\lambda_3^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2-m_1^2+i\epsilon}\cdot\frac{i}{(p_1+p_2-k)^2-m_2^2+i\epsilon}$$
$$\underset{d=4}{\rightarrow}\frac{\lambda_3^2\mu^{4-d}}{8\pi^2}\Big(\frac{1}{d-4}+\frac{1}{2}(\gamma+\int_0^1\mathrm{d}z\ln\Big(\frac{|z(1-z)s-zm_1^2-(1-z)m_2^2|}{4\pi\mu^2}\Big))\Big) \tag{52}$$

and

$$-i\lambda_3^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2-m_1^2+i\epsilon}\cdot\frac{i}{(p_1-p_4-k)^2-m_2^2+i\epsilon}$$
$$\underset{d=4}{\rightarrow}\frac{\lambda_3^2\mu^{4-d}}{8\pi^2}\Big(\frac{1}{d-4}+\frac{1}{2}(\gamma+\int_0^1\mathrm{d}z\ln\Big(\frac{|z(1-z)u-zm_1^2-(1-z)m_2^2|}{4\pi\mu^2}\Big))\Big). \tag{53}$$

5.2. One loop renormalization

5.2.1. Lagrangian density with counterterms and new Feynman rules

The first step in the renormalization program is to rewrite the original Lagrangian density (1) by splitting it into two parts with identical structure

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 \\ &\quad - \frac{1}{2}m_1^2\phi_1^2 - \frac{1}{2}m_2^2\phi_2^2 - \frac{\lambda_1}{4!}\phi_1^4 - \frac{\lambda_2}{4!}\phi_2^4 - \frac{\lambda_3}{4}\phi_1^2\phi_2^2 \\ &= \frac{1}{2}\partial_\mu\varphi_1\partial^\mu\varphi_1 + \frac{1}{2}\partial_\mu\varphi_2\partial^\mu\varphi_2 + \frac{\delta_{Z_1}}{2}\partial_\mu\varphi_1\partial^\mu\varphi_1 + \frac{\delta_{Z_2}}{2}\partial_\mu\varphi_2\partial^\mu\varphi_2 \\ &\quad - \frac{1}{2}M_1^2\varphi_1^2 - \frac{1}{2}M_2^2\varphi_2^2 - \frac{\Lambda_1}{4!}\varphi_1^4 - \frac{\Lambda_2}{4!}\varphi_2^4 - \frac{\Lambda_3}{4}\varphi_1^2\varphi_2^2 \\ &\quad - \frac{\delta_{M_1}}{2}\varphi_1^2 - \frac{\delta_{M_2}}{2}\varphi_2^2 - \frac{\delta_{\Lambda_1}}{4!}\varphi_1^4 - \frac{\delta_{\Lambda_2}}{4!}\varphi_2^4 - \frac{\delta_{\Lambda_3}}{4}\varphi_1^2\varphi_2^2.\end{aligned} \tag{54}$$

Here M_1, M_2, Λ_1, Λ_2 and Λ_3 are new parameters of the Lagrangian density and the terms containing δ_{Z_1}, δ_{Z_2}, δm_1, δm_2, $\delta\lambda_1$, $\delta\lambda_2$ and $\delta\lambda_3$ are called *counterterms*. The fields φ_1 and φ_2 have different normalization than the fields ϕ_1 and ϕ_2. The parameters m_1, m_2, λ_1, λ_2 and λ_3 of the original Lagrangian density (1) are called *bare* parameters and the following relations hold

$$\phi_1 = \sqrt{Z_{\varphi_1}}\varphi_1, \quad \phi_2 = \sqrt{Z_{\varphi_2}}\varphi_2, \quad Z_{\phi_1} = 1 + \delta_{Z_1}, \quad Z_{\phi_2} = 1 + \delta_{Z_2},$$
$$m_1^2 = \frac{M_1^2 + \delta_{M_1}}{Z_{\varphi_1}}, \quad m_2^2 = \frac{M_2^2 + \delta_{M_2}}{Z_{\varphi_2}}, \tag{55}$$
$$\lambda_1 = \frac{\Lambda_1 + \delta_{\Lambda_1}}{Z_{\varphi_1}^2}, \quad \lambda_2 = \frac{\Lambda_2 + \delta_{\Lambda_2}}{Z_{\varphi_2}^2}, \quad \lambda_3 = \frac{\Lambda_3 + \delta_{\Lambda_3}}{Z_{\varphi_1} Z_{\varphi_2}}.$$

Splitting of the Lagrangian density, Eq. (54) introduces new rules for the Feynman diagrams, which are shown in Table 2, which have to be used in calculation of the diagrams.

Before proceeding we have to specify, what we mean by the *physical* coupling constant. From Subsection 5.1.2, we know that at the order of one loop the coupling constants do depend on the Mandelstam variables s, t and u, so the definition of the *physical* coupling constant has to be taken at certain values of these variables. In a determination of the *physical* coupling constant we must use the Feynman rules from Table 2. A possible choice of the Mandelstam variables is

$$\begin{aligned}
&\text{for } \Lambda_1\text{:} \quad s = 4M_1^2, \qquad t = 0, \quad u = 0, \\
&\text{for } \Lambda_2\text{:} \quad s = 4M_2^2, \qquad t = 0, \quad u = 0, \\
&\text{for } \Lambda_3\text{:} \quad s = 4M_1 M_2, \quad t = 0, \quad u = 0.
\end{aligned} \tag{56}$$

We will consider other choices when we discuss the renormalization group equations.

5.2.2. Renormalization of the coupling constants

Using Eq. (48), prescription (56) and the Feynman rules from Table 2 we get for the Λ_1 four point Green's function, which should be finite at the one loop level

$$\Lambda_1 \mu^{4-d} + \frac{\Lambda_1^2 \mu^{4-d}}{16\pi^2}\Big(\frac{3}{d-4} + \frac{3}{2}\gamma + \frac{1}{2}\big(F_\Sigma(4M_1^2, 0, 0, M_1^2, \mu^2)\big)\Big) + \frac{\Lambda_3^2 \mu^{4-d}}{16\pi^2}\Big(\frac{3}{d-4} + \frac{3}{2}\gamma + \frac{1}{2}\big(F_\Sigma(4M_1^2, 0, 0, M_2^2, \mu^2)\big)\Big) + \delta_{\Lambda_1}, \quad (57)$$

so in order to cancel the infinite terms in Eq. (57) we choose δ_{Λ_1} to be equal

$$\delta_{\Lambda_1} = -\frac{\Lambda_1^2 \mu^{4-d}}{16\pi^2}\Big(\frac{3}{d-4} + \frac{3}{2}\gamma + \frac{1}{2}\big(F_\Sigma(4M_1^2, 0, 0, M_1^2, \mu^2)\big)\Big) - \frac{\Lambda_3^2 \mu^{4-d}}{16\pi^2}\Big(\frac{3}{d-4} + \frac{3}{2}\gamma + \frac{1}{2}\big(F_\Sigma(4M_1^2, 0, 0, M_2^2, \mu^2)\big)\Big). \quad (58)$$

Analogously we obtain

$$\delta_{\Lambda_2} = -\frac{\Lambda_2^2 \mu^{4-d}}{16\pi^2}\Big(\frac{3}{d-4} + \frac{3}{2}\gamma + \frac{1}{2}\big(F_\Sigma(4M_2^2, 0, 0, M_2^2, \mu^2)\big)\Big) - \frac{\Lambda_3^2 \mu^{4-d}}{16\pi^2}\Big(\frac{3}{d-4} + \frac{3}{2}\gamma + \frac{1}{2}\big(F_\Sigma(4M_2^2, 0, 0, M_1^2, \mu^2)\big)\Big). \quad (59)$$

For Λ_3 four point Green's function we have

$$\Lambda_3\mu^{4-d} + \frac{\Lambda_1\Lambda_3\mu^{4-d}}{16\pi^2}\Big(\frac{1}{d-4} + \frac{1}{2}(\gamma + F(4M_1M_2, M_1^2, \mu^2))\Big)$$
$$+ \frac{\Lambda_2\Lambda_3\mu^{4-d}}{16\pi^2}\Big(\frac{1}{d-4} + \frac{1}{2}(\gamma + F(4M_1M_2, M_2^2, \mu^2))\Big)$$
$$+ \frac{\Lambda_3^2\mu^{4-d}}{8\pi^2}\Big(\frac{1}{d-4} + \frac{1}{2}(\gamma + \int_0^1 \mathrm{d}z \ln\Big(\frac{|z(1-z)4M_1M_2 - zM_1^2 - (1-z)M_2^2|}{4\pi\mu^2}\Big))\Big)$$
$$+ \frac{\Lambda_3^2\mu^{4-d}}{8\pi^2}\Big(\frac{1}{d-4} + \frac{1}{2}(\gamma + \int_0^1 \mathrm{d}z \ln\Big(\frac{|zM_1^2 + (1-z)M_2^2|}{4\pi\mu^2}\Big))\Big) \quad (60)$$

and δ_{Λ_3} is equal

$$\delta_{\Lambda_3} = -\frac{\Lambda_1\Lambda_3\mu^{4-d}}{16\pi^2}\Big(\frac{1}{d-4} + \frac{1}{2}(\gamma + F(4M_1M_2, M_1^2, \mu^2))\Big)$$
$$- \frac{\Lambda_2\Lambda_3\mu^{4-d}}{16\pi^2}\Big(\frac{1}{d-4} + \frac{1}{2}(\gamma + F(4M_1M_2, M_2^2, \mu^2))\Big)$$
$$- \frac{\Lambda_3^2\mu^{4-d}}{8\pi^2}\Big(\frac{1}{d-4} + \frac{1}{2}(\gamma + \int_0^1 \mathrm{d}z \ln\Big(\frac{|z(1-z)4M_1M_2 - zM_1^2 - (1-z)M_2^2|}{4\pi\mu^2}\Big))\Big)$$
$$- \frac{\Lambda_3^2\mu^{4-d}}{8\pi^2}\Big(\frac{1}{d-4} + \frac{1}{2}(\gamma + \int_0^1 \mathrm{d}z \ln\Big(\frac{|zM_1^2 + (1-z)M_2^2|}{4\pi\mu^2}\Big))\Big). \quad (61)$$

Putting all previous results together and taking the limit $d \to 4$ we obtain the following result for the one loop amplitude of the process of elastic scattering of two particles of the type 1

$$\Lambda_1 + \frac{\Lambda_1^2}{32\pi^2}\big(F_\Sigma(s,t,u,M_1^2,\mu^2) - F_\Sigma(4M_1^2,0,0,M_1^2,\mu^2)\big) \\ + \frac{\Lambda_3^2}{32\pi^2}\big(F_\Sigma(s,t,u,M_2^2,\mu^2) - F_\Sigma(4M_1^2,0,0,M_2^2,\mu^2)\big). \quad (62)$$

The coupling constants Λ_1, Λ_2 and Λ_3 are chosen to be finite. From Eqs. (55) it then follows that the bare coupling constants λ_1, λ_2 and λ_3 must have a pole at $d = 4$.

The one loop amplitude for the process of elastic scattering of two particles of the type 2 is equal

$$\Lambda_2 + \frac{\Lambda_2^2}{32\pi^2}\big(F_\Sigma(s,t,u,M_2^2,\mu^2) - F_\Sigma(4M_2^2,0,0,M_2^2,\mu^2)\big) \\ + \frac{\Lambda_3^2}{32\pi^2}\big(F_\Sigma(s,t,u,M_1^2,\mu^2) - F_\Sigma(4M_2^2,0,0,M_1^2,\mu^2)\big) \quad (63)$$

and the one loop amplitude for the process of elastic scattering of two particles, one of type

1 and the other of type 2 is equal

$$\Lambda_3 + \frac{\Lambda_1\Lambda_3}{16\pi^2}(F(s,M_1^2,\mu^2) - F(4M_1M_2,M_1^2,\mu^2))$$
$$+ \frac{\Lambda_2\Lambda_3}{16\pi^2}(F(s,M_2^2,\mu^2) - F(4M_1M_2,M_2^2,\mu^2))$$
$$+ \frac{\Lambda_3^2}{16\pi^2}\Big(\int_0^1 \mathrm{d}z \ln\Big(\frac{|z(1-z)s - zM_1^2 - (1-z)M_2^2|}{|z(1-z)4M_1M_2 - zM_1^2 - (1-z)M_2^2|}\Big)\Big)$$
$$+ \frac{\Lambda_3^2}{16\pi^2}\Big(\int_0^1 \mathrm{d}z \ln\Big(\frac{|z(1-z)u - zM_1^2 - (1-z)M_2^2|}{|zM_1^2 + (1-z)M_2^2|}\Big)\Big). \quad (64)$$

5.2.3. Renormalization of the masses

The propagator of particle 1 is equal (at one loop $\delta_{Z_1} = 0$ and $\delta_{Z_2} = 0$)

$$\frac{i}{p^2 - M_1^2 - \delta_{M_1} - \Sigma_1 - \Sigma_3^1}. \quad (65)$$

Here we were using the Feynman rules given in Table 2 and Σ_1 an Σ_3^1 are equal

$$\Sigma_1 = \frac{\Lambda_1 M_1^2}{16\pi^2}\Big(\frac{1}{d-4} + \frac{1}{2}(\gamma - 1 + \ln\Big(\frac{M_1^2}{4\pi\mu^2}\Big))\Big), \quad (66a)$$
$$\Sigma_3^1 = \frac{\Lambda_3 M_2^2}{16\pi^2}\Big(\frac{1}{d-4} + \frac{1}{2}(\gamma - 1 + \ln\Big(\frac{M_2^2}{4\pi\mu^2}\Big))\Big). \quad (66b)$$

The propagator has a pole at the value of the physical mass, which is equal to M_1, so from Eq. (65) we obtain the value of δ_{M_1}

$$\delta_{M_1} = -\Sigma_1 - \Sigma_3^1 \tag{67}$$

and also for δ_{M_2}, after the interchange of the indices $1 \leftrightarrow 2$

$$\delta_{M_2} = -\Sigma_2 - \Sigma_3^2, \tag{68}$$

where Σ_2 and Σ_3^2 are equal

$$\Sigma_2 = \frac{\Lambda_2 M_2^2}{16\pi^2}\Big(\frac{1}{d-4} + \frac{1}{2}\big(\gamma - 1 + \ln\Big(\frac{M_2^2}{4\pi\mu^2}\Big)\big)\Big), \tag{69a}$$

$$\Sigma_3^2 = \frac{\Lambda_3 M_1^2}{16\pi^2}\Big(\frac{1}{d-4} + \frac{1}{2}\big(\gamma - 1 + \ln\Big(\frac{M_1^2}{4\pi\mu^2}\Big)\big)\Big). \tag{69b}$$

The corrections δ_{M_1} and δ_{M_2} have poles at $d = 4$, so from Eq. (55) it follows that the bare masses m_1^2 and m_2^2 must also have poles at $d = 4$.

One should mention here that at the next order of the perturbation theory and also at two loops, there is an additional type of the divergent diagram of the type shown in Fig. 11. This divergence is removed by the parameters $\delta_{Z_1} \neq 0$ and $\delta_{Z_2} \neq 0$ and the normalization of the fields ϕ_1 and ϕ_2 is modified. It will be discussed in Sec. 7.

6. One loop renormalization group equations

6.1. Derivation of equations

Renormalization group equations (RGE) in field theory [4,5] provide a method for the study of the asymptotic behavior of the Green's functions. The first step in the derivation of the RGEs is the demonstration that the Green's functions do not depend on the cut-off parameter μ and the determination of the equations for the Green's functions. From the split form of the Lagrangian density (54) it follows that we have two sets of the Green's functions: one set for the original Lagrangian density, dependent on the parameters m_1, m_2, λ_1, λ_2, λ_3 and the other set of Green's functions obtained from the Lagrangian density after splitting, dependent on M_1, M_2, Λ_1, Λ_2, Λ_3 and μ. These two sets of Green's functions describe the same theory, so they have to be equal and we have the following relations between these Green's functions in the space-time dimension d

$$G^{(n)}_{i_1 \dots i_n}(p_1, \dots p_n, \lambda_1, \lambda_2, \lambda_3, m_1, m_2, d) \\ = Z_{\phi_1}^{-n_1/2} Z_{\phi_2}^{-n_2/2} \mathbf{G}^{(n)}_{i_1 \dots i_n}(p_1, \dots, p_n, \Lambda_1, \Lambda_2, \Lambda_3, M_1, M_2, d, \mu). \quad (70)$$

Here the function $G^{(n)}_{i_1 \dots i_n}$ is the Green's function before the splitting of the Lagrangian density and $\mathbf{G}^{(n)}_{i_1 \dots i_n}$ is the Green's function after the splitting, which is finite. The n_1 and n_2 denote the number of fields in the Green's function of type 1 and type 2, respectively, $(n_1 + n_2 = n)$.

The left-hand side of Eq. (70) does not depend on μ, so the right-hand side cannot depend on μ either. If we differentiate Eq. (70) with respect to μ then we obtain

$$\Bigg[\mu\frac{\partial}{\partial\mu}+\mu\frac{\partial\Lambda_1}{\partial\mu}\frac{\partial}{\partial\Lambda_1}+\mu\frac{\partial\Lambda_2}{\partial\mu}\frac{\partial}{\partial\Lambda_2}+\mu\frac{\partial\Lambda_3}{\partial\mu}\frac{\partial}{\partial\Lambda_3}$$
$$+\mu\frac{\partial M_1}{\partial\mu}\frac{\partial}{\partial M_1}+\mu\frac{\partial M_2}{\partial\mu}\frac{\partial}{\partial M_2}-\mu\frac{n_1}{2}\frac{\partial\ln Z_{\varphi_1}}{\partial\mu}$$
$$-\mu\frac{n_2}{2}\frac{\partial\ln Z_{\varphi_2}}{\partial\mu}\Bigg]\mathbf{G}^{(n)}_{i_1\dots i_n}(p_1,\dots,p_n,\Lambda_1,\Lambda_2,\Lambda_3,M_1,M_2,d,\mu)=0. \quad (71)$$

Eq. (71) is the condition that the Green's functions have to fulfill that the physical predictions of the theory do not depend on the choice of the renormalization point μ. Conventionally one introduces the notation

$$\begin{aligned}
\beta_{\Lambda_i}(\Lambda_1,\Lambda_2,\Lambda_3,\frac{M_1}{\mu},\frac{M_2}{\mu},d)&=\mu\frac{\partial\Lambda_i}{\partial\mu}, && i=1,2,3,\\
\gamma_{d_i}(\Lambda_1,\Lambda_2,\Lambda_3,\frac{M_1}{\mu},\frac{M_2}{\mu},d)&=\frac{\mu}{2}\frac{\partial\ln Z_{\varphi_i}}{\partial\mu}, && i=1,2,\\
\gamma_{M_i}(\Lambda_1,\Lambda_2,\Lambda_3,\frac{M_1}{\mu},\frac{M_2}{\mu},d)&=\frac{\mu}{2}\frac{\partial\ln M_i^2}{\partial\mu}, && i=1,2.
\end{aligned} \quad (72)$$

Next, one considers the Green's function with the momenta scaled by the factor t: $\mathbf{G}^{(n)}_{i_1\dots i_n}(tp_1,\dots,tp_n,\Lambda_1,\Lambda_2,\Lambda_3,M_1,M_2,d,\mu)$, which has the dimension $4-n+\frac{(4-d)(n-2)}{2}$

and fulfills the following scaling equation

$$\left[\mu\frac{\partial}{\partial\mu}+t\frac{\partial}{\partial t}+M_1\frac{\partial}{\partial M_1}+M_2\frac{\partial}{\partial M_2}-\left(4-n+\frac{(4-d)(n-2)}{2}\right)\right]$$
$$\mathbf{G}^{(n)}_{i_1\dots i_n}(tp_1,\dots,tp_n,\Lambda_1,\Lambda_2,\Lambda_3,M_1,M_2,d,\mu)=0. \quad (73)$$

After subtracting Eq. (73) from Eq. (71) and taking the limit $d\to 4$ we obtain an equation that describes the scaling properties of the Green's functions

$$\Bigg[-t\frac{\partial}{\partial t}+\beta_{\Lambda_1}\frac{\partial}{\partial\Lambda_1}+\beta_{\Lambda_2}\frac{\partial}{\partial\Lambda_2}+\beta_{\Lambda_3}\frac{\partial}{\partial\Lambda_3}+\Big(\gamma_{M_1}-1\Big)M_1\frac{\partial}{\partial M_1}$$
$$+\Big(\gamma_{M_2}-1\Big)M_2\frac{\partial}{\partial M_2}-n_1\gamma_{d_1}-n_2\gamma_{d_2}+4-n\Bigg]$$
$$\mathbf{G}^{(n)}_{i_1\dots i_n}(tp_1,\dots,tp_n,\Lambda_1,\Lambda_2,\Lambda_3,M_1,M_2,4,\mu)=0. \quad (74)$$

The solution of Eq. (74) can be expressed in the following way

$$\mathbf{G}^{(n)}_{i_1\dots i_n}(tp_1,\dots,tp_n,\Lambda_1,\Lambda_2,\Lambda_3,M_1,M_2,4,\mu)$$
$$=t^{4-n}\exp\Big[-n_1\int_1^t\frac{\gamma_{M_1}(\tau)\mathrm{d}\tau}{\tau}\Big]\exp\Big[-n_2\int_1^t\frac{\gamma_{M_2}(\tau)\mathrm{d}\tau}{\tau}\Big]$$
$$\times\,\mathbf{G}^{(n)}_{i_1\dots i_n}(p_1,\dots,p_n,\Lambda_1(t),\Lambda_2(t),\Lambda_3(t),M_1(t),M_2(t),4,\mu). \quad (75)$$

Here $\Lambda_1(t)$, $\Lambda_2(t)$, $\Lambda_3(t)$, $M_1(t)$, $M_2(t)$ are the *running* coupling constants and masses that fulfill the following equations and initial conditions

$$t\frac{\mathrm{d}\Lambda_i(t)}{\mathrm{d}t} = \beta_{\Lambda_i}(\Lambda_1(t), \Lambda_2(t), \Lambda_3(t), M_1(t), M_2(t)), \quad i = 1, 2, 3, \tag{76a}$$

$$t\frac{\mathrm{d}M_i(t)}{\mathrm{d}t} = M_i(t)(\gamma_{M_i}(\Lambda_1(t), \Lambda_2(t), \Lambda_3(t), M_1(t), M_2(t)) - 1),\ i = 1, 2. \tag{76b}$$

$$\Lambda_1(0) = \Lambda_1,\ \Lambda_2(0) = \Lambda_2,\ \Lambda_3(0) = \Lambda_3,\ M_1(0) = M_1,\ M_2(0) = M_2. \tag{76c}$$

Eq. (75) is the key relation of the renormalization group method: it relates the Green's functions at scaled momenta and unscaled coupling constants with Green's functions at unscaled momenta and scaled coupling constants. Eqs. (76) are called the *renormalization group* equations for the *running* coupling constants and masses. The solutions of these equations, with the help of Eq. (75), give the information about the asymptotic behavior of the theory. The right-hand side of Eqs. (76), the functions $\beta_{\Lambda_i}(\Lambda_1(t), \Lambda_2(t), \Lambda_3(t), M_1(t), M_2(t))$ and $\gamma_{M_i}(\Lambda_1(t), \Lambda_2(t), \Lambda_3(t), M_1(t), M_2(t))$, can only be determined perturbatively and we will find now their form at order of one loop.

6.2. Calculation of the β functions

The definition of the β and γ functions is given in Eqs. (72), which show that in general they depend on the coupling constants Λ_i and the masses M_i. The β_{Λ_i} functions contain the derivatives of Λ_i with respect to μ, the mass parameter in the dimensional renormalization. The calculation of this derivative is not a straightforward matter, because we do not know the dependence of Λ_i on μ, but we know the dependence of the *bare* coupling constants λ_i as functions of the renormalized coupling constants Λ_i.

Let us start with the determination of the β functions, which are analytic at $d = 4$. It means that the expansion around point $d = 4$ has the form

$$\beta_{\Lambda_i}(\Lambda_1, \Lambda_2, \Lambda_3, d, \mu) = \sum_{l=0}^{\infty} b_l^{\Lambda_i}(\Lambda_1, \Lambda_2, \Lambda_3, \mu)(d-4)^l \tag{77}$$

and at $d = 4$

$$\mu \frac{\partial \Lambda_i}{\partial \mu}\bigg|_{d=4} = \beta_{\Lambda_i}(\Lambda_1, \Lambda_2, \Lambda_3, 4, \mu) = b_0^{\Lambda_i}, \tag{78}$$

so we have to calculate $b_0^{\Lambda_i}$.

The unrenormalized coupling constant λ_1 which is independent of the cut off μ has a Laurent expansion around point $d = 4$

$$\lambda_1 = \frac{\Lambda_1 + \delta_{\lambda_1}}{Z_{\varphi_1}^2} = \mu^{4-d}\left[\Lambda_1 + \sum_{l=1}^{\infty} \frac{a_l^{\Lambda_1}(\Lambda_1, \Lambda_2, \Lambda_3)}{(d-4)^l}\right]. \tag{79}$$

The right hand side of Eq. (79) depends on μ explicitly through the factor μ^{4-d} and implicitly through the coupling constants Λ_i. If we differentiate Eq. (79) with respect to μ we obtain

$$
\begin{aligned}
0 = \mu\frac{\partial\lambda_1}{\partial\mu} &= (4-d)\mu^{4-d}\left[\Lambda_1 + \sum_{l=1}^{\infty}\frac{a_l^{\Lambda_1}}{(d-4)^l}\right] \\
&+ \mu^{4-d}\mu\frac{\partial\Lambda_1}{\partial\mu}\left[1 + \sum_{l=1}^{\infty}\frac{\partial a_l^{\Lambda_1}}{\partial\Lambda_1}\frac{1}{(d-4)^l}\right] + \mu^{4-d}\mu\frac{\partial\Lambda_2}{\partial\mu}\sum_{l=1}^{\infty}\frac{\partial a_l^{\Lambda_1}}{\partial\Lambda_2}\frac{1}{(d-4)^l} \\
&+ \mu^{4-d}\mu\frac{\partial\Lambda_3}{\partial\mu}\sum_{l=1}^{\infty}\frac{\partial a_l^{\Lambda_1}}{\partial\Lambda_3}\frac{1}{(d-4)^l} = (4-d)\mu^{4-d}\left[\Lambda_1 + \sum_{l=1}^{\infty}\frac{a_l^{\Lambda_1}}{(d-4)^l}\right] \\
&+ \mu^{4-d}\beta_{\Lambda_1}\left[1 + \sum_{l=1}^{\infty}\frac{\partial a_l^{\Lambda_1}}{\partial\Lambda_1}\frac{1}{(d-4)^l}\right] + \mu^{4-d}\beta_{\Lambda_2}\sum_{l=1}^{\infty}\frac{\partial a_l^{\Lambda_1}}{\partial\Lambda_2}\frac{1}{(d-4)^l} \\
&+ \mu^{4-d}\beta_{\Lambda_3}\sum_{l=1}^{\infty}\frac{\partial a_l^{\Lambda_1}}{\partial\Lambda_3}\frac{1}{(d-4)^l}.
\end{aligned}
\tag{80}
$$

The coefficients at various powers of $(d-4)$ in Eq. (80) are equal

$$-a_1^{\Lambda_1} + b_0^{\Lambda_1} + \sum_{l=1}^{\infty}\left(b_l^{\Lambda_1}\frac{\partial a_l^{\Lambda_1}}{\partial \Lambda_1} + b_l^{\Lambda_2}\frac{\partial a_l^{\Lambda_1}}{\partial \Lambda_2} + b_l^{\Lambda_3}\frac{\partial a_l^{\Lambda_1}}{\partial \Lambda_3}\right) = 0 \quad \text{at } (d-4)^0 \tag{81a}$$

$$-\Lambda_1 + b_1^{\Lambda_1} + \sum_{l=1}^{\infty}\left(b_{l+1}^{\Lambda_1}\frac{\partial a_l^{\Lambda_1}}{\partial \Lambda_1} + b_{l+1}^{\Lambda_2}\frac{\partial a_l^{\Lambda_1}}{\partial \Lambda_2} + b_{l+1}^{\Lambda_3}\frac{\partial a_l^{\Lambda_1}}{\partial \Lambda_3}\right) = 0 \quad \text{at } (d-4)^1 \tag{81b}$$

$$b_k^{\Lambda_1} + \sum_{l=1}^{\infty}\left(b_{l+k}^{\Lambda_1}\frac{\partial a_l^{\Lambda_1}}{\partial \Lambda_1} + b_{l+k}^{\Lambda_2}\frac{\partial a_l^{\Lambda_1}}{\partial \Lambda_2} + b_{l+k}^{\Lambda_3}\frac{\partial a_l^{\Lambda_1}}{\partial \Lambda_3}\right) = 0 \quad \text{at } (d-4)^k,\ k \geq 2 \tag{81c}$$

$$-a_{k+1} + \sum_{l=1}^{\infty}\left(b_l^{\Lambda_1}\frac{\partial a_{l+k}^{\Lambda_1}}{\partial \Lambda_1} + b_l^{\Lambda_2}\frac{\partial a_{l+k}^{\Lambda_1}}{\partial \Lambda_2} + b_l^{\Lambda_3}\frac{\partial a_{l+k}^{\Lambda_1}}{\partial \Lambda_3}\right) = 0 \quad \text{at } (d-4)^{-k},\ k \geq 1. \tag{81d}$$

Similar equations hold for $\mu\frac{\partial \lambda_2}{\partial \mu}$ and $\mu\frac{\partial \lambda_3}{\partial \mu}$. Eqs. (81) are the recursive equations for the coefficients $b_k^{\Lambda_1}$ of expansion of the beta functions in terms of the coefficients $a_k^{\Lambda_i}$ that are calculated in the renormalization process. From Eqs. (81c), which are linear homogeneous equations it follows that

$$b_k^{\Lambda_i} = 0 \quad \text{for } k \geq 2 \text{ and } i = 1, 2, 3. \tag{82}$$

It means that we have the following equations for the coefficients $b_0^{\Lambda_i}$ and $b_1^{\Lambda_i}$

$$\left.\begin{aligned} -a_1^{\Lambda_i} + b_0^{\Lambda_i} + \left(b_1^{\Lambda_1}\frac{\partial a_1^{\Lambda_i}}{\partial \Lambda_1} + b_1^{\Lambda_2}\frac{\partial a_1^{\Lambda_i}}{\partial \Lambda_2} + b_1^{\Lambda_3}\frac{\partial a_1^{\Lambda_i}}{\partial \Lambda_3}\right) = 0 \\ -\Lambda_i + b_1^{\Lambda_i} = 0 \end{aligned}\right\} \quad i = 1, 2, 3, \tag{83}$$

which immediately give

$$\beta_{\Lambda_i} = b_0^{\Lambda_i} = a_1^{\Lambda_i} - \Big(\Lambda_1 \frac{\partial a_1^{\Lambda_i}}{\partial \Lambda_1} + \Lambda_2 \frac{\partial a_1^{\Lambda_i}}{\partial \Lambda_2} + \Lambda_3 \frac{\partial a_1^{\Lambda_i}}{\partial \Lambda_3}\Big). \tag{84}$$

Let us calculate now the functions β_{Λ_i} at one loop. From Eqs. (55) and (58) we have ($Z_{\phi_i} = 1$ at one loop)

$$\lambda_1 = \Lambda_1 + \delta_{\lambda_1} = \Lambda_1 - \frac{\Lambda_1^2 \mu^{4-d}}{16\pi^2}\Big(\frac{3}{d-4} + \frac{3}{2}\gamma + \frac{1}{2}\big(F_\Sigma(4M_1^2, 0, 0, M_1^2, \mu^2)\big)\Big) \\ - \frac{\Lambda_3^2 \mu^{4-d}}{16\pi^2}\Big(\frac{3}{d-4} + \frac{3}{2}\gamma + \frac{1}{2}\big(F_\Sigma(4M_1^2, 0, 0, M_2^2, \mu^2)\big)\Big) \tag{85}$$

and we see that $a_1^{\Lambda_1}$ is equal

$$a_1^{\Lambda_1} = -\frac{3(\Lambda_1^2 + \Lambda_3^2)}{16\pi^2}, \tag{86}$$

so β_{Λ_1} is equal

$$\beta_{\Lambda_1} = \frac{3(\Lambda_1^2 + \Lambda_3^2)}{16\pi^2}. \tag{87}$$

β_{Λ_2} is obtained by the exchange of the indices

$$\beta_{\Lambda_2} = \frac{3(\Lambda_2^2 + \Lambda_3^2)}{16\pi^2}. \tag{88}$$

From Eq. (61) we obtain the value of $a_1^{\Lambda_3}$

$$a_1^{\Lambda_3} = -\frac{(\Lambda_1 + \Lambda_2)\Lambda_3}{16\pi^2} - \frac{\Lambda_3^2}{4\pi^2} \tag{89}$$

and β_{Λ_3}

$$\beta_{\Lambda_3} = \frac{(\Lambda_1 + \Lambda_2)\Lambda_3}{16\pi^2} + \frac{\Lambda_3^2}{4\pi^2}. \tag{90}$$

The one loop renormalization group equations have thus the form

$$t\frac{\partial \Lambda_1}{\partial t} = \frac{3(\Lambda_1^2 + \Lambda_3^2)}{16\pi^2}, \tag{91a}$$

$$t\frac{\partial \Lambda_2}{\partial t} = \frac{3(\Lambda_2^2 + \Lambda_3^2)}{16\pi^2}, \tag{91b}$$

$$t\frac{\partial \Lambda_3}{\partial t} = \frac{(\Lambda_1 + \Lambda_2)\Lambda_3}{16\pi^2} + \frac{\Lambda_3^2}{4\pi^2}. \tag{91c}$$

We will now find the renormalization group equations for the masses M_i. The procedure is similar to the derivation of the renormalization group equations for the coupling constant. The functions γ_{M_i} from Eq. (72) are analytic at $d = 4$ and have the expansion

$$\gamma_{M_i}(\Lambda_1, \Lambda_2, \Lambda_3, d, \mu) = \sum_{l=0}^{\infty} g_l^{M_i}(\Lambda_1, \Lambda_2, \Lambda_3, \mu)(d-4)^l \tag{92}$$

and at $d = 4$

$$\frac{\mu}{2}\frac{\partial M_i^2}{\partial \mu}\bigg|_{d=4} = \gamma_{M_i}(\Lambda_1, \Lambda_2, \Lambda_3, 4, \mu) = g_0^{M_i}. \tag{93}$$

We also have the relation between m_i^2 and M_i^2

$$m_i^2 = M_1^2 \sum_{l=0}^{\infty} \frac{b_l^{M_i}(\Lambda_1, \Lambda_2, \Lambda_3)}{(d-4)^l} + M_2^2 \sum_{l=0}^{\infty} \frac{c_l^{M_i}(\Lambda_1, \Lambda_2, \Lambda_3)}{(d-4)^l}, \tag{94}$$

$$b_0^{M_1} = 1, \quad b_0^{M_2} = 0, \quad c_0^{M_1} = 0, \quad c_0^{M_2} = 1.$$

At one loop Eq. (94) for m_1^2 reads

$$m_1^2 = M_1^2 = \frac{M_1^2 + \delta_{m_1}}{Z_{\phi_1}} = M_1^2 - \frac{\Lambda_1 M_1^2 + \Lambda_3 M_2^2}{16\pi^2(d-4)}. \tag{95}$$

The bare masses m_i do not depend on μ, so differentiating Eq. (94) with respect to μ we obtain

$$0 = \mu\frac{\partial m_1^2}{\partial \mu} = \mu\frac{\partial M_1^2}{\partial \mu} - \frac{1}{16\pi^2(d-4)}\Big(M_1^2\mu\frac{\partial \Lambda_1}{\partial \mu} + M_2^2\mu\frac{\partial \Lambda_3}{\partial \mu}\Big). \tag{96}$$

The derivatives $\mu(\partial\Lambda_i)/(\partial\mu)$ at order $(d-4)$ are equal

$$\mu\frac{\partial \Lambda_i}{\partial \mu} = (d-4)\Lambda_i, \tag{97}$$

so we obtain

$$\mu\frac{\partial M_1^2}{\partial\mu} = \frac{\Lambda_1 M_1^2 + \Lambda_3 M_2^2}{16\pi^2}. \tag{98}$$

The functions γ_{M_i} are thus equal

$$\gamma_{M_1} = \frac{\Lambda_1 M_1^2 + \Lambda_3 M_2^2}{32\pi^2 M_1^2}, \quad \gamma_{M_2} = \frac{\Lambda_2 M_2^2 + \Lambda_3 M_1^2}{32\pi^2 M_2^2} \tag{99}$$

and the one loop renormalization group equations for the masses are equal

$$t\frac{\partial M_1^2}{\partial t} = \frac{\Lambda_1 M_1^2 + \Lambda_3 M_2^2}{32\pi^2} \tag{100a}$$

$$t\frac{\partial M_2^2}{\partial t} = \frac{\Lambda_2 M_2^2 + \Lambda_3 M_1^2}{32\pi^2}. \tag{100b}$$

7. Renormalization at two loops

The complexity of calculations in field theory is rapidly increasing with the order of the perturbation theory and the number of loops. The reason is that the number of the Feynman diagrams grows very fast at each order and its structure becomes more involved. To overcome these difficulties there have been developed various methods to generate the diagrams and the corresponding analytical expressions together with the explicit calculation of the integrals. In this paper we will consider the *FeynArts* [6, 7], which is the *Mathematica* package for the determination of the Feynman diagrams of the QED and *Standard Model* and some of

Generation of the model files scalars2.gen and scalars2.mod

```
FR$Parallel = False;
$FeynRulesPath = SetDirectory[$UserBaseDirectory <> "/Applications/FeynRules"];
<< FeynRules`;
SetDirectory[NotebookDirectory[]];
LoadModel["scalars2.fr"];
WriteFeynArtsOutput[L, Output → "scalars2"];
```

Figure 12: Listing of the *Mathematica* program for generation of the *driver* files for the *FeynArts* package.

```
t4 = CreateTopologies[2, 1 → 1, ExcludeTopologies → Reducible, Adjacencies → 4];
Paint[t4, ColumnsXRows → {2, 1}, Numbering → None, SheetHeader → None, ImageSize → 384];
```

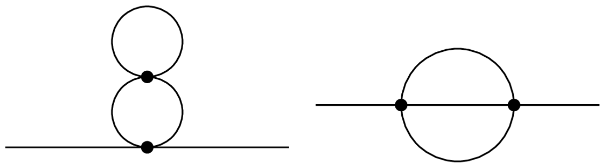


Figure 13: Command to generate and draw the two loop topology diagrams, using the *FeynArts* package of *Mathematica*. The full program is given in Appendix **??**.

```
f5 = InsertFields[t4, S[1] → S[1], GenericModel → scalars2,
    Model → scalars2, InsertionLevel → {Particles}];
Paint[f5, ColumnsXRows → {3, 2}, Numbering → None, SheetHeader → None, ImageSize → 512];
```

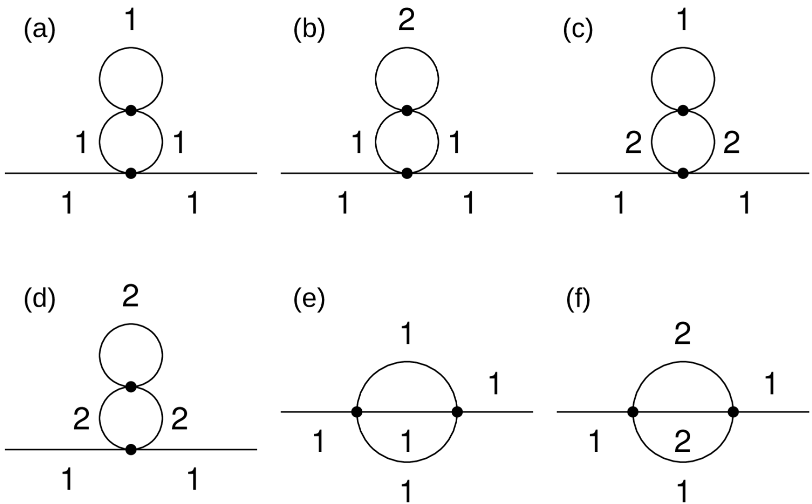


Figure 14: Feynman diagrams at two loops for the propagator of the particle 1. The full program is given in Appendix **??**. The labels (a)–(f) were added after evaluation by *Mathematica*.

```
t5 = CreateCTTopologies[2, 1 → 1, ExcludeTopologies → Reducible, Adjacencies → 4];
f6 = InsertFields[t5, S[1] → S[1], GenericModel → scalars2,
   Model → scalars2, InsertionLevel → {Particles}];
Paint[f6, ColumnsXRows → {4, 1}, Numbering → None, SheetHeader → None, ImageSize → 640];
```

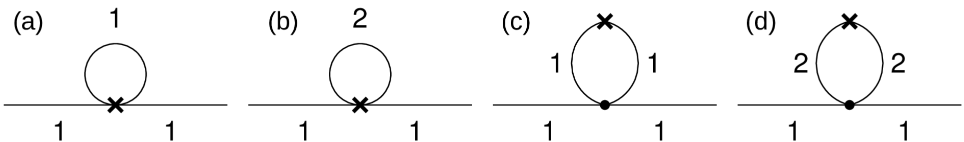


Figure 15: Counter term Feynman diagrams at two loops for the propagator of the particle 1. The labels (a)–(d) were added after evaluation by *Mathematica*. The full program is given in Appendix **??**.

its extensions. For each of the considered models there is a special *driver* file (with the extension ".mod"), which contains the information about the Feynman rules. The package also contains a *generic* file, with the information about the kinematic (Lorentz) structure of the fields and of the vertices. The model of two interacting scalar fields is not included in the *FeynArts*, so we have prepared the corresponding *driver* file, using the *FeynRules* [8–10], which is another *Mathematica* package dedicated to the generation of such *driver* files directly from the Lagrangian density. The prepared *FeynRules* file, called `scalar2.fr` is given in Appendix **??**. The listing of the *Mathematica* program to generate the files `scalars.gen` and `scalars.mod` is given in Fig. 12 and the listings of the files `scalars2.gen` and `scalars2.mod` thus obtained are given in Appendix **??**.

With the help of the *driver* files for our model we can generate the Feynman diagrams. For this purpose we use the *Mathematica* program which is given in Appendix **??**. First, we execute the command, which generates the topology diagrams for two loop propagator, which are shown in the listing shown in Fig. 13. The next step is to attach particles and vertices to the topology diagrams. This is shown in Fig. 14. We see that at two loops we have 6 diagrams for the propagator of particle 1.

The analytical expressions for the two loop diagrams (a)–(f) are

$$i(-i\Lambda_1)^2 \frac{1}{4} \int \frac{\mathrm{d}^4 k_1}{(2\pi)^4} \frac{i}{k_1^2 - M_1^2 + i\epsilon} \int \frac{\mathrm{d}^4 k_2}{(2\pi)^4} \frac{(i)^2}{(k_2^2 - M_1^2 + i\epsilon)^2}, \tag{101a}$$

$$i(-i\Lambda_1)(-i\Lambda_3) \frac{1}{4} \int \frac{\mathrm{d}^4 k_1}{(2\pi)^4} \frac{i}{k_1^2 - M_2^2 + i\epsilon} \int \frac{\mathrm{d}^4 k_2}{(2\pi)^4} \frac{(i)^2}{(k_2^2 - M_1^2 + i\epsilon)^2}, \tag{101b}$$

$$i(-i\Lambda_3)^2 \frac{1}{4} \int \frac{\mathrm{d}^4 k_1}{(2\pi)^4} \frac{i}{k_1^2 - M_1^2 + i\epsilon} \int \frac{\mathrm{d}^4 k_2}{(2\pi)^4} \frac{(i)^2}{(k_2^2 - M_2^2 + i\epsilon)^2}, \tag{101c}$$

$$i(-i\Lambda_2)(-i\Lambda_3) \frac{1}{4} \int \frac{\mathrm{d}^4 k_1}{(2\pi)^4} \frac{i}{k_1^2 - M_2^2 + i\epsilon} \int \frac{\mathrm{d}^4 k_2}{(2\pi)^4} \frac{(i)^2}{(k_2^2 - M_2^2 + i\epsilon)^2}, \tag{101d}$$

$$i(-i\Lambda_1)^2 \frac{1}{6} \int \frac{\mathrm{d}^4 k_1}{(2\pi)^4} \int \frac{\mathrm{d}^4 k_2}{(2\pi)^4} \frac{i}{k_1^2 - M_1^2 + i\epsilon} \frac{i}{k_2^2 - M_1^2 + i\epsilon} \times \frac{i}{(p - k_1 + k_2)^2 - M_1^2 + i\epsilon}, \tag{101e}$$

$$i(-i\Lambda_3)^2 \frac{1}{4} \int \frac{\mathrm{d}^4 k_1}{(2\pi)^4} \int \frac{\mathrm{d}^4 k_2}{(2\pi)^4} \frac{i}{k_1^2 - M_2^2 + i\epsilon} \frac{i}{k_2^2 - M_2^2 + i\epsilon} \times \frac{i}{(p - k_1 + k_2)^2 - M_1^2 + i\epsilon}. \tag{101f}$$

The integrals in Eqs. (101) are divergent, so their singularities have to be removed by regularization and renormalization.

For a complete renormalization program we also need the Feynman diagrams with counter terms, which are shown in Fig. 15. The analytical expressions corresponding to the diagrams in Fig. 15 are the following

$$i(-i\delta_{\Lambda_1})\frac{1}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2-M_1^2+i\epsilon}, \tag{102a}$$

$$i(-i\delta_{\Lambda_3})\frac{1}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2-M_2^2+i\epsilon}, \tag{102b}$$

$$i(-i\Lambda_1)(-i\delta_{M_1})\frac{1}{2}\int\frac{\mathrm{d}^4k_2}{(2\pi)^4}\frac{(i)^2}{(k_2^2-M_1^2+i\epsilon)^2}, \tag{102c}$$

$$i(-i\Lambda_3)(-i\delta_{M_2})\frac{1}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{(i)^2}{(k^2-M_2^2+i\epsilon)^2}. \tag{102d}$$

In our analysis we will limit ourselves only to the discussion of the divergent terms. The evaluation of the diagrams displayed in Fig. 14 (a)-(d) with the corresponding formulas given in Eqs. (101) (a)-(d) is simple and the result is the following

$$i(-i\Lambda_1\mu^{4-d})^2\frac{1}{4}i(-i)i\frac{\Gamma(1-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}\Gamma(1)}\frac{1}{(M_1^2)^{1-\frac{d}{2}}}\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}\Gamma(2)}\frac{1}{(M_1^2)^{2-\frac{d}{2}}}, \tag{103a}$$

$$i(-i\Lambda_1\mu^{4-d})(-i\Lambda_3\mu^{4-d})\frac{1}{4}i(-i)i\frac{\Gamma(1-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}\Gamma(1)}\frac{1}{(M_2^2)^{1-\frac{d}{2}}}\frac{\Gamma(1-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}\Gamma(1)}\frac{1}{(M_1^2)^{1-\frac{d}{2}}}, \tag{103b}$$

$$i(-i\Lambda_3\mu^{4-d})^2\frac{1}{4}i(-i)i\frac{\Gamma(1-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}\Gamma(1)}\frac{1}{(M_1^2)^{1-\frac{d}{2}}}\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}\Gamma(2)}\frac{1}{(M_2^2)^{2-\frac{d}{2}}}, \tag{103c}$$

$$i(-i\Lambda_2\mu^{4-d})(-i\Lambda_3\mu^{4-d})\frac{1}{4}i(-i)i\frac{\Gamma(1-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}\Gamma(1)}\frac{1}{(M_2^2)^{1-\frac{d}{2}}}\frac{\Gamma(1-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}\Gamma(1)}\frac{1}{(M_2^2)^{1-\frac{d}{2}}}. \tag{103d}$$

Next we calculate the diagrams corresponding to the couterterms given in Fig. 15 (a)-(d).

Using the formulas from Eqs. (102) we get

$$i(-\delta_{\Lambda_1})i(-i)\frac{1}{2}\frac{\Gamma(1-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}\Gamma(1)}\frac{1}{(M_1^2)^{1-\frac{d}{2}}}, \tag{104a}$$

$$i(-\delta_{\Lambda_2})i(-i)\frac{1}{2}\frac{\Gamma(1-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}\Gamma(1)}\frac{1}{(M_2^2)^{1-\frac{d}{2}}}, \tag{104b}$$

$$i(-i\Lambda_1)(-i\delta_{M_1})\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}\Gamma(2)}\frac{1}{(M_1^2)^{2-\frac{d}{2}}}, \tag{104c}$$

$$i(-i\Lambda_3)(-i\delta_{M_2})\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}\Gamma(2)}\frac{1}{(M_2^2)^{2-\frac{d}{2}}}. \tag{104d}$$

The following step is to find the sum of all diagrams from Figs. 14 (a)-(d) and 15 and calculate the divergent contributions for the spacetime dimension $d = 4$. It turns out that all divergent contributions cancel at this order [6].

We will now consider the remaining two *setting sun* diagrams in Figs. 14 (e) and (f). These diagrams are an example of the *overlapping divergencies* and such diagrams are more difficult to calculate [11, 12]. From Eqs. (101e) and (101f) one can see that they are quadratically divergent. The analytic structure of both equations is identical, but Eq. (101f) is more general, because it contains two different masses. Such a diagram is not present for the case of one scalar field. Another important property of these diagrams is their dependence on the external momentum p of the particle. The integral in Eq. (101f) converted to the Euclidean

[6] It should be noticed that such a cancellation does not occur for the case of only one scalar field

space in d dimensions has the form

$$I(p^2) = \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \int \frac{\mathrm{d}^d k_2}{(2\pi)^d} \frac{1}{k_1^2 + M_2^2} \frac{1}{k_2^2 + M_2^2} \frac{1}{(p - k_1 + k_2)^2 + M_1^2} \tag{105}$$

and it is the function of p^2, so its Taylor expansion in powers of p^2 is

$$I(p^2) = I(0) + p^2 \Big(\frac{p^\mu}{p^2} \frac{\partial I(p^2)}{\partial p^\mu} \Big) \Big|_{p^2=0} + \frac{1}{2!} p^4 \Big(\frac{p^\mu p^\nu}{p^4} \frac{\partial^2 I(p^2)}{\partial p^\mu \partial p^\nu} \Big) |_{p^2=0} + \cdots \tag{106}$$

The first term $I(0)$ is quadratically divergent and the next term, linear in p^2 is logarithmically divergent. The third term and all higher terms are convergent. The fact that the second term is divergent is important, because it has to be included in the renormalization procedure and it gives contribution to the term δ_{Z_1} in Eq. (54).

The integral for the first term in Eq. (106) is equal

$$I(0) = \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \int \frac{\mathrm{d}^d k_2}{(2\pi)^d} \frac{1}{k_1^2 + M_2^2} \frac{1}{k_2^2 + M_2^2} \frac{1}{(k_1 - k_2)^2 + M_1^2}. \tag{107}$$

The first step in the calculation of this integral is to apply the scheme of t'Hooft and Veltman [3] which consists in inserting the following expression into the integral

$$1 = \frac{1}{2d} \Big(\frac{\partial (k_1)_\mu}{\partial (k_1)_\mu} + \frac{\partial (k_2)_\mu}{\partial (k_2)_\mu} \Big)$$

and then integrating by parts. After such an operation one obtains

$$
\begin{aligned}
I(0) = -\frac{1}{2d}\Big(& -6I(0) \\
& + 2M_1^2 \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \int \frac{\mathrm{d}^d k_2}{(2\pi)^d} \frac{1}{k_1^2+M_2^2} \frac{1}{k_2^2+M_2^2} \frac{1}{((k_1-k_2)^2+M_1^2)^2} \\
& + 4M_2^2 \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \int \frac{\mathrm{d}^d k_2}{(2\pi)^d} \frac{1}{(k_1^2+M_2^2)^2} \frac{1}{k_2^2+M_2^2} \frac{1}{(k_1-k_2)^2+M_1^2} \Big)
\end{aligned} \tag{108}
$$

so $I(0)$ becomes

$$
\begin{aligned}
I(0) = & -\frac{1}{d-3} \\
& \times \Big(M_1^2 \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \int \frac{\mathrm{d}^d k_2}{(2\pi)^d} \frac{1}{k_1^2+M_2^2} \frac{1}{k_2^2+M_2^2} \frac{1}{((k_1-k_2)^2+M_1^2)^2} \\
& + 2M_2^2 \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \int \frac{\mathrm{d}^d k_2}{(2\pi)^d} \frac{1}{(k_1^2+M_2^2)^2} \frac{1}{k_2^2+M_2^2} \frac{1}{(k_1-k_2)^2+M_1^2} \Big).
\end{aligned} \tag{109}
$$

The second integral in Eq. (109) is calculated as follows

$$
\int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \int \frac{\mathrm{d}^d k_2}{(2\pi)^d} \frac{1}{(k_1^2+M_2^2)^2} \frac{1}{k_2^2+M_2^2} \frac{1}{(k_1-k_2)^2+M_1^2} \tag{110a}
$$

$$
= \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \int \frac{\mathrm{d}^d k_2}{(2\pi)^d} \int_0^1 \mathrm{d}x \frac{1}{(k_1^2+M_2^2)^2} \frac{1}{(x(k_2^2+M_2^2)+(1-x)((k_1-k_2)^2+M_1^2))^2} \tag{110b}
$$

$$= \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \int \frac{\mathrm{d}^d k_2}{(2\pi)^d} \int_0^1 \mathrm{d}x \frac{1}{(k_1^2 + M_2^2)^2}$$
$$\times \frac{1}{(k_2^2 - 2(1-x)k_1 k_2 + (1-x)k_1^2 + xM_2^2 + (1-x)M_1^2)^2} \tag{110c}$$
$$= \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \int \frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}} \int_0^1 \mathrm{d}x \frac{1}{(k_1^2 + M_2^2)^2} \frac{1}{(x(1-x)k_1^2 + xM_2^2 + (1-x)M_1^2)^{2-\frac{d}{2}}} \tag{110d}$$
$$= \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \int \frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}} \int_0^1 \mathrm{d}x \frac{1}{(k_1^2 + M_2^2)^2} \frac{(x(1-x))^{\frac{d}{2}-2}}{\left(k_1^2 + \frac{xM_2^2+(1-x)M_1^2}{x(1-x)}\right)^{2-\frac{d}{2}}} \tag{110e}$$
$$= \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \frac{\Gamma(4-\frac{2}{d})}{(4\pi)^{\frac{d}{2}}} \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \frac{(x(1-x))^{\frac{d}{2}-2} y^{1-\frac{d}{2}}(1-y)}{\left((1-y)(k_1^2+M_2^2) + y\left(k_1^2 + \frac{xM_2^2+(1-x)M_1^2}{x(1-x)}\right)\right)^{4-\frac{d}{2}}} \tag{110f}$$
$$= \frac{\Gamma(4-d)}{(4\pi)^d} \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \frac{(x(1-x))^{\frac{d}{2}-2} y^{1-\frac{d}{2}}(1-y)}{\left((1-y)M_2^2 + y\left(\frac{xM_2^2+(1-x)M_1^2}{x(1-x)}\right)\right)^{4-d}} \tag{110g}$$
$$= -\frac{\Gamma(4-d)}{(4\pi)^d(2-\frac{d}{2})} \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y (x(1-x))^{\frac{d}{2}-2} y^{2-\frac{d}{2}}$$
$$\times \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{(1-y)}{\left((1-y)M_2^2 + y\left(\frac{xM_2^2+(1-x)M_1^2}{x(1-x)}\right)\right)^{4-d}}\right). \tag{110h}$$

Let us explain each step in Eqs. (110)

Eq. (110a)→ (110b) Feynman prescription.

Eq. (110b)→ (110c) Transformation of the denominator under the integral.

Eq. (110c)→ (110d) Integration over k_2.

Eq. (110d)→ (110e) Transformation of the fraction under the integral.

Eq. (110e)→ (110f) Feynman prescription.

Eq. (110f)→ (110g) Integration over k_1.

Eq. (110g)→ (110h) Integration by parts, using the identity

$$y^{1-\frac{d}{2}} = \frac{1}{2-\frac{d}{2}} \frac{\mathrm{d}y^{2-\frac{d}{2}}}{\mathrm{d}y}.$$

The integrated function in Eq. (110h) has no singularity at $d = 4$ and it can be expanded in the Taylor series at $d = 4$. The first two terms of this expansion can be explicitly integrated and the final result for the integral (110h) at this order is

$$\int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \int \frac{\mathrm{d}^d k_2}{(2\pi)^d} \frac{1}{(k_1^2+M_2^2)^2} \frac{1}{k_2^2+M_2^2} \frac{1}{(k_1-k_2)^2+M_1^2} \\ = \frac{\Gamma(4-d)}{(4\pi)^d(2-\frac{d}{2})} \big(1+(d-4)\big(-\frac{1}{2}+\ln M_2^2\big)\big). \quad (111)$$

In analogy with the first integral in Eq. (109) we obtain

$$\int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \int \frac{\mathrm{d}^d k_2}{(2\pi)^d} \frac{1}{k_1^2+M_2^2} \frac{1}{k_2^2+M_2^2} \frac{1}{((k_1-k_2)^2+M_1^2)^2} \\ = \frac{\Gamma(4-d)}{(4\pi)^d(2-\frac{d}{2})} \big(1+(d-4)\big(-\frac{1}{2}+\ln M_1^2\big)\big). \quad (112)$$

We will now calculate the second term in Eq. (106). It is equal

$$\frac{p^\mu}{p^2}\frac{\partial}{\partial p^\mu}\int\frac{\mathrm{d}^dk_1}{(2\pi)^d}\int\frac{\mathrm{d}^dk_2}{(2\pi)^d}\frac{1}{k_1^2+M_2^2}\frac{1}{k_2^2+M_2^2}\frac{1}{(p-k_1+k_2)^2+M_1^2}\bigg|_{p^2=0}. \tag{113}$$

Let us calculate the integral in Eq. (113). After making the same transformations as in the previous integral we obtain

$$-\frac{p^\mu}{2p^2}\frac{\Gamma(3-d)}{(4\pi)^d}\frac{\partial}{\partial p^\mu}\int_0^1\mathrm{d}(x(1-x))^{\frac{d}{2}-2}$$
$$\times\int_0^1\mathrm{d}yy^{1-\frac{d}{2}}\;\frac{1}{\left(p^2y(1-y)+(1-y)M_2^2+y\left(\frac{xM_2^2+(1-x)M_1^2}{x(1-x)}\right)\right)^{3-d}}\Bigg|_{p^2=0} \tag{114a}$$

$$=\frac{(3-d)\Gamma(3-d)}{(4\pi)^d}\int_0^1\mathrm{d}(x(1-x))^{\frac{d}{2}-2}\int_0^1\mathrm{d}yy^{2-\frac{d}{2}}(1-y)$$
$$\times\frac{1}{\left(p^2y(1-y)+(1-y)M_2^2+y\left(\frac{xM_2^2+(1-x)M_1^2}{x(1-x)}\right)\right)^{4-d}}\Bigg|_{p^2=0} \tag{114b}$$

$$=\frac{\Gamma(4-d)}{(4\pi)^d}\int_0^1\mathrm{d}(x(1-x))^{\frac{d}{2}-2}\int_0^1\mathrm{d}yy^{2-\frac{d}{2}}(1-y)$$
$$\times\frac{1}{\left((1-y)M_2^2+y\left(\frac{xM_2^2+(1-x)M_1^2}{x(1-x)}\right)\right)^{4-d}}. \tag{114c}$$

The singular term at $d = 4$ of the integral in Eq. (114c) is equal

$$\frac{\Gamma(4-d)}{2(4\pi)^d}. \tag{115}$$

The contribution of the same type of the integral in Fig. 14 (f) is the same as for the diagram in Fig. 14 (e) and the sum of the all divergent contributions to the diagrams in Figs. 14 and 15 is equal

$$\begin{aligned}&\frac{3M_1^2(\Lambda_1\mu^{4-d})^2}{6(d-3)}\frac{\Gamma(4-d)}{(4\pi)^d(2-\frac{d}{2})}\big(1+(d-4)\big(-\frac{1}{2}+\ln M_1^2\big)\big)\\ &\qquad+\frac{(\Lambda_3\mu^{4-d})^2}{6(d-3)}\frac{\Gamma(4-d)}{(4\pi)^d(2-\frac{d}{2})}\times\Big(M_1^2\big(1+(d-4)\big(-\frac{1}{2}+\ln M_1^2\big)\big)\\ &\qquad+2M_2^2\big(1+(d-4)\big(-\frac{1}{2}+\ln M_2^2\big)\big)\Big)-p^2\frac{\Gamma(4-d)}{2(4\pi)^d}\Big(\frac{(\Lambda_1\mu^{4-d})^2}{6}+\frac{(\Lambda_3\mu^{4-d})^2}{4}\Big).\end{aligned} \tag{116}$$

To obtain the counterterms one has to expand the formula in Eq. (116) around the point

$d = 4$ and keep only the divergent terms. The result of this procedure is

$$\frac{1}{2(d-4)^2}(2M_1^2\hat{\Lambda}_1^2 + (M_1^2 + 2M_2^2)\hat{\Lambda}_3^2)$$
$$+ \frac{1}{4(d-4)}(M_1^2(2\hat{\Lambda}_1^2 + \hat{\Lambda}_3^2)(-3 + 2\gamma + 2\ln\left(\frac{M_1^2}{4\pi\mu^2}\right)))$$
$$+ \frac{1}{4(d-4)}(2M_2^2(2\hat{\Lambda}_3^2 + \hat{\Lambda}_3^2)(-3 + 2\gamma + 2\ln\left(\frac{M_2^2}{4\pi\mu^2}\right)))$$
$$+ \frac{p^2}{24(d-4)}(2\hat{\Lambda}_1^2 + 3\hat{\Lambda}_3^2). \quad (117)$$

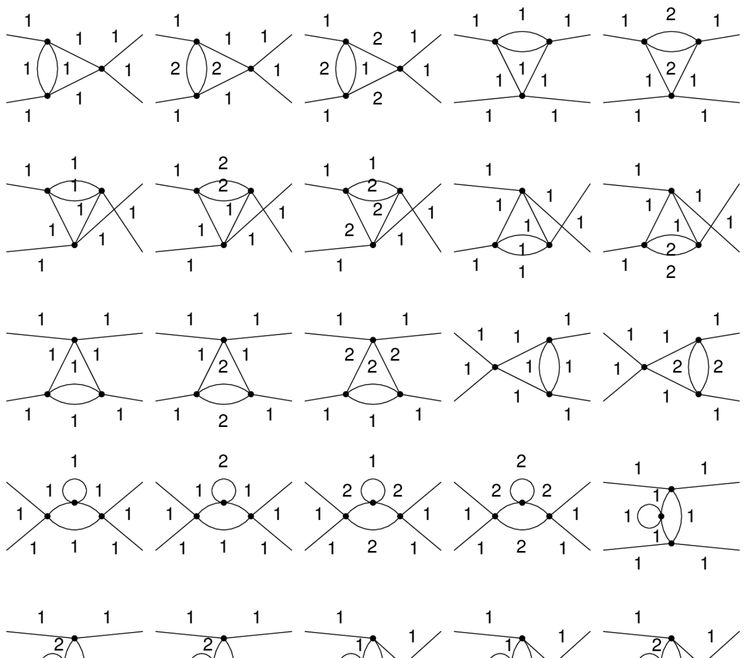
Here we introduced the notation

$$\hat{\Lambda}_i = \frac{\Lambda_i}{(4\pi)^2}.$$

Equation (117) defines the two loop counterterms for the propagator of the particle 1. It consists of two different types of terms. The first type does not depend on the external momentum p^2 and it gives the two loop contribution to δ_{M_1}. The term proportional to p^2 cannot be included in the counterterm δ_{M_1} and it is included in the counterterm δ_{Z_1}. The results for the propagator of the particle 2 are obtained from those for the particle 1 by a simple exchange of the indices for the masses and coupling constants.

To complete the renormalization program at two loops one has to consider the renormalization of the coupling constants. The necessary diagrams generated by the *FeynArts* program are

```
t6 = CreateTopologies[2, 2 → 2, ExcludeTopologies → Reducible, Adjac
f7 = InsertFields[t6, {S[1], S[1]} → {S[1], S[1]},
   GenericModel → scalars2, Model → scalars2, InsertionLevel → {Pa
Paint[f7, ColumnsXRows → {6, 7}, Numbering → None, SheetHeader → None
```



Technically, the diagrams shown in this figure do not present any new difficulties and they can be calculated using the methods described above. For this reason we will not calculate these diagrams here.

8. Conclusions

We have presented here a detailed description of the renormalization of the quantum field theory of two interacting scalar fields with an interaction described by the 4-th order homogeneous polynomial. Such a theory is an extension of the quantum field theory of one scalar field with a potential equal to ϕ^4. The theory with one scalar field is discussed in many textbooks (see for example [11, 13–16]), but the most complete discussion can be found in the book by Hagen Kleinert and Verena Schulte-Frohlinde [12]. The theory with two scalar fields shows many similarities with one scalar field model, but there are some differences. The most important one is the cancellation of the divergencies in Figs. 14 (a)-(d) and 15. Such cancellation does not occur for one scalar field. Another important feature of the theory with two scalar fields is the fact that it is a step forward in the path to study possible symmetries under the finite symmetry group transformations of systems of several scalar fields. Such symmetries may introduce some new and unexpected properties of these systems.

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Appendices

A. Wick's theorem

The calculation of the matrix elements of the time ordered products of the fields is done with the help of the Wick's theorem that transforms the time ordered product into the normal product of the fields denoted by $:\cdots:$. Let us consider for simplicity that we have only one scalar field $\phi(x)$. For the product of n such scalar fields the Wick's theorem states:

$$
\begin{aligned}
T(\phi(x_1)\phi(x_2)\cdots\phi(x_n)) =& :\phi(x_1)\phi(x_2)\cdots\phi(x_n): + :\wick{\c1\phi(x_1)\c1\phi(x_2)}\cdots\phi(x_n): \\
&+ :\wick{\c1\phi(x_1)\phi(x_2)\c1\phi(x_3)}\cdots\phi(x_n): + \cdots + :\wick{\c1\phi(x_1)\phi(x_2)\cdots\c1\phi(x_n)}: \\
&+ :\phi(x_1)\wick{\c1\phi(x_2)\c1\phi(x_3)}\cdots\phi(x_n): + \cdots + :\wick{\c1\phi(x_1)\c2\phi(x_2)\c1\phi(x_3)\c2\phi(x_4)}\cdots\phi(x_n): \\
&+ \cdots + :\wick{\c1\phi(x_1)\c2\phi(x_2)\c1\phi(x_3)\phi(x_4)\cdots\c2\phi(x_n)}: + \cdots
\end{aligned}
\qquad \text{(A-1)}
$$

Here the symbol of contraction $\contraction{}{\phi}{(x_i)}{\phi}\phi(x_i)\phi(x_j)$ means that the contracted pair of the fields has to be substituted by

$$\contraction{}{\phi}{(x_i)}{\phi}\phi(x_i)\phi(x_j) \rightarrow \langle 0|T(\phi(x_i)\phi(x_j))|0\rangle = i\Delta(x_i - x_j) \tag{A-2}$$

and the contractions have to be taken over all possible pairs of fields. If there are more kinds of fields then the contractions have to be taken only for those pairs of fields that do not commute. The remaining diagrams obtained from Fig. 3 are also logarithmically divergent.

B. Calculation of $G^{(4)}_{1111}$

Let us introduce the shorthand notation

$$\phi^{\rm I}_i(x_k) \rightarrow \phi^k_i \text{ and } \phi^{\rm I}_i(z) \rightarrow \phi^z_i.$$

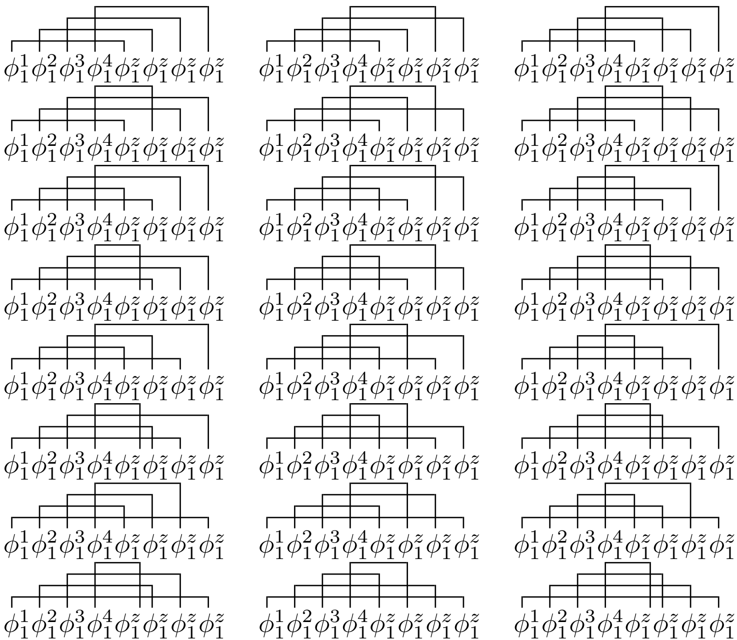
Using this notation and the explicit form of $\mathcal{V}(\phi^{\rm I}_1(z), \phi^{\rm I}_2(z))$ we have to calculate three types of the terms

$$T(\phi^1_1, \phi^2_1, \phi^3_1, \phi^4_1, \phi^z_1, \phi^z_1, \phi^z_1, \phi^z_1) \tag{B-1a}$$
$$T(\phi^1_1, \phi^2_1, \phi^3_1, \phi^4_1, \phi^z_2, \phi^z_2, \phi^z_2, \phi^z_2) \tag{B-1b}$$
$$T(\phi^1_1, \phi^2_1, \phi^3_1, \phi^4_1, \phi^z_1, \phi^z_1, \phi^z_2, \phi^z_2). \tag{B-1c}$$

All of the Wick's contractions for Eq. (B-1a) giving connected diagrams are shown in Table 3 and each term in this equation gives the same contribution which is equal

$$T(\phi^{\rm I}_1(x_1), \phi^{\rm I}_1(z))T(\phi^{\rm I}_1(x_2), \phi^{\rm I}_1(z))T(\phi^{\rm I}_1(x_3), \phi^{\rm I}_1(z))T(\phi^{\rm I}_1(x_4), \phi^{\rm I}_1(z)). \tag{B-2}$$

Table 3: All the Wick's contractions for Eq. (B-1a) giving connected diagrams.



The time ordered products in Eqs. (B-1b) and (B-1c) lead to the disconnected diagrams and do not contribute.

Concluding, there are 24 equal terms in Eq. (B-1a) so we see that the factor 1/4! in $\mathcal{V}(\phi_1^{\mathrm{I}}(z), \phi_2^{\mathrm{I}}(z))$ is canceled and the final result is

$$T(\phi_1^{\mathrm{I}}(x_1), \phi_1^{\mathrm{I}}(x_2), \phi_1^{\mathrm{I}}(x_3), \phi_1^{\mathrm{I}}(x_4), \mathcal{V}(\phi_1^{\mathrm{I}}(z), \phi_2^{\mathrm{I}}(z))) \\ = \lambda_1 T(\phi_1^{\mathrm{I}}(x_1)\phi_1^{\mathrm{I}}(z)) T(\phi_1^{\mathrm{I}}(x_2)\phi_1^{\mathrm{I}}(z)) T(\phi_1^{\mathrm{I}}(x_3)\phi_1^{\mathrm{I}}(z)) T(\phi_1^{\mathrm{I}}(x_4)\phi_1^{\mathrm{I}}(z)). \quad \text{(B-3)}$$

Similarly we obtain

$$T(\phi_2^{\mathrm{I}}(x_1), \phi_2^{\mathrm{I}}(x_2), \phi_2^{\mathrm{I}}(x_3), \phi_2^{\mathrm{I}}(x_4), \mathcal{V}(\phi_1^{\mathrm{I}}(z), \phi_2^{\mathrm{I}}(z))) \\ = \lambda_2 T(\phi_2^{\mathrm{I}}(x_1), \phi_2^{\mathrm{I}}(z)) T(\phi_2^{\mathrm{I}}(x_2), \phi_2^{\mathrm{I}}(z)) T(\phi_2^{\mathrm{I}}(x_3), \phi_2^{\mathrm{I}}(z)) T(\phi_2^{\mathrm{I}}(x_4), \phi_2^{\mathrm{I}}(z)) \quad \text{(B-4)}$$

and this demonstrates that there is no factor 1/4! in the Feynman diagrams with coupling constants λ_1 and λ_2 in Table 1.

C. Calculation of $G^{(4)}_{1122}$

This time we have to calculate three types of terms

$$T(\phi_1^1, \phi_1^2, \phi_2^3, \phi_2^4, \phi_1^z, \phi_1^z, \phi_1^z, \phi_1^z) \quad \text{(C-1a)}$$

$$T(\phi_1^1, \phi_1^2, \phi_2^3, \phi_2^4, \phi_2^z, \phi_2^z, \phi_2^z, \phi_2^z) \quad \text{(C-1b)}$$

$$T(\phi_1^1, \phi_1^2, \phi_2^3, \phi_2^4, \phi_1^z, \phi_1^z, \phi_2^z, \phi_2^z). \quad \text{(C-1c)}$$

The time ordered products in Eqs. (C-1a) and (C-1b) lead to disconnected diagrams so they do not contribute to the $G^{(4)}_{1122}$.

All the Wick's contractions for Eq. (C-1c) giving connected diagrams are shown in Table 4 and each term in this equation gives the same contribution which is equal

$$T(\phi_1^{\mathrm{I}}(x_1), \phi_1^{\mathrm{I}}(z))T(\phi_1^{\mathrm{I}}(x_2), \phi_1^{\mathrm{I}}(z))T(\phi_1^{\mathrm{I}}(x_3), \phi_1^{\mathrm{I}}(z))T(\phi_1^{\mathrm{I}}(x_4), \phi_1^{\mathrm{I}}(z)). \tag{C-2}$$

The final result is thus

$$\begin{aligned} &T(\phi_1^{\mathrm{I}}(x_1), \phi_1^{\mathrm{I}}(x_2), \phi_2^{\mathrm{I}}(x_3), \phi_2^{\mathrm{I}}(x_4), \mathcal{V}(\phi_1^{\mathrm{I}}(z), \phi_2^{\mathrm{I}}(z))) \\ &\qquad = \lambda_3 T(\phi_1^{\mathrm{I}}(x_1), \phi_1^{\mathrm{I}}(z))T(\phi_1^{\mathrm{I}}(x_2), \phi_1^{\mathrm{I}}(z))T(\phi_2^{\mathrm{I}}(x_3), \phi_2^{\mathrm{I}}(z))T(\phi_2^{\mathrm{I}}(x_4), \phi_2^{\mathrm{I}}(z)) \end{aligned} \tag{C-3}$$

and this demonstrates that there is no factor 1/4 in the Feynman diagrams with coupling constants λ_3 in Table 1.

D. The symmetry factors for the diagrams in Fig. 2

The time ordered product in Eq. (13a) is

$$\frac{\lambda_1}{4!} \langle 0| T(\phi_1^{\mathrm{I}}(x_1), \phi_1^{\mathrm{I}}(x_2), \phi_1^{\mathrm{I}}(z_1)\phi_1^{\mathrm{I}}(z_1)\phi_1^{\mathrm{I}}(z_1)\phi_1^{\mathrm{I}}(z_1))|0\rangle \,. \tag{D-1}$$

The Wick's contractions in Eq. (D-1) are given in Table 5

Table 4: All the Wick's contractions for Eq. (C-1c) giving connected diagrams.

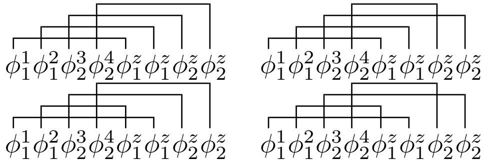
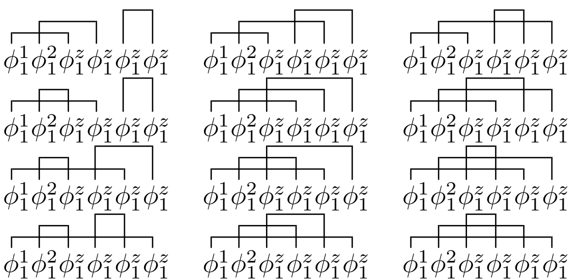


Table 5: All Wick's contractions for Eq. (D-1) giving connected diagrams.



Every contraction in Table 5 gives the same contribution and there are 12 such terms, so 1/4! gets multiplied by 12 and there remains overall factor 1/2. The symmetry factor is the inverse of this overall factor and for the diagram (a) in Fig. 2 it is equal $S = 2$.

The time ordered product in Eq. (13b) is

$$\frac{\lambda_3}{4} \langle 0|T(\phi_1^{\mathrm{I}}(x_1), \phi_1^{\mathrm{I}}(x_2), \phi_1^{\mathrm{I}}(z_1)\phi_1^{\mathrm{I}}(z_1)\phi_2^{\mathrm{I}}(z_1)\phi_2^{\mathrm{I}}(z_1))|0\rangle \,. \tag{D-2}$$

The Wick's contractions in Eq. (D-2) are given in Table 6.

Both contractions in Table 6 give the same contribution and there are 2 such terms, so 1/4 gets multiplied by 2 and there remains overall factor 1/2. The symmetry factor is the inverse of this overall factor and for the diagram (b) in Fig. 2 we get $S = 2$.

E. The symmetry factors for the diagrams in Fig. 3

From Eq. (16a) we obtain three Feynman diagrams in Fig. 16. To calculate the symmetry factor of the Feynman diagram in Fig. 3 (a) we have to calculate all contractions in the following expression

$$\phi_1^1\phi_1^2\phi_1^3\phi_1^4\phi_1^{z_1}\phi_1^{z_1}\phi_1^{z_1}\phi_1^{z_1}\phi_1^{z_2}\phi_1^{z_2}\phi_1^{z_2}\phi_1^{z_2}$$

such that there are two contractions between the ϕ_1^1, ϕ_1^2 and $\phi_1^{z_1}$, $\phi_1^{z_1}$, two contractions between the ϕ_1^3, ϕ_1^4 and $\phi_1^{z_2}$, $\phi_1^{z_2}$ and two contractions between the $\phi_1^{z_1}$, $\phi_1^{z_1}$ and $\phi_1^{z_2}$, $\phi_1^{z_2}$ and

Table 6: All Wick's contractions for Eq. (D-2) giving connected diagrams.

$$\phi_1^1\phi_1^2\phi_1^z\phi_1^z\phi_2^z\phi_2^z \qquad \phi_1^1\phi_1^2\phi_1^z\phi_1^z\phi_2^z\phi_2^z$$

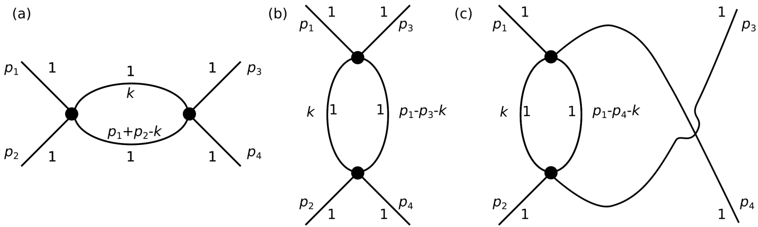


Figure 16: Three Feynman diagrams obtained from Eq. (16a). $p_1, \dots p_4$ are external momenta and k is the loop momentum. The momentum is conserved at each vertex. All three diagrams have the same symmetry factor.

Table 7: symmetry factors for the Feynman diagram patterns in Fig. 3.

Pattern in Fig. 3	symmetry factor S
(a)–(f)	2
(g)	1

then it has to be multiplied by 2, because the vertices can be permuted. This gives

$$S^{-1} = \underbrace{\left(\frac{1}{4!}\right)^2 \frac{1}{2!}}_{\text{factor in Eq. (16a)}} \cdot \underbrace{(4\cdot 3)}_{\substack{\phi_1^1,\ \phi_1^2 \text{ and } \phi_1^{z1},\ \phi_1^{z1} \\ \text{contractions}}} \cdot \underbrace{(4\cdot 3)}_{\substack{\phi_1^3,\ \phi_1^4 \text{ and } \phi_1^{z2},\ \phi_1^{z2} \\ \text{contractions}}} \cdot \underbrace{2}_{\substack{\phi_1^{z1},\ \phi_1^{z1} \text{ and } \phi_1^{z2},\ \phi_1^{z2} \\ \text{contractions}}} \cdot \underbrace{2}_{\text{permutations of vertices}} = \frac{1}{2}. \quad \text{(E-1)}$$

The symmetry factor is thus 2 and this means that the factor in front of this diagram is 1/2. The symmetry factors for the remaining patterns in Fig. 3 are given in Table 7.