Quantum formalism in lower-dimensional real and complex Hilbert spaces

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Key concepts of quantum formalism, including quantum measurement, entanglement, Bell and CSHS inequalities, circuit complexity, open systems, are presented using simple systems such as light polarization, spin-$1/2$, two-level atoms, colors and more.

Inspired by

- Nielsen M. A., Dowling M. R., Gu M., Doherty A. C. : Quantum Computation as Geometry, Science, **311**, 1133–1135 (2006).
- Nielsen M. A., Dowling M. R. : The geometry of quantum computation, quant-ph/0701004.
- Breuer H. P., Petruccione F. : *The Theory of Open Quantum Systems*, Oxford Univ. Press (2007).
- Chruściński D. : Time inhomogeneous quantum dynamical maps, Scientific Reports (2022) 12 :21223 (2022) ; (and references therein)
- Bergeron H., Curado E., G. J.-P. , Rodrigues L. : Orientations in the Plane as Quantum States, Brazilian Journal of Physics **49** 391–401 (2019) ; arXiv :2108.04086 [quant-ph]
- Beneduci R., Frion E., G. J.-P. : Quantum description of angles in the plane. Acta Polytechnica, 62 (1), 8–15 (2022) ; https ://doi.org/10.14311/AP.2022.62.0008
- Beneduci R., Frion E., G. J.-P., Perri A. : Quantum formalism on the plane : POVM-Toeplitz quantization, Naimark theorem and linear polarisation of the light, Annals of Physics **447** 169134 (2022) ; arXiv :2108.04086v3 [quant-ph]
- Curado E., Faci S., G. J.-P., Koide T., Maioli A., Noguera D : Quantum circuit complexity for linearly polarised light, Phys. Rev. A 111, 032208 (2025) ; arXiv :2410.03391 [quant-ph]

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Quantum formalism on the plane: POVM-Toeplitz quantization, Naimark theorem and linear polarization of the light

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ABSTRACT

We investigate two aspects of the elementary example of POVMs on the Euclidean plane, namely their status as quantum observables and their role as quantizers in the integral quantization procedure. The compatibility of POVMs in the ensuing quantum formalism is discussed, and a Naimark dilation is found for the quantum operators. The relation with Toeplitz quantization is explained. A physical situation is discussed, where we describe the linear polarization of the light with the use of Stokes parameters. In particular, the case of sequential measurements in a real bidimensional Hilbert space is addressed. An interpretation of the Stokes parameters in the framework of unsharp or fuzzy observables is given. Finally, a necessary condition for the compatibility of two dichotomic fuzzy observables which provides a condition for the approximate joint measurement of two incompatible sharp observables is found.

PHYSICAL REVIEW A **111**, 032208 (2025)

Quantum circuit complexity for linearly polarized light

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In this study, we explore a form of quantum circuit complexity that extends to open systems. To illustrate our methodology, we focus on a basic model where the projective Hilbert space of states is depicted by the set of orientations in the Euclidean plane. Specifically, we investigate the dynamics of mixed quantum states as they undergo interactions with a sequence of gates. The latter aim to accurately adjust the path from referent to target, aligning it as closely as possible with the path we have chosen. Our approach involves the analysis of sequences of real 2×2 density matrices. This mathematical model is physically exemplified by the Stokes density matrices, which delineate the linear polarization of a quasimonochromatic light beam, and the gates, which are viewed as quantum polarizers, whose states are also real 2×2 density matrices. The interaction between polarizer-linearly polarized light is construed within the context of this quantum formalism. Each density matrix for the light evolves in an approach analogous to a Gorini-Kossakowski-Lindblad-Sudarshan (GKLS) process during the time interval between consecutive gates. Notably, when considering an upper limit for the tolerance or accuracy, we unearth that the number of gates follows a power-law relationship which gives an upper bound of the complexity.

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OUTLINE

HISTORICAL PREAMBLE

(A quantum formalism with **NO** Planck constant $\hbar$)

1808 : Quantum formalism was already at work with Malus !

- In 1808, using a calcite crystal, Malus discovered that natural incident light became polarised when it was reflected by a glass surface, and that the light reflected close to an angle of incidence of 57° could be extinguished when viewed through the crystal. He then proposed that natural light consisted of the "s- and p-polarisations", which were perpendicular to each other.

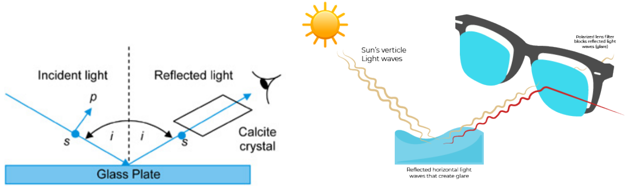


Figure – Left : Malus' law, from https ://spie.org/publications/fg05_p03_maluss_law ?SSO=1. Right : Polarized Sunglasses.

- Malus : since the intensity of the reflected light varied from a maximum to a minimum as the crystal was rotated, the **amplitude** of the reflected beam should be $A = A_0 \cos\theta$, and by squaring it the **intensity** of the reflected polarised light is

$$I(\theta) = I_0 \cos^2\theta \quad \text{(Malus' Law)} \quad I_0 = A_0^2$$

1852 : Quantum formalism with Stokes

- The Stokes parameters are a set of values that describe the polarisation state of electromagnetic radiation. They were defined by George Gabriel Stokes in 1852
- Electric field for a propagating quasi-monochromatic e.m. wave along the z-axis : $\overrightarrow{\mathrm{E}}(t) = \overrightarrow{\mathrm{E}_0}(t)\, e^{\mathrm{i}\omega t} = \mathrm{E}_x \widehat{\boldsymbol{\imath}} + \mathrm{E}_y \widehat{\boldsymbol{\jmath}} = (\mathrm{E}_\alpha)$.
- $\overrightarrow{\mathrm{E}_0}(t)$ determines the polarisation, has slow temporal variation and is measured through Nicol prisms [1], or other devices, by measuring the intensity of the light yielded by mean values of $\propto \mathrm{E}_\alpha \mathrm{E}_\beta$, $\mathrm{E}_\alpha \mathrm{E}_\beta^*$ and conjugates, where $\alpha, \beta \in \{x, y\}$.

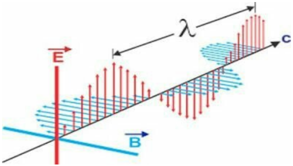


Figure – How is light made up of electric fields and magnetic fields ? (Maxwell, 1861)

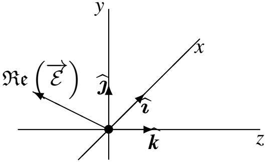
1. The Nicol prism consists of two specially cut calcite prisms bonded together with an adhesive known as Canada balsam. This prism transmits waves vibrating in one direction only and thus produces a plane-polarised beam from ordinary light. From https ://www.britannica.com/technology/prism-optics

1852 : Quantum formalism with Stokes (continued)

- Due to rapidly oscillating factors and so null temporal average $\langle\cdot\rangle_t$, partially polarised light is described by the 2×2 Hermitian matrix (Stokes parameters)

$$\frac{1}{J}\begin{pmatrix} \langle \mathsf{E}_{0x}\mathsf{E}_{0x}^*\rangle_t & \left\langle \mathsf{E}_{0x}\mathsf{E}_{0y}^*\right\rangle_t \\ \langle \mathsf{E}_{0y}\mathsf{E}_{0x}^*\rangle_t & \left\langle \mathsf{E}_{0y}\mathsf{E}_{0y}^*\right\rangle_t \end{pmatrix} = \frac{1}{J}\left[a_+\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + a_-\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}\right]$$

- where $a_\pm = \frac{1}{2}\left(\langle \mathsf{E}_{0x}\mathsf{E}_{0x}^*\rangle_t \pm \left\langle \mathsf{E}_{0y}\mathsf{E}_{0y}^*\right\rangle_t\right)$; $b = \mathrm{Re}\left\langle \mathsf{E}_{0x}\mathsf{E}_{0y}^*\right\rangle_t$; $c = -\mathrm{Im}\left\langle \mathsf{E}_{0x}\mathsf{E}_{0y}^*\right\rangle_t$
  $J = \sum_\alpha J_{\alpha\alpha} = \langle |\mathsf{E}_{0x}|^2\rangle_t + \langle |\mathsf{E}_{0y}|^2\rangle_t$: wave intensity.



1927 with Pauli : matrices

Zur Quantenmechanik des magnetischen Elektrons.

Von **W. Pauli jr.** in Hamburg.

Man kann diese Relationen auch in der symbolischen Matrizenform schreiben:

$$\mathbf{s}_x(\psi) = \begin{pmatrix} 0, & 1 \\ 1, & 0 \end{pmatrix} \cdot \psi; \quad \mathbf{s}_y(\psi) = \begin{pmatrix} 0, & -i \\ i, & 0 \end{pmatrix} \cdot \psi; \quad \mathbf{s}_z(\psi) = \begin{pmatrix} 1, & 0 \\ 0, & -1 \end{pmatrix} \cdot \psi. \quad (3')$$

- Many works in quantum information, quantum measurement, quantum foundations, ..., are illustrated with manipulations of the two real Pauli matrices and their tensor products

$$\sigma_x \equiv \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_z \equiv \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- No complex numbers, i.e. no $\sigma_y \equiv \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$, just the Euclidean plane and its Cartesian or tensor products : $\mathbb{R}^2 \oplus \mathbb{R}^2 \cong \mathbb{R}^2 \times \mathbb{R}^2 \cong \mathbb{R}^2 \otimes \mathbb{R}^2$.
- With quantum circuit complexity let us show that the 2d real Hilbert space with its mathematical and interpretative resources is not just a toy model...

CIRCUIT COMPLEXITY RESULTS (SUMMARY)

Our motivation : quantum circuit complexity

- Algorithm **complexity** = amount of resources required to run it.
- Algorithm is **efficient** if the number of steps increases **polynomially** with the size of the input.
- For a quantum algorithm the complexity of preparing a **target** state $|\psi_T\rangle = U|\psi_R\rangle \in$ Hilbert space $\mathcal{H}$ from a **reference** state through a unitary transformation U is related to the **number** of quantum gates applied to get to the target state.
- **Tolerance** ϵ allows to judge the transformation to be successful when

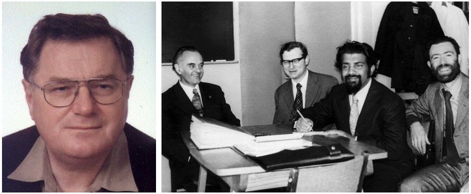
$$\| |\psi_T\rangle - U|\psi_R\rangle \| \leq \epsilon \,.$$

- **Complexity** of the state $|\psi_T\rangle$ = **minimum** number of gates required to produce the transformation $|\psi_T\rangle = U|\psi_R\rangle$ within tolerance, i.e., the number of elementary gates in the **optimal or shortest circuit**.
- **Challenge** : Identify this optimal circuit from amongst an infinite number of possibilities. [2]

2. In Nielsen M. A., Dowling M. R., Gu M., Doherty A. C. : Quantum Computation as Geometry, Science, **311**, 1133–1135 (2006) ; Nielsen M. A., Dowling M. R. : The geometry of quantum computation, quant-ph/0701004., the question of interest was to find the minimal size quantum circuit required to exactly implement a specified n-qubit unitary operation U.

Quantum circuit complexity : Outline continued

- Inspired by the Nielsen's ideas, **the work** [3], concerns a type of quantum circuit complexity for **open** systems
- Precisely we examine the evolution of **mixed** quantum states submitted to interact with a sequence of gates. The fact that the system is open means that we depart from the unitary paradigm of Equation $|\psi_T\rangle = U|\psi_R\rangle$.
- Instead we examine a sequence $(\rho_n)_{0\leq n\leq N-1}$ of N **density matrices**, each one evolving along a **Gorini-Kossakowski-Lindblad-Sudarshan** (GKLS) equation, possibly inhomogeneous GKLS, during the time interval between two gates.



Left : Göran Lindblad (1940-2022) (Wikipedia). Right : from left to right, Roman Ingarden (1920-2011), Andrzej Kossakowski (1938-2021), George Sudarshan (1932-2018) and Vittorio Gorini (uploaded by Chruściński Darius)

3. Curado E., Faci S., G. J.-P., Koide T., Maioli A., Noguera D. : Quantum circuit complexity for linearly polarised light, Phys. Rev. A 111, 032208 (2025) ; arXiv :2410.03391 [quant-ph]

Quantum circuit complexity : Outline continued

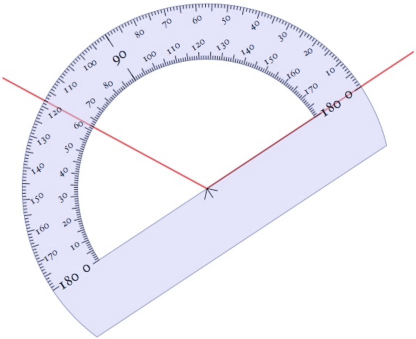
- We illustrate our approach with the elementary situation where Hilbert space of states is the **Euclidean plane** $\mathbb{R}^2$ and mixed states ρ have the following parametrisation :

$$\rho_{r,\phi} = \frac{1}{2}(\mathbb{1}_2 + r\sigma_{2\phi})\,,\; \sigma_{2\phi} = \begin{pmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{pmatrix}\,,\; r \in [0,1]\,,\; \phi \in [0,\pi)\,.$$

- **Pure states** correspond to $r = 1$, $\rho_{1,\phi} = |\phi\rangle\langle\phi|$ and $r = 0$ is for totally **mixed states**, i.e., $\mathbb{1}_2/2$.
- Linearly polarised light is described by such states (Stokes parameters, 1852))
- Advantage : All analytic calculations and/or numerical simulations are **easily implemented**.
- Starting from the reference state ρ_{r_R,ϕ_R} and aiming at reaching the target state ρ_{r_T,ϕ_T} through a sequence of gates which modify parameters r, ϕ and leave the mixed state interact with its **dissipative environment** till the next gate.
- We introduce an **interaction "quantum polariser"-linearly polarised light**, and geometrical (e.g. distances) criteria for comparing various paths in the space of such density matrices.

Quantum states in the Euclidean plane as a protractor : quantum kindergarten

$$\begin{aligned}\rho_{r,\phi} &= \frac{1}{2}(\mathbb{1}_2 + r\sigma_{2\phi})\,,\ \sigma_{2\phi} = \begin{pmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{pmatrix}\,,\ r \in [0,1]\,,\ \phi \in [0,\pi) \\ &= \frac{1+r}{2}P_\phi + \frac{1-r}{2}P_{\phi+\pi/2}\,, \quad P_\phi = \begin{pmatrix} \cos^2\phi & \sin\phi\cos\phi \\ \sin 2\phi\cos\phi & \sin 2\phi \end{pmatrix}\,.\end{aligned}$$

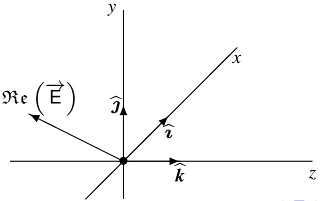


Polarization tensor of light ∼ density matrix, a reminder

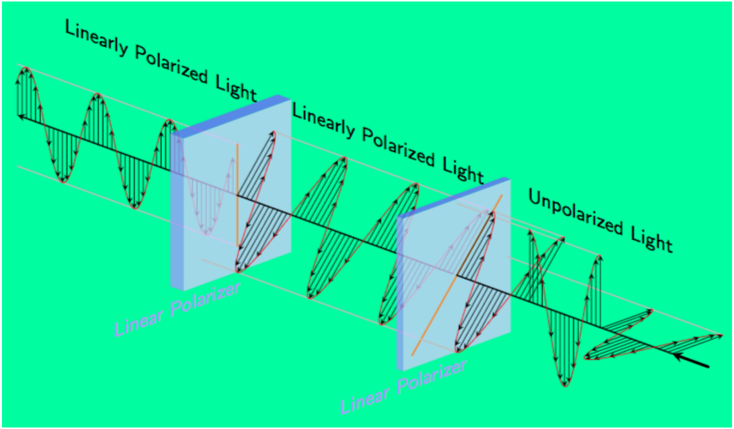
- Electric field for a propagating quasi-monochromatic e.m. wave along the z-axis :
$\vec{\mathsf{E}}(t) = \vec{\mathsf{E}}_0(t)\, e^{\mathrm{i}\omega t} = \mathsf{E}_x \hat{\imath} + \mathsf{E}_y \hat{\jmath} = (\mathsf{E}_\alpha)$,
- $\vec{\mathsf{E}}_0(t)$ determines the polarisation, has slow temporal variation and is measured through Nicol prisms, or other devices, by measuring the intensity of the light yielded by mean values $\propto \mathsf{E}_\alpha \mathsf{E}_\beta$, $\mathsf{E}_\alpha \mathsf{E}^*_\beta$ and conjugates
- Due to rapidly oscillating factors and so null temporal average $\langle\cdot\rangle_t$, partially polarised light is described by the 2×2 Hermitian matrix (**Stokes** parameters, 1852)

$$\frac{1}{J}\begin{pmatrix} \langle \mathsf{E}_{0x}\mathsf{E}^*_{0x}\rangle_t & \left\langle \mathsf{E}_{0x}\mathsf{E}^*_{0y}\right\rangle_t \\ \langle \mathsf{E}_{0y}\mathsf{E}^*_{0x}\rangle_t & \left\langle \mathsf{E}_{0y}\mathsf{E}^*_{0y}\right\rangle_t \end{pmatrix} \equiv \rho_{r,\phi} + \frac{A}{2}\sigma_2 = \frac{1+r}{2}P_\phi + \frac{1-r}{2}P_{\phi+\pi/2} + \mathrm{i}\frac{A}{2}\tau_2$$

J : wave intensity ; $0 \le r \le 1$: linear polarisation ; $-1 \le A \le 1$: circular polarisation ;
Real matrix $\tau_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\tau_2^2 = -\mathbb{1}_2$, generates rotations : $\exp \theta\tau_2 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

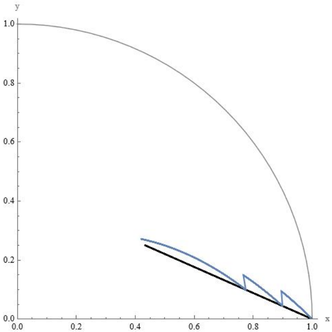


Slits ∼ Passing Directions for linearly polarised light



If we send polarised linearly light ($|\phi\rangle$) into 2d polaroid, at an angle $\phi - \eta$ to the passing direction (here $\eta = \pi/2$), the amplitude which comes out of the polaroid is only the $\cos(\phi - \eta)$ part, the $\sin(\phi - \eta)$ being absorbed. The energy or intensity of light which passes through the polaroid is proportional to $\cos^2(\phi - \eta)$ (Malus' law, 1808)

Results



The black line represents the geodesic related to the trace distance and the blue curves represent the trajectory of the quantum state evolved accordingly with GLKS equations. **When the distance reaches a chosen accuracy** $\epsilon = 0.05$, we place a quantum polariser to change the angle of the polarisation to be the same as the one at the geodesics. The action of the polarization can be viewed as the abruptly change in the quantum state path.
It is possible to observe via the parameters of the linear fit $\log N_g = m + n \log \epsilon$, that in both cases (a,b and c,d) the circuits obey a power law

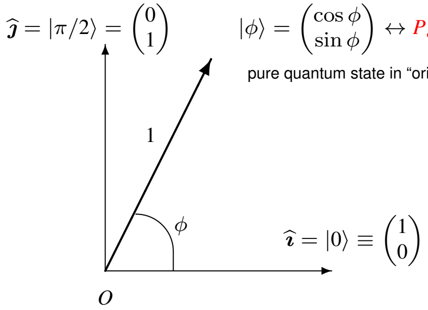
$$\begin{aligned} N_g &\approx 10^{-1.05} \times \epsilon^{-1.09} \quad \text{examples } (a, b), \\ N_g &\approx 10^{-0.17} \times \epsilon^{-1.04} \quad \text{examples } (c, d). \end{aligned}$$

QUANTUM ORIENTATIONS IN $\mathbb{R}^2$

Euclidean plane as Hilbert space of quantum states

$$|\phi\rangle = \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \leftrightarrow P_\phi = |\phi\rangle\langle\phi|$$

pure quantum state in "orientation representation"



A pure state in the horizontal-vertical representation can be decomposed as

$$|\phi\rangle = \cos\phi\,|0\rangle + \sin\phi\,\left|\frac{\pi}{2}\right\rangle\,, \quad \langle 0|\phi\rangle = \cos\phi\,, \qquad \left\langle\frac{\pi}{2}\right|\phi\rangle = \sin\phi\,,$$

To $|\phi\rangle$ corresponds the orthogonal projector $P_\phi = |\phi\rangle\langle\phi| = \begin{pmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{pmatrix}$

Mixed states ∼ density matrices

- **Density matrices** are non-negative unit trace matrices with spectral decomposition

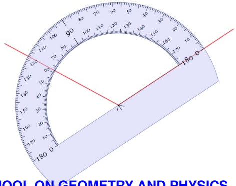
$$\rho = \left(\frac{1+r}{2}\right) P_\phi + \left(\frac{1-r}{2}\right) P_{\phi+\pi/2}\,, \quad 0 \leq r \leq 1\,, \quad P_\phi = |\phi\rangle\langle\phi|$$

Parameter r encodes the distance of ρ to the pure state $P_\phi = |\phi\rangle\langle\phi|$ while $1-r$ measures the degree of "mixing".

- With polar coordinates (r, ϕ) for the upper half unit disk,

$$\rho \equiv \rho_{r,\phi} = \frac{1}{2}\mathbb{1}_2 + \frac{r}{2}\mathcal{R}(\phi)\sigma_3\mathcal{R}(-\phi) = \begin{pmatrix} \frac{1}{2} + \frac{r}{2}\cos 2\phi & \frac{r}{2}\sin 2\phi \\ \frac{r}{2}\sin 2\phi & \frac{1}{2} - \frac{r}{2}\cos 2\phi \end{pmatrix}$$

where $\mathcal{R}(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} = \exp \phi\tau_2$, with $\tau_2 = -\mathrm{i}\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$



Metric structure of space of density matrices

- *Trace norm* $\|\rho\|_{\mathrm{TN}} = \mathrm{tr}\sqrt{\rho\rho^\dagger} = \mathrm{tr}\sqrt{\rho^2} = 1.$

- *Trace distance* :

$$d_{TN}(\rho_{r,\phi}, \rho_{r',\phi'}) = \frac{1}{2}\mathrm{tr}\sqrt{(\rho_{r,\phi} - \rho_{r',\phi'})^\dagger(\rho_{r,\phi} - \rho_{r',\phi'})} = \frac{1}{2}\sum_{i=1}^{2}|\lambda_i|\,,$$

where λ_i is the i-th eigenvalue of $(\rho_{r,\phi} - \rho_{r',\phi'})$.

$$d_{\mathrm{TN}}(\rho_{r,\phi}, \rho_{r',\phi'}) = \frac{1}{2}\sqrt{r^2 + r'^2 - 2rr'\cos(2\phi - 2\phi')} = \frac{1}{2}\|(r, 2\phi) - (r', 2\phi')\|_{\mathbb{R}^2}\,,$$

where $\|(r,\theta) - (r',\theta')\|_{\mathbb{R}^2}$ is the Euclidean distance in the plane between vectors in polar coordinates.

- Corresponding **geodesics** are straight lines described in polar coordinates by :

$$r = \frac{1}{C_3\cos\phi + C_4\sin\phi}\,.$$

Metric structure of space of density matrices (continued)

- (Uhlmann-Jozsa) ***Fidelity*** :
  Fidelity measures the similarity between two quantum states, ρ and σ. It is defined for a pair of quantum states defined on a Hilbert space $\mathcal{H}$ by [4] :

$$F(\rho,\sigma) = \left(\mathrm{Tr}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right)^2 = \|\sqrt{\rho}\sqrt{\sigma}\|^2_{\mathrm{TN}} = \mathrm{Tr}\,(\sqrt{\rho\sigma})^2$$

  Properties :
  - $F(\rho,\sigma) = F(\sigma,\rho)$
  - $0 \leq F(\rho,\sigma) \leq 1$
  - $F(\rho,\sigma) = 0$ if and only if $\rho \perp \sigma$.
  - $F(\rho,\sigma) = 1$ if and only if $\rho = \sigma$.
- In our two-dimensional case :

$$F(\rho_{r,\theta},\sigma_{s,\phi}) = \frac{1}{2}\left[1 + r\sigma\cos 2(\theta-\phi) + \sqrt{(1-r^2)(1-\sigma^2)}\right]$$

- ***Bures Distance*** : The Bures-Helstrom [5] distance is a metric related to fidelity :

$$D_B(\rho,\sigma) = \sqrt{2(1-\sqrt{F(\rho,\sigma)})}$$

4. A Simplified Expression for Quantum Fidelity Adrian Müller, arXiv :2309.10565v4 [quant-ph]
5. D. Bures(1969). Trans. AMS 135, AMS 199 ; C.W. Helstrom, C.W. (1967). Phys. Lett. A 25

Hilbertian metric on space of density matrices

- Any density operator ρ is Hilbert–Schmidt and so has norm $\|\rho\|_{\mathrm{HS}} = \sqrt{\mathrm{tr}\rho\rho^\dagger} = \sqrt{\mathrm{tr}\rho^2}$.
- In our two-dimensional framework, we use the relation (involving the rotation matrix $\mathcal{R}(\theta)$) :

$$\rho_{r,\phi}\,\rho_{r',\phi'} = \frac{1}{2}\left(\rho_{r,\phi} + \rho_{r',\phi'} + \frac{rr'}{2}\mathcal{R}(2(\phi-\phi')) - \frac{\mathbb{1}_2}{2}\right)\,,$$

to get

$$\mathrm{tr}\left(\rho_{r,\phi}\rho_{r',\phi'}\right) = \frac{1}{2}\left(1 + rr'\cos 2(\phi-\phi')\right) = \frac{1-rr'}{2} + rr'\cos^2(\phi-\phi')\,,$$

- In particular,

$$\|\rho_{r,\phi}\|_{\mathrm{HS}} = \sqrt{\frac{1+r^2}{2}}\,.$$

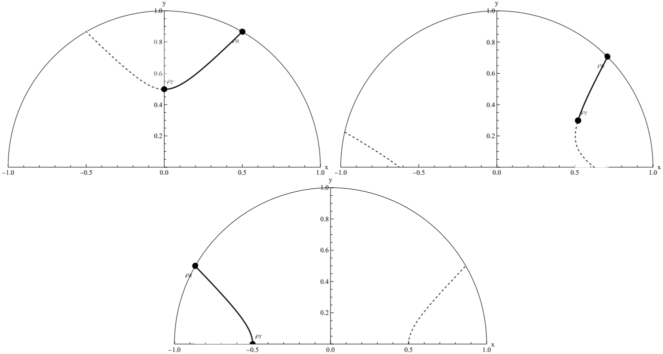
- The associated distance

$$d_{\mathrm{HS}}\left(\rho_{r,\phi},\rho_{r',\phi'}\right) = \|\rho_{r,\phi} - \rho_{r',\phi'}\|_{\mathrm{HS}}$$

is given by

$$d_{\mathrm{HS}}\left(\rho_{r,\phi},\rho_{r',\phi'}\right) = \frac{1}{\sqrt{2}}\sqrt{r^2 + r'^2 - 2rr'\cos(2\phi - 2\phi')} = \frac{1}{\sqrt{2}}\|(r,2\phi)-(r',2\phi')\|_{\mathbb{R}^2}$$

Geodesics for Hilbert-Schmidt metric



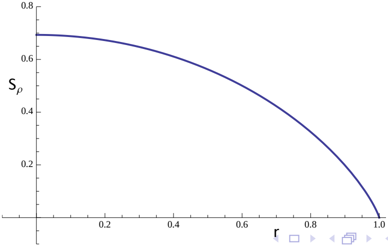
Dashed curves are quartic plane functions and the full curves are geodesics between reference state ρ_R and target state ρ_T. (Top left) $r_R = 1$, $\phi_R = \pi/3$, $r_T = 0.5$, $\phi_T = \pi/2$. (Top right) $r_R = 1$, $\phi_R = \pi/4$, $r_T = 0.6$, $\phi_T = \pi/6$. (Bottom) $r_R = 1$, $\phi_R = 5\pi/6$, $r_T = 0.5$, $\phi_T = \pi$.

von Neumann entropy of states

- In $\rho_{r,\phi} = \begin{pmatrix} \frac{1}{2} + \frac{r}{2}\cos 2\phi & \frac{r}{2}\sin 2\phi \\ \frac{r}{2}\sin 2\phi & \frac{1}{2} - \frac{r}{2}\cos 2\phi \end{pmatrix}$ the parameter r encodes the distance of ρ to the pure state $P_\phi = |\phi\rangle\langle\phi|$ while $1 - r$ measures the degree of "mixing".
- A statistical interpretation is made possible through the **von Neumann entropy** (Boltzmann-Gibbs or Shannon entropy in the case of two possibilities) :

$$S_\rho := -\mathrm{Tr}(\rho \ln \rho) = -\frac{1+r}{2}\ln\frac{1+r}{2} - \frac{1-r}{2}\ln\frac{1-r}{2}\,.$$

- As a function of $r \in [0,1]$ S_ρ is nonnegative, concave and symmetric with respect to its maximum value $\log 2$ at $r = 0$, i.e., $\rho_0 \equiv \mathbb{1}_2/2$ (completely random orientations) .



TIME EVOLUTION

Closed systems

- Heisenberg-Dirac equation when applied to a possibly time-dependent quantum observable $A(t)$,

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}A_H = [A_H, H(t)] + U(t,t_0)^\dagger \left(\mathrm{i}\frac{\partial}{\partial t}A\right) U(t,t_0)\,,$$

where $A_H(t) := U(t,t_0)^\dagger A(t) U(t,t_0)$ and Hamiltonian H is self-adjoint.

- In the present context the allowed form of H is

$$H(t) = \begin{pmatrix} 0 & -\mathrm{i}\mathcal{E}(t) \\ \mathrm{i}\mathcal{E}(t) & 0 \end{pmatrix} = \mathcal{E}(t)\sigma_2 = \mathrm{i}\mathcal{E}(t)\tau_2\,, \quad \tau_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\,.$$

- It results the evolution operator as the rotation in the plane, as expected!

$$U(t,t_0) = \exp \mathrm{i}\int_{t_0}^{t} H(t')\mathrm{d}t' = \mathcal{R}\left(\int_{t_0}^{t}\mathcal{E}(t')\,\mathrm{d}t'\right)\,.$$

(Baby) Majorana equation

- Since our Heisenberg-Dirac equation is homogeneous in the imaginary "i", we can describe this dynamics in terms of real numbers only. Introduce the real pseudo-Hamiltonian

$$\widetilde{H}(t) = \begin{pmatrix} 0 & -\mathcal{E}(t) \\ \mathcal{E}(t) & 0 \end{pmatrix} = \mathcal{E}(t)\tau_2 \,, \quad \tau_2 = -\mathrm{i}\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \,,$$

- one gets

$$\frac{\mathrm{d}A_H}{\mathrm{d}t} = [A_H, \widetilde{H}(t)] + U(t,t_0)^{\dagger} \left(\frac{\partial A}{\partial t}\right) U(t,t_0) \,,$$

- They are **Majorana**-like equations, where only real numbers are involved.
- This real quantum dynamics involves the antisymmetric matrix τ_2, with $\tau_2^2 = -\mathbb{1}$. playing the role of the imaginary "i". [6]

6. Together with σ_1 and σ_3 these 3 matrices generate the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ of the group $\mathrm{SL}(2,\mathbb{R})$ of 2×2 real matrices with determinant equal to 1.

$$[\sigma_1, \tau_2] = 2\sigma_3 \,, \quad [\tau_2, \sigma_3] = 2\sigma_1 \,, \quad [\sigma_3, \sigma_1] = -2\tau_2 \,.$$

GKLS Open systems for $\mathbb{R}^2$

- Now consider an extension to a type of quantum master equation, namely the Gorini-Kossakowski-Lindblad-Sudarshan (GKLS) equation describing the time evolution of the density matrix $\rho_{r,\phi}$ for an open system. When applied to the evolution of a quantum observable A, this equation in its diagonal form (which does not restrict its validity) reads :

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \mathrm{i}\,[H, A] + \sum_k h_k \left[L_k A L_k^\dagger - \frac{1}{2}\left(A L_k^\dagger L_k + L_k^\dagger L_k A\right)\right] \equiv \mathcal{L}(A)\,,$$

  where the L_k's with the identity form an arbitrary basis of operators and the coefficients h_k are non-negative constants.

- In the present framework with no imaginary i the three matrices σ_1, τ_2, σ_3, together with the identity form a basis for the vector space of real 2×2 matrices ; then the above equation, when applied to $\rho_{r,\phi}$, becomes

$$\begin{aligned}\frac{\mathrm{d}\rho_{r,\phi}}{\mathrm{d}t} &= [\rho_{r,\phi}, \widetilde{H}] + h_1\left(\sigma_1\rho_{r,\phi}\sigma_1 - \rho_{r,\phi}\right) + h_2\left(-\tau_2\rho_{r,\phi}\tau_2 + \rho_{r,\phi}\right) + h_3\left(\sigma_3\rho_{r,\phi}\sigma_3 - \rho_{r,\phi}\right)\\ &= r\left(\frac{\mathcal{E}(t)}{\hbar}\,\sin 2\phi - h_1\,\cos 2\phi\right)\sigma_3 + r\left(-\frac{\mathcal{E}(t)}{\hbar}\,\cos 2\phi - h_3\,\sin 2\phi\right)\sigma_1 + h_2\,\mathbb{1}_2\,.\end{aligned}$$

Resulting dynamical system for $\mathbb{R}^2$

- Since $\frac{\mathrm{d}\rho_{r,\phi}}{\mathrm{d}t} \equiv \dot{\rho}_{r,\phi} = \frac{1}{2}(\dot{r}\cos 2\phi - 2r\sin 2\phi\,\dot{\phi})\sigma_3 + \frac{1}{2}(\dot{r}\sin 2\phi + 2r\cos 2\phi\,\dot{\phi})\sigma_1$, we get by identification $h_2 = 0$ and the first-order differential (dynamical) system

$$\dot{\phi} = \alpha \sin 4\phi - \mathcal{E}(t)\,, \quad \alpha := \frac{h_1 - h_3}{2} \in \mathbb{R}\,, \tag{1}$$

$$\frac{\dot{r}}{r} = -2\alpha\, \cos 4\phi - \beta\,, \quad \beta := h_1 + h_3 > 0\,. \tag{2}$$

- Eq. (1) leads to a kind of Ricatti equation. Having in hand a solution $\phi(t)$, Equation (2) is easily integrated and we find

$$r(t) = r_0 \exp\left[-\beta(t - t_0) - 2\alpha \int_{t_0}^{t} \cos 4\phi(t')\,\mathrm{d}t'\right]\,.$$

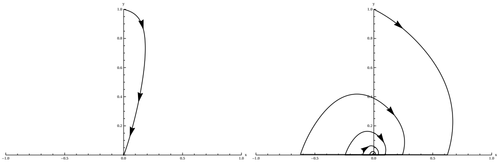
- Due to $\beta > 0$, $r \to 0$, i.e., $\rho_{r,\phi} \to \mathbb{1}_2/2$ as $t \to \infty$: the von Neumann entropy of the open system tends to its maximum at large time. For $\alpha = 0$ the angle ϕ behaves like for a closed system, $\phi(t) - \phi(t_0) = -\int_{t_0}^{t} \mathrm{d}t'\,\mathcal{E}(t')$ whereas $r(t) = r_0 e^{-2h_1(t-t_0)}$.

Examples : Solutions of Lindblad system at constant $\mathcal{E}$

- **Case $\mathcal{E}^2 > \alpha^2$** With $\omega := \sqrt{\mathcal{E}^2 - \alpha^2}$,

$$\phi(t) = \frac{1}{2}\arctan\left[\frac{-\omega_1}{\mathcal{E}}\tan(2\omega(t-t_0)+c_0) + \frac{\alpha}{\mathcal{E}}\right] + \frac{n\pi}{2},$$

where n must be carefully chosen to maintain the continuity of $\phi(t)$ in the domain $[0, \pi)$.



Trajectory of the density matrix in the half-plane $\mathbf{r}(t) = (r(t)\cos\phi(t), r(t)\sin\phi(t))$ for two representative cases, with initial conditions $r(0) = 1$ and $\phi(0) = \pi/2$ and $\mathcal{E} = 10$, but different parameters $\alpha = -9$, $\beta = 20$ (left), $\alpha = 0.5$, and $\beta = 3$(right). Different parameter values result in a distinct number of cycles over the half-plane. This connection becomes explicit when one recognizes that the constant ω is an angular frequency. Then, the period T of the ϕ cycle is $2\omega T = \pi$, where π is the periodicity of the tangent function.

- **Case $\mathcal{E}^2 < \alpha^2$** With $\gamma = \sqrt{\alpha^2 - \mathcal{E}^2}$,

$$\phi(t) = \frac{1}{2}\arctan\left[\frac{\alpha}{\mathcal{E}} + \frac{\gamma}{\mathcal{E}}\coth(2\gamma\Delta t + c_2)\right] + \frac{n\pi}{2}, \tag{3}$$

where n is an integer that is responsible for the continuity of $\phi(t)$ in the half-plane. In this case, the parameter γ can be interpreted as an attenuation constant for the ϕ variable, because of the behavior of the $\coth$ function which tends to unity for long times.

POLARIZATION OF THE LIGHT

through a von Neumann quantum interaction

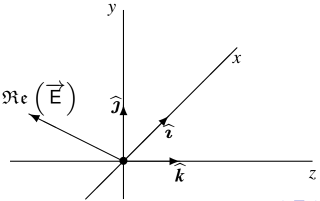
Polarization tensor of light ∼ density matrix. A reminder

- Electric field for a propagating quasi-monochromatic e.m. wave along the z-axis : $\overrightarrow{\mathsf{E}}(t) = \overrightarrow{\mathsf{E}_0}(t)\, e^{\mathrm{i}\omega t} = \mathsf{E}_x \widehat{\boldsymbol{\imath}} + \mathsf{E}_y \widehat{\boldsymbol{\jmath}} = (\mathsf{E}_\alpha)$,
- $\overrightarrow{\mathsf{E}_0}(t)$ determines the polarisation, has slow temporal variation and is measured through Nicol prisms, or other devices, by measuring the intensity of the light yielded by mean values $\propto \mathsf{E}_\alpha \mathsf{E}_\beta$, $\mathsf{E}_\alpha \mathsf{E}^*_\beta$ and conjugates
- Due to rapidly oscillating factors and so null temporal average $\langle\cdot\rangle_t$, partially polarised light is described by the 2×2 Hermitian matrix (**Stokes** parameters)

$$\frac{1}{J}\begin{pmatrix} \langle \mathsf{E}_{0x}\mathsf{E}^*_{0x}\rangle_t & \left\langle \mathsf{E}_{0x}\mathsf{E}^*_{0y}\right\rangle_t \\ \langle \mathsf{E}_{0y}\mathsf{E}^*_{0x}\rangle_t & \left\langle \mathsf{E}_{0y}\mathsf{E}^*_{0y}\right\rangle_t \end{pmatrix} \equiv \rho_{r,\phi} + \frac{A}{2}\sigma_2 = \frac{1+r}{2}P_\phi + \frac{1-r}{2}P_{\phi+\pi/2} + \mathrm{i}\frac{A}{2}\tau_2$$

J : wave intensity ; $0 \leq r \leq 1$: linear polarisation ; $-1 \leq A \leq 1$: circular polarisation ;

Real matrix $\tau_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\tau_2^2 = -\mathbb{1}_2$, generates rotations : $\exp\theta\tau_2 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$



von Neumann interaction polariser-partially linear polarised light

- Two planes $\mathbb{R}^2$ and their tensor product $\mathbb{R}^2 \otimes \mathbb{R}^2 \sim \mathbb{R}^4 \sim \mathrm{M}_2(\mathbb{R})$: the first $\mathbb{R}^2$ is the Hilbert space on which act the states $\rho^M_{s,\theta}$ of a **quantum polariser**.
- The second $\mathbb{R}^2$ is the Hilbert space on which act the **partially linearized polarisation states** $\rho^L_{r,\phi}$ of the plane wave crossing the polariser.
- Its spectral decomposition corresponds to the incoherent superposition of two completely linearly polarised waves

$$\rho^L_{r,\phi} = \frac{1+r}{2} P_\phi + \frac{1-r}{2} P_{\phi+\pi/2}\,, \quad P_\phi := |\phi\rangle\langle\phi|\,.$$

von Neumann interaction polariser-partially linear polarised light (continued)

- Action of the polariser on the light states : quantum observable A^L with spectral decomposition

$$A^L = \lambda_{\parallel}\, P_\gamma + \lambda_\perp\, P_{\gamma+\pi/2}\,,$$

where the direction P_γ is the filter's orientation, while total absorption occurs along the direction $\gamma+\pi/2$.

- The interaction polariser-light generates (weak or with no collapse) measurement whose time duration is the interval $I_M=(t_M-\eta, t_M+\eta)$ centred at t_M is described by the (pseudo-) Hamiltonian operator

$$\widetilde{H}_{\text{int}}(t) = g^\eta_M(t)\tau_2\otimes A^L\,, \quad \tau_2 = \begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}\,,$$

where g^η_M is an approximation of a Dirac peak and has support in I_M, *i.e.*, $\lim_{\eta\to 0}\int_{-\infty}^{+\infty}\mathrm{d}t\, f(t)\, g^\eta_M(t) = f(t_M)$

Interaction polariser-partially linear polarised light as a quantum measurement (continued)

- The unitary evolution operator is defined for $t_0 < t_M - \eta$ as

$$U(t,t_0) = \exp\left[\int_{t_0}^{t} \mathrm{d}t'\, g_M^{\eta}(t')\, \tau_2 \otimes A^L\right] = \exp\left[G_M^{\eta}(t)\, \tau_2 \otimes A^L\right] \,,$$

with $G_M^{\eta}(t) = \int_{t_0}^{t} \mathrm{d}t' \;\; g_M^{\eta}(t')$.

For $t_0 < t_M - \eta$ and $t > t_M + \eta$, we obtain

$$U(t,t_0) = \mathcal{R}\left(\lambda_{\parallel}\right) \otimes P_{\gamma} + \mathcal{R}\left(\lambda_{\perp}\right) \otimes P_{\gamma+\pi/2} \,.$$

Interaction polariser-partially linear polarised light as a quantum measurement (continued II)

- The evolution of the initial state for the system polariser-light reads for $t > t_M + \eta$

$$U(t,t_0)\,\rho^M_{s_0,\theta_0}\otimes\rho^L_{r_0,\phi_0}\,U(t,t_0)^\dagger$$
$$= \rho^M_{s_0,\theta_0+\lambda_\parallel}\otimes\frac{1+r_0\cos 2(\gamma-\phi_0)}{2}\,P_\gamma + \rho^M_{s_0,\theta_0+\lambda_\perp}\otimes\frac{1-r_0\cos 2(\gamma-\phi_0)}{2}\,P_{\gamma+\pi/2}$$
$$+\frac{1}{4}\left(\mathcal{R}\left(\lambda_\parallel-\lambda_\perp\right)+s_0\sigma_{2\theta_0+1}\right)\otimes r_0\sin 2(\gamma-\phi_0)\,P_\gamma\tau_2$$
$$-\frac{1}{4}\left(\mathcal{R}\left(\lambda_\perp-\lambda_\parallel\right)+s_0\sigma_{2\theta_0+1}\right)\otimes r_0\sin 2(\gamma-\phi_0)\,\tau_2 P_\gamma\,.$$

- Probability for the detector to rotate by $\lambda_\parallel$, $\sim$ polarisation along γ :

$$\mathrm{Tr}\left[\left(U(t,t_0)\,\rho^M_{s_0,\theta_0}\otimes\rho^L_{r_0,\phi_0}\,U(t,t_0)^\dagger\right)(\mathbb{1}\otimes P_\gamma)\right]$$
$$=\mathrm{Tr}\left[\rho^L_{r_0,\phi_0}P_\gamma\right]=\frac{1-r_0}{2}+r_0\cos^2(\gamma-\phi_0)\,,$$

and along $\gamma+\pi/2$:

$$\mathrm{Tr}\left[\left(U(t,t_0)\,\rho^M_{s_0,\theta_0}\otimes\rho^L_{r_0,\phi_0}\,U(t,t_0)^\dagger\right)(\mathbb{1}\otimes P_{\gamma+\pi/2})\right]=\frac{1-r_0}{2}+r_0\sin^2(\gamma-\phi_0)\,,$$

- For the completely linear polarisation of the light, i.e. $r_0=1$, we recover the familiar **Malus** (1808) laws, $\cos^2(\gamma-\phi_0)$ and $\sin^2(\gamma-\phi_0)$ respectively.

After interaction polariser-partially linearly polarised light

- After interaction light polarisation is described by the density matrix obtained by tracing out the polariser part :

$$\rho^L_{r',\phi'} = \mathrm{Tr}_M \left[U(t,t_0)\, \rho^M_{s_0,\theta_0} \otimes \rho^L_{r_0,\phi_0}\, U(t,t_0)^\dagger \right] = \frac{1+r'}{2} P_{\phi'} + \frac{1-r'}{2} P_{\phi'+\pi/2} \,.$$

- New parameters for light state :

$$r' = r_0 \sqrt{\cos^2 2(\gamma-\phi_0) + \cos^2(\lambda_\parallel - \lambda_\perp) \sin^2 2(\gamma-\phi_0)} \leq r_0$$

$$\tan 2\phi' = \frac{\tan 2\gamma - \cos(\lambda_\parallel - \lambda_\perp) \tan 2(\gamma-\phi_0)}{1+\cos(\lambda_\parallel - \lambda_\perp) \tan 2(\gamma-\phi_0) \tan 2\gamma} \,.$$

- Observe that the initial state $\rho^L_{r_0,\phi_0}$ of the light is not modified (up to a right angle) if the polariser aligns along the linear polarisation of the light (up to a right angle) :

$$\gamma = \phi_0 \quad \text{or} \quad \phi_0 \pm \pi/2 \Rightarrow r' = r_0 \,, \quad \text{and} \quad \phi' = \phi_0 \quad \text{or} \quad \phi_0 \pm \pi/2$$

After interaction polariser-partially linearly polarised light (continued)

- In order to physically represent the rôle of the polariser as forcing the light to be partially polarised along its eigendirection γ, the angle ϕ' of the light state after the action of the polariser must be the same as γ.
- From this constraint we obtain

$$\tan 2\phi' = \frac{\tan 2\gamma - \cos(\lambda_{||} - \lambda_{\perp})\tan 2(\gamma - \phi_0)}{1 + \cos(\lambda_{||} - \lambda_{\perp})\tan 2(\gamma - \phi_0)\tan 2\gamma} = \tan 2\gamma\,,$$

which holds if $\lambda_{||} - \lambda_{\perp} = (2k+1)\frac{\pi}{2}$

- It follows for the parameter r' the attenuation :

$$r' = r_0\sqrt{\cos^2 2(\gamma - \phi_0) + \cos^2(\lambda_{||} - \lambda_{\perp})\sin^2 2(\gamma - \phi_0)} = r_0\left|\cos 2(\gamma - \phi_0)\right|\,,$$

which gives the **Malus law, 1808** :

$$\frac{1 + r'}{2} = \frac{1 - r_0}{2} + r_0\cos^2(\gamma - \phi_0) \quad \text{for} \quad \gamma \leq \phi_0 + \frac{\pi}{2}\,.$$

RESULTS

Our program : algorithm for minimal complexity

- Consider N gates (i.e. quantum polarisers), $\{\Gamma_0 = \Gamma_{\text{ref}}\,,\,\Gamma_1\,,\,\ldots\,,\,\Gamma_{N-1}\}$ through the path followed by the state $\rho_{r,\phi}$ from the referent $\rho_{r_\text{R},\phi_\text{R}}$ to the target $\rho_{r_\text{T},\phi_\text{T}}$.
- Our aim is to approximate the straight lines geodesics with some trajectories with **constant parameters**.
- We consider an isotropic environment, where the decay rates are constant $h_1 = h_3 = 1 \Leftrightarrow \alpha = 0\,,\,\beta = 2$, and we choose a constant Hamiltonian with $\mathcal{E} = -2$.
- With no action of a polariser, the system evolves according to the Lindblad equations, where it will naturally change the polarisation while depolarising it. Therefore, we apply a sequence of quantum polarisers in order to keep the state close to the geodesics.
- **Then, rather than calculating the distance from the trajectory to the geodesic at each moment, we compare the geodesic state with the time-evolving state at the same radial coordinate r. It results the simplified expression for the trace distance**

$$d_{\text{geo}}\left(\rho_{r,\phi}, \rho_{r'=r,\phi_{\text{geo}}}\right) = r\left|\sin(\phi - \phi_{\text{geo}})\right|$$

Algorithm for minimal complexity (continued)

- The objective is to identify the minimum number of gates Γ_i necessary to maintain the constant parameter path **within a defined accuracy threshold** ϵ

$$d_{\text{geo}}\left(\rho_{r,\phi}, \rho_{r,\phi_{\text{geo}}}\right) \leq \epsilon.$$

- Therefore, the strategy consists of applying the polariser every time the state reaches this maximum distance.
- We choose the following representative cases.

Example	ϕ_R	r_R	ϕ_T	r_T
a	0	1	$\pi/6$	0.5
b	$\pi/3$	1	$\pi/2$	0.5
c	$\pi/4$	1	$\pi/12$	0.5
d	$11\pi/12$	1	$3\pi/4$	0.5

Table – Values for the reference state variables ϕ_R, r_R (all are for pure states), and the target state variables ϕ_T, r_T.

Algorithm for minimising loss function (continued)

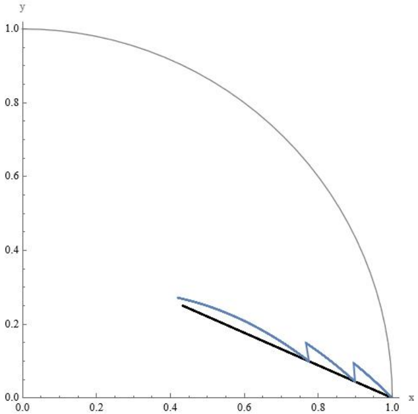


Figure – The black line represents the geodesic related to the trace distance and for example (a) of Table 1, and the blue curves represent the trajectory of the quantum state evolved accordingly with GLKS equations. **When the distance reaches a chosen accuracy** $\epsilon = 0.05$, we place a quantum polariser to change the angle of the polarisation to be the same as the one at the geodesics. The action of the polarization can be viewed as the abruptly change in the quantum state path.

Algorithm for minimising loss function (continued)

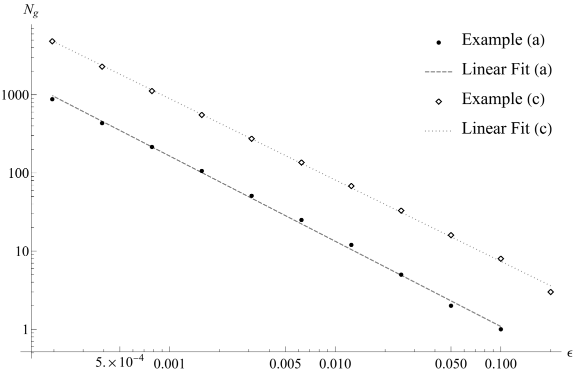


Figure – The Log-Log plot of the number of polarisers for each accuracy related to examples (a) and (c) of the table 1. The metric here is defined by the trace distance. The pairs of examples (a) and (b), and (c) and (d) have the same data. Therefore, they share the same results for the estimates of the linear fit : (a) $m = -1.05469$, $n = -1.09012$, (c) $m = -0.171026$, $n = -1.04029$. The examples (b) and (d) are omitted in the figure.

Observations

- In examples (a) and (b) the same number of gates is necessary to maintain the path close to the geodesics for any given accuracy parameter. (c) and (d) behave analogously.
- The difference between the pairs of examples (a,b) and (c,d) exists due to the variation of ϕ of the geodesics and the one with constant parameters path.
- In other words, in examples (a,b) both the geodesics and the trajectories given by Lindblad equations have $\dot{\phi} > 0$, while the geodesics of the examples (c,d) have $\dot{\phi} < 0$.
- It is possible to observe via the parameters of the linear fit $\log N_g = m + n \log \epsilon$, that in both cases (a,b and c,d) the circuits obey a power law

$$\begin{aligned} N_g &\approx 10^{-1.05} \times \epsilon^{-1.09} \quad \text{examples (a, b)}, \\ N_g &\approx 10^{-0.17} \times \epsilon^{-1.04} \quad \text{examples (c, d)}. \end{aligned}$$

Conclusion

- Our aim was to obtain the orbit of a dynamical map with Lindblad dynamical map acting on a reference state ρ_R to a target state ρ_T in the upper half-unit disk.
- The path of **minimum complexity** was identified as the **geodesic** (i.e., straight line segment) in the half-unit disk for the trace distance metric.
- The quantum circuit which approximates this dynamical map is a finite sequence of quantum operations or gates, that we identified as **quantum polarisers**, such that the orbit of the circuit in the half-disk is close to the geodesic up to a certain tolerance.
- The chosen gates for the quantum circuit come from solutions of the Lindblad with constant parameters defining the control functions for the circuit.
- The number of gates scales with the accuracy follows a **power law** $N_g = \mathcal{O}(\epsilon^{-1})$. This behavior aligns with the typical patterns seen in **efficient** algorithms, avoiding exponential increases.

HIGHER DIMENSIONS

Comparing with $\mathbb{C}^2$ (spin $1/2$ or other two-level quantum systems)

- Generic quantum states now read (with $\ell(\phi) = \begin{pmatrix} e^{\mathrm{i}\phi/2} & 0 \\ 0 & e^{-\mathrm{i}\phi/2} \end{pmatrix}$:

$$\rho_{r,\theta,\phi} = \frac{1}{2}\begin{pmatrix} 1 + r\cos\theta & r\sin\theta\, e^{\mathrm{i}\phi} \\ r\sin\theta\, e^{-\mathrm{i}\phi} & 1 - r\cos\theta \end{pmatrix} = \frac{1}{2}\left(\mathbb{1}_2 + r\ell(\phi)\sigma_\theta\ell(-\phi)\right) .$$

- Note the extreme cases, $r = 1$ (pure state $\equiv$ spin coherent state), and $r = 0$ (totally random state) :

$$\rho_{1,\theta,\phi} = |\theta,\phi\rangle\langle\theta,\phi| \, , \quad r_0 = \frac{1}{2}\mathbb{1}_2 \, . \tag{4}$$

Like for the $\mathbb{R}^2$ case, the parameter r can be viewed as the inverse of a “noise” temperature $r \propto 1/T$.

- Lindblad equation

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{i}}{\hbar}[H,A] + \sum_k h_k \left[L_k A L_k^\dagger - \frac{1}{2}\left(A L_k^\dagger L_k + L_k^\dagger L_k A\right)\right] ,$$

with $A = \rho_r(\theta,\phi)$. The Hamiltonian $H = H(t) = \begin{pmatrix} h_{00} & h_{01} \\ \overline{h_{01}} & h_{11} \end{pmatrix}$ is now is a Hermitian 2×2-matrix and the basic operators L_k are chosen as

$$L_1 = \sigma_1 \, , \; L_2 = -\mathrm{i}\sigma_2 = \tau_2 = e_2 \, , \; L_3 = \sigma_3 \, ,$$
$$L_4 = \mathrm{i}\sigma_1 = e_1 \, , \; L_5 = \sigma_2 \, , \; L_6 = \mathrm{i}\sigma_3 = e_3 \, .$$

Comparing with $\mathbb{C}^2$ (spin $1/2$ or other two-level quantum systems) continued

- We obtain (with $\hbar = 1$) :

$$\frac{\mathrm{d}\rho_{r,\theta,\phi}}{\mathrm{d}t} = \mathrm{i}\,\left[H, \rho_{r,\theta,\phi}\right] + 2h_1\left(\sigma_1\rho_{r,\theta,\phi}\sigma_1 - \rho_{r,\theta,\phi}\right) + \\ + 2h_2\left(\sigma_2\rho_{r,\theta,\phi}\sigma_2 - \rho_{r,\theta,\phi}\right) + 2h_3\left(\sigma_3\rho_{r,\theta,\phi}\sigma_3 - \rho_{r,\theta,\phi}\right)\,.$$

- and the corresponding dynamical system with $\theta \neq 0, \pi$:

$$\dot{\phi} = \delta - (l_1 - l_2)\sin 2\phi - \cot\theta(\xi\cos\phi + \eta\sin\phi)\,,$$
$$\dot{\theta} = (1 + \sin^2\theta)(-\xi\sin\phi + \eta\cos\phi) + \sin 2\theta\left(l_1 + l_2\sin^2\phi - l_3(1 + \sin^2\phi)\right)\,,$$
$$\frac{\dot{r}}{r} = -2l_1\cos^2\theta - 2l_2(\sin^2\theta\cos^2\phi + \cos^2\theta) - 2l_3\sin^2\theta(1 + \sin^2\phi)\,.$$

with

$$\delta = \frac{1}{2}(h_{00} - h_{11}) \quad \xi = \mathrm{Re}h_{01}\,, \quad \eta = \mathrm{Im}h_{01}\,,$$
$$l_1 = h_1 + h_4 \geq 0\,, \quad l_2 = h_2 + h_5 \geq 0\,, \quad l_3 = h_3 + h_6 \geq 0\,.$$

Quantum states in $\mathbb{R}^n$ (real "Qudit")

- A pure state is represented by an element of $\mathbb{S}^{n-1}/\mathbb{Z}_2$, and a mixed state by the general density matrix

$$\rho \equiv \rho_{\boldsymbol{\eta},\boldsymbol{\phi}} = \frac{1}{n}\mathbb{1}_n + \mathcal{R}(\boldsymbol{\phi})\,\mathsf{D}(\boldsymbol{\eta})\,{}^t\mathcal{R}(\boldsymbol{\phi})\,,$$

- Symbol D stands for the diagonal matrix

$$\mathsf{D}(\boldsymbol{\eta}) = \begin{pmatrix} \eta_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \eta_n \end{pmatrix}\,,$$

  where the vector $\boldsymbol{\eta} = {}^t(\eta_1, \eta_2, \cdots, \eta_n)$ lies in the simplex defined by the conditions $\sum_{i=1}^n \eta_i = 0\,, \quad -\frac{1}{n} \leq \eta_i \leq 1 - \frac{1}{n}$

- $\mathcal{R}(\boldsymbol{\phi}) \in \mathrm{SO}(n)$ and $\boldsymbol{\phi}$ stands for the $n(n-1)/2$-uple of (angular) parameters of SO(n). (we could use parameters $r_i = n\eta_i$ to stick better with the notations of the $2d$ case.)
- Thus, the manifold of density matrices is real with dimension equal to $(n-1)(n+2)/2$.

Quantum orientations in $\mathbb{R}^n$

- In the non-degenerate case, the matrix $\mathcal{R}(\boldsymbol{\phi}) \in \mathrm{SO}(n)$ reads as the row of n orthonormal vectors forming a basis of $\mathbb{R}^n$,

$$\mathcal{R}(\boldsymbol{\phi}) = (\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_n) \,.$$

- By construction, the vector $\boldsymbol{u}_i$ is eigenvector of $\rho_{\boldsymbol{\eta},\boldsymbol{\phi}}$ for the eigenvalue $\lambda_i = 1/n + \eta_i$, or, equivalently, eigenvector of the matrix

$$\sigma_{\boldsymbol{u},\boldsymbol{\phi}} = \mathcal{R}(\boldsymbol{\phi})\, \mathsf{D}(\boldsymbol{\eta})\, {}^t\mathcal{R}(\boldsymbol{\phi}) \,,$$

for the eigenvalue η_i.

- Hence we have the decompositions in terms of pure states :

$$\rho_{\boldsymbol{\eta},\boldsymbol{\phi}} = \sum_{i=1}^{n} \left(\frac{1}{n} + \eta_i\right) |\boldsymbol{u}_i\rangle\langle\boldsymbol{u}_i| \,, \quad \sigma_{\boldsymbol{u},\boldsymbol{\phi}} = \sum_{i=1}^{n} \eta_i \, |\boldsymbol{u}_i\rangle\langle\boldsymbol{u}_i| \,.$$

- In the present context, these pure states are viewed as quantum orientations in $\mathbb{R}^n$.

GKLS equation in $\mathbb{R}^n$

- Let $M_{ij} = -M_{ji}$, $1 \leq i < j \leq n$, be the $n(n-1)/2$ generators of the rotations $\mathcal{R}(\boldsymbol{\phi}) \in \mathrm{SO}(n)$. M_{ij} is for rotation in the 2-plane (i,j), and its matrix elements are given by
$$(M_{ij})_{kl} = \delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}\,.$$
- They span the Lie algebra $\mathfrak{so}(n)$, and obey the commutation rules
$$[M_{ij}, M_{kl}] = \delta_{ik}M_{jl} + \delta_{jl}M_{ik} - \delta_{il}M_{jk} - \delta_{jk}M_{il}\,.$$
- Hence, any rotation can be written as
$$\mathcal{R}(\boldsymbol{\phi}) = \exp\left[\sum_{1\leq i<j\leq n} \phi_{ij} M_{ij}\right]\,.$$
- The GKLS equation for $\rho_{\boldsymbol{\eta},\boldsymbol{\phi}}$ generalises to
$$\frac{\mathrm{d}\rho_{\boldsymbol{\eta},\boldsymbol{\phi}}}{\mathrm{d}t} = \frac{1}{\hbar}\,[\rho_{\boldsymbol{\eta},\boldsymbol{\phi}}, \widetilde{H}] + \sum_{1\leq k\leq n^2-1} h_k \left[L_k \rho_{\boldsymbol{\eta},\boldsymbol{\phi}}\, {}^{\mathrm{t}}L_k - \frac{1}{2}\left(\rho_{\boldsymbol{\eta},\boldsymbol{\phi}} L_k\,{}^{\mathrm{t}}L_k + L_k\,{}^{\mathrm{t}}L_k \rho_{\boldsymbol{\eta},\boldsymbol{\phi}}\right)\right]\,,$$
- in which the pseudo-Hamiltonian $\widetilde{H}$ is a "magnetic-like" element of $\mathfrak{so}(n)$:
$$\widetilde{H} = \sum_{1\leq i<j\leq n} B_{ij} M_{ij}\,,$$
and the $n^2 - 1$ L_k's form with the identity $\mathbb{1}_n$ a basis of the Lie algebra $\mathfrak{sl}(\mathbb{R}, n)$. All coefficients h_k are non-negative.

GKLS equation in $\mathbb{R}^n$ (continued)

- Now, from the Cartan decomposition of $\mathfrak{sl}(\mathbb{R}, n)$,

$$\mathfrak{sl}(\mathbb{R}, n) = \mathfrak{so}(n) \oplus \mathfrak{p}(n)\,,$$

where $\mathfrak{p}(n)$ is the vector subspace of all traceless symmetric matrices.

- We know that

$$[\mathfrak{so}(n), \mathfrak{so}(n)] = \mathfrak{so}(n)\,, \quad [\mathfrak{so}(n), \mathfrak{p}(n)] \subset \mathfrak{p}(n)\,, \quad [\mathfrak{p}(n), \mathfrak{p}(n)] \subset \mathfrak{so}(n)\,.$$

- A basis for $\mathfrak{p}(n)$ can be chosen as the set of $\frac{(n-1)(n+2)}{2}$ matrices S_i, $0 \leq i \leq n-1$, S_{ij}, $0 \leq i < j \leq n$, with respective matrix elements :

$$(S_i)_{kl} = \delta_{kl}\left(\delta_{ki} - \delta_{ki+1}\right)\,, \quad (S_{ij})_{kl} = \delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}\,.$$

- Hence the $n^2 - 1$ L_k's can be picked in the set $\{M_{ij}\,, S_i\,, \,, S_{ij}\}$.

Quantum orientations in $\mathbb{R}^3$ ("Qutrit")

- The case $n = 3$ is remarkable since it concerns our daily experience : it could be applied to the mixture of the three basic colours, red, blue, green, red, or the three symbols of Morse alphabet, or the three quark colors, or the three components of the angular momentum vector, or to any system whose description request three parameters.
- Let us stick to orientations in our space, viewed as the quantum states by the states

$$\rho_{\boldsymbol{\eta},\boldsymbol{\phi}} = \frac{1}{3}\mathbb{1}_3 + \mathcal{R}(\boldsymbol{\phi})\,\mathsf{D}(\boldsymbol{\eta})\,{}^t\mathcal{R}(\boldsymbol{\phi})\,,$$

- The three generators of a rotation have respective matrix representations :

$$M_{12} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad M_{31} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

- By putting $M_1 \equiv M_{23}$, $M_2 \equiv M_{31}$, $M_3 \equiv M_{12}$, one checks

$$\left(\sum_{i=1}^{3} v_i M_i\right)\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \boldsymbol{v} \times \boldsymbol{w}\,,$$

- i.e. vector $\boldsymbol{v}$ is like a magnetic field and $\sum_{i=1}^{3} v_i M_i$ is its representation as the magnetic part of the electromagnetic tensor field $T^{\mu\nu}$.

QUANTUM MEASUREMENT
PV, POVM, NAIMARK, AND ALL THAT

Definition of POVM

- A normalized Positive-Operator Valued measure (POVM) is a map $F : \mathcal{B}(\Omega) \to \mathcal{L}_s^+(\mathcal{H})$ from the Borel σ-algebra of a topological space Ω to the space of linear positive self-adjoint operators such that :

$$F\left(\bigcup_{n=1}^{\infty} \Delta_n\right) = \sum_{n=1}^{\infty} F(\Delta_n) \ , \tag{5}$$

$$F(\Omega) = \mathbb{1} \ , \tag{6}$$

  where $\{\Delta_n\}$ is a countable family of disjoint sets in $\mathcal{B}(\Omega)$ and the series converges in the weak operator topology.
- The POVM is said to be real if $\Omega = \mathbb{R}$.
- A projection-valued measure (PVM) is a POVM such that $F(\Delta)$ is a projection operator for every $\Delta \in \mathcal{B}(\Omega)$.

Characterization of quantum measurement

- Quantum framework : a complex and separable Hilbert space $\mathcal{H}$ is associated with each system, and the states are represented by density operators, *i.e.* non-negative, bounded self-adjoint operators with trace 1.
- (Holevo) There is a one-to-one correspondence between POVMs $F : \mathcal{B}(\Omega) \to \mathcal{L}_s^+(\mathcal{H})$ and affine maps $S(\mathcal{H}) \mapsto \mathcal{M}_+(\Omega)$ from states to probability measures which is given by

$$\mu(\Delta) = Tr(\rho F(\Delta)) \,,$$

$$\Delta \in \mathcal{B}(\Omega) \,, \quad \rho \in S(\mathcal{H}) \,, \quad \mu \in \mathcal{M}_+(\Omega)$$

- More in P. Busch et al. Quantum measurement. Vol. 23. Springer, 2016

Naimark theorem and compatibility of POVMs

- (Naimark, 1943) Let $F : \mathcal{B}(\Omega) \to \mathcal{L}_s^+(\mathcal{H})$ be a POVM in a Hilbert space $\mathcal{H}$. Then, there is an extended Hilbert space $\mathcal{H}^+$ and a projector-valued (PV) measure $E^+ : \mathcal{B}(\Omega) \to \mathcal{L}_s(\mathcal{H}^+)$ such that

$$PE^+(\Delta)\psi = F(\Delta)\psi, \quad \psi \in \mathcal{H}, \quad \Delta \in \mathcal{B}(\Omega)\,,$$

  where P is the projection operator onto $\mathcal{H}$.

- Naimark's theorem provides a necessary and sufficient condition for the compatibility of two POVMs

- **Definition** : two POVMs $F_1 : \mathcal{B}(\Omega_1) \to \mathcal{L}_s^+(\mathcal{H})$ and $F_2 : \mathcal{B}(\Omega_2) \to \mathcal{L}_s^+(\mathcal{H})$ are compatible if there is a third POVM $F : \mathcal{B}(\Omega_1 \times \Omega_2) \to \mathcal{L}_s^+(\mathcal{H})$ of which F_1 and F_2 are the marginals, *i.e.*,

$$F_1(\Delta_1) = F(\Delta_1 \times \Omega_2), \quad F_2(\Delta_2) = F(\Omega_1 \times \Delta_2).$$

- **Theorem** Two POVMs $F_1 : \mathcal{B}(\Omega_1) \to \mathcal{L}_s^+(\mathcal{H})$ and $F_2 : \mathcal{B}(\Omega_2) \to \mathcal{L}_s^+(\mathcal{H})$ are *compatible* if and only if there are two Naimark extensions $E_1^+ : \mathcal{B}(\Omega_1) \to \mathcal{L}_s(\mathcal{H}^+)$ and $E_2^+ : \mathcal{B}(\Omega_2) \to \mathcal{L}_s(\mathcal{H}^+)$ such that $[E_1^+, E_2^+] = 0$.

- More in R. Beneduci. "Joint measurability through Naimark's dilation theorem", Reports on Mathematical Physics 79.2 (2017)

INTEGRAL QUANTIZATION

Integral quantization

- Integral quantization [7] of $f(x)$ on a measure space (X, ν) is the linear map :

$$f \mapsto A_f = \int_X \mathsf{M}(x) f(x)\, \mathrm{d}\nu(x)\,, \tag{7}$$

where the family of operators $\mathsf{M}(x)$ solves the identity as

$$X \ni x \mapsto \mathsf{M}(x)\,, \quad \int_X \mathsf{M}(x)\, \mathrm{d}\nu(x) = \mathbb{1}\,. \tag{8}$$

- If the $\mathsf{M}(x)$ are non-negative, the quantum operator related to the characteristic function on Δ, $A(\chi_\Delta)$, defines a POVM through the quantization map

$$F(\Delta) := A(\chi_\Delta) = \int_X \mathsf{M}(x)\, \chi_\Delta(x)\, \mathrm{d}\nu(x) = \int_\Delta \mathsf{M}(x)\, \mathrm{d}\nu(x)\,, \tag{9}$$

- Therefore, two key roles of POVMs : they are the mathematical representatives of observables and they provide a quantization procedure.

7. J.-P. G. & H. Bergeron, Integral quantizations with two basic examples, Annals of Physics 344 (2014)

Semi-classical portraits

- A quantum model for a system is expected to have a classical or semi-classical counterpart. We say there exists a *dequantization* map acting on an operator A_f to give back the original $f(x)$, in general with some corrections.
- Given two families $\mathsf{M}_\mathrm{a}(x)$ (for "analysis") and $\mathsf{M}_\mathrm{r}(x)$ (for "reconstruction") resolving the identity, one defines

$$A_f \mapsto \check{f}(x) := \mathrm{tr}(\mathsf{M}_\mathrm{r}(x)\, A_f) = \int_X \mathrm{tr}(\mathsf{M}_\mathrm{r}(x)\mathsf{M}_\mathrm{a}(x'))\, f(x')\, \mathrm{d}\nu(x')\,, \tag{10}$$

- Choosing $\mathsf{M}_\mathrm{a}(x) = \rho_\mathrm{a}(x)$ and $\mathsf{M}_\mathrm{r}(x) = \rho_\mathrm{r}(x)$ to be density matrices, the map $x' \mapsto \mathrm{tr}(\rho_\mathrm{r}(x)\rho_\mathrm{a}(x'))$ defines a probability distribution $\mathrm{tr}(\rho_\mathrm{r}(x)\rho_\mathrm{a}(x'))$ on the measure space $(X, \mathrm{d}\nu(x'))$
- Thus, the expectation value of the operator A_f, $\check{f}(x)$, is built from the analysis POVMs based on the $\mathsf{M}_\mathrm{a}(x)$'s and given by :

$$f(x) \mapsto \check{f}(x) = \int_X f(x')\, \mathrm{tr}(\rho_\mathrm{r}(x)\rho_\mathrm{a}(x'))\, \mathrm{d}\nu(x')\,. \tag{11}$$

- The map (11) represents in general a regularization of the original, possibly extremely singular, f. Standard cases are for $\rho_\mathrm{a} = \rho_\mathrm{r}$, in particular for rank-one density matrices (coherent states).

Integral quantization with 2×2 real density matrices

- The measure space $(X, \mathrm{d}\nu(x))$ for the Euclidean plane is the unit circle with its uniform (Lebesgue) measure :

$$X = \mathbb{S}^1\,, \quad \mathrm{d}\nu(x) = \frac{\mathrm{d}\phi}{\pi}\,, \quad \phi \in [0, 2\pi)\,.$$

- Resolution of the identity by the density matrices $\rho_{r,\phi+\phi_0}$, ϕ_0 arbitrary,

$$\int_0^{2\pi} \rho_{r,\phi+\phi_0}\, \frac{\mathrm{d}\phi}{\pi} = \mathbb{1}_2\,.$$

- Quantization of a function (or distribution) $f(\phi)$ on the circle.

$$f \mapsto A_f = \int_0^{2\pi} f(\phi)\rho_{r,\phi+\phi_0}\, \frac{\mathrm{d}\phi}{\pi} = \begin{pmatrix} \langle f\rangle + \frac{r}{2} C_c\left(R_{\phi_0} f\right) & \frac{r}{2} C_s\left(R_{\phi_0} f\right) \\ \frac{r}{2} C_s\left(R_{\phi_0} f\right) & \langle f\rangle - \frac{r}{2} C_c\left(R_{\phi_0} f\right) \end{pmatrix}$$
$$= \langle f\rangle\, \mathbb{1} + \frac{r}{2}\left[C_c\left(R_{\phi_0} f\right)\, \sigma_3 + C_s\left(R_{\phi_0} f\right)\, \sigma_1\right]\,,$$

where $\langle f\rangle := \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\, \mathrm{d}\phi$ is the average of f on the unit circle and $R_{\phi_0}(f)(\phi) := f(\phi - \phi_0)$.

- C_c and C_s : cosine and sine doubled angle Fourier coefficients of f,

$$C_c(f) = \int_0^{2\pi} f(\phi) \cos 2\phi\, \frac{\mathrm{d}\phi}{\pi}\,, \quad C_s(f) = \int_0^{2\pi} f(\phi) \sin 2\phi\, \frac{\mathrm{d}\phi}{\pi}\,.$$

An outcome as typical of quantum-mechanical ensembles

- As a consequence, representation of a given mixed state as a continuous superposition of mixed states :

$$\rho_{s,\theta} = \int_0^{2\pi} \underbrace{\left[\frac{1}{2} + \frac{s}{r}\,\cos 2\phi\right]}_{f(\phi)} \rho_{r,\phi+\theta}\,\frac{\mathrm{d}\phi}{\pi}\,,$$

- It is convex for $r \geq 2s$ and $\phi \mapsto \left[\frac{1}{2} + \frac{s}{r}\,\cos 2\phi\right]$ is a probability distribution.
- It provides one more illustration of a *typical property of quantum-mechanical ensembles in comparison with their classical counterparts.*

A non-commutative reading of $\mathbb{R}^3$

- Let us identify $\mathbb{R}^3$ with the subspace in $L^2(\mathbb{S}^1, \mathrm{d}\phi/\pi)$,

$$\mathbb{R}^3 \sim V_3 = \mathrm{Span}\left\{ e_0(\phi) := \frac{1}{\sqrt{2}}, e_1(\phi) := \cos 2\phi, e_2(\phi) := \sin 2\phi \right\}$$

- Then, the integral quantization map with $\rho_{r,\phi+\phi_0}$ yields a non-commutative version of $\mathbb{R}^3$

$$\begin{aligned} A_{e_0} &= \frac{\mathbb{1}}{\sqrt{2}}\,, \\ A_{e_1} &= \frac{r}{2}[\cos 2\phi_0\, \sigma_3 + \sin 2\phi_0\, \sigma_1] \equiv \frac{r}{2}\sigma_{2\phi_0}\,, \\ A_{e_2} &= \frac{r}{2}[-\sin 2\phi_0\, \sigma_3 + \cos 2\phi_0\, \sigma_1] \equiv \frac{r}{2}\sigma_{2\phi_0+\pi/2}\,, \end{aligned}$$

- Non-zero commutation rule

$$[A_{e_1}, A_{e_2}] = -\frac{r^2}{2}\tau_2\,, \quad \tau_2 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\mathrm{i}\sigma_2\,.$$

Integral quantization on the circle and Toeplitz-Naimark formalism

- Remind (Naimark, 1943) Let $F : \mathcal{B}(\Omega) \rightarrow \mathcal{L}_s^+(\mathcal{H})$ be a POVM in a Hilbert space $\mathcal{H}$. Then, there is an extended Hilbert space $\mathcal{H}^+$ and a projector-valued (PV) measure $E^+ : \mathcal{B}(\Omega) \rightarrow \mathcal{L}_s(\mathcal{H}^+)$ such that

$$PE^+(\Delta)\psi = F(\Delta)\psi, \quad \psi \in \mathcal{H}, \quad \Delta \in \mathcal{B}(\Omega)\,,$$

  where P is the projection operator onto $\mathcal{H}$.
- The quantization of the circle offers a nice illustration of this theorem in combination with Toeplitz quantization.
- One shows [8] that there exist orthogonal projectors from $L^2(\mathbb{S}^1, \mathrm{d}\phi/\pi)$ to $\mathbb{R}^2$ such that for a function $f(\phi)$ the multiplication operator on $L^2(\mathbb{S}^1, \mathrm{d}\phi/\pi)$ defined by

$$v \mapsto M_f v = fv\,.$$

  map M_f to A_f.

8. Beneduci et al, AOP, 2022

Sequential measurements and dichotomic POVM

Summary of our other results in our AOP article (2022) :

- A dichotomic POVM F is a pair $F = \{A, \mathbb{1} - A\}$ with A an effect, *i.e.* a symmetric operator A such that $0 \leq A \leq \mathbb{1}$, but not a projection operator (it is a multiple of a projection operator).
- We show that two measurements in which a light ray goes first through an oblique polarizer before passing through a vertical polarizer is described by a dichotomic POVM, while a measurement in the reverse order is described by another dichotomic POVM, showing the incompatibility of the measurement procedures.
- We also find the necessary condition for the compatibility of two dichotomic POVMs in a real bidimensional Hilbert space.
- Finally we relate the density matrix's parameters to the Stokes parameters defining a polarization tensor for linearly polarized light. It turns out that we can identify the degree of mixing of a density matrix with the fuzziness of a quantum observable.
- In conclusion compatibility conditions of two POVMs can be expressed in terms of Stokes parameters of the polarization matrix.

ENTANGLEMENT ISOMORPHISMS AND BELL INEQUALITIES

$2 \times 2 = 2 + 2$

- Quantum entanglement of states is a logical consequence of the construction of tensor products of Hilbert spaces for describing quantum states of composite systems.
- In the present case, we are in presence of a remarquable sequence of vector space isomorphisms due to the fact that $2 \times 2 = 2 + 2$ [9] :

$$\mathbb{R}^2 \otimes \mathbb{R}^2 \cong \mathbb{R}^2 \times \mathbb{R}^2 \cong \mathbb{R}^2 \oplus \mathbb{R}^2 \cong \mathbb{C}^2 \cong \mathbb{H}\,,$$

  where $\mathbb{H}$ is the field of quaternions.
- It is straightforward to transpose into the present setting the 1964 analysis and result presented by Bell in his discussion about the EPR paper and about the subsequent Bohm's approaches based on the assumption of hidden variables.
- Just replace the Bell spin one-half particles with the horizontal (i.e., $+1$) and vertical (i.e., -1) quantum orientations in the plane as the only possible issues of the observable $\sigma_\phi = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix}$ supposing that there exists a pointer device designed for measuring such orientations with outcomes ± 1 only.

9. Remind that $\dim(V \otimes W) = \dim V \dim W$ while $\dim(V \times W) = \dim V + \dim W$ for 2 finite-dimensional vector spaces V and W

Bell states

- Let us first write the (canonical) orthonormal basis of the tensor product $\mathbb{R}^2_A \otimes \mathbb{R}^2_B$, the first factor being for system "A" and the other for system "B", as

$$|0\rangle_A \otimes |0\rangle_B\,, \quad \left|\frac{\pi}{2}\right\rangle_A \otimes \left|\frac{\pi}{2}\right\rangle_B\,, \quad |0\rangle_A \otimes \left|\frac{\pi}{2}\right\rangle_B\,, \quad \left|\frac{\pi}{2}\right\rangle_A \otimes |0\rangle_B\,.$$

- Note that this material is relevant to previous quantum measurement
- Qubits $|0\rangle$, $\left|\frac{\pi}{2}\right\rangle$ pertaining to A or to B, can be associated to a pointer measuring the horizontal (resp. vertical) direction or polarisation described by the state $|0\rangle$ (resp. $\left|\frac{\pi}{2}\right\rangle$).
- Celebrated Bell pure states in $\mathbb{R}^2_A \otimes \mathbb{R}^2_B$: orthonormal basis of $\mathbb{R}^2_A \otimes \mathbb{R}^2_B$:

$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_A \otimes |0\rangle_B \pm \left|\frac{\pi}{2}\right\rangle_A \otimes \left|\frac{\pi}{2}\right\rangle_B\right)\,,$$

$$|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_A \otimes \left|\frac{\pi}{2}\right\rangle_B \pm \left|\frac{\pi}{2}\right\rangle_A \otimes |0\rangle_B\right)\,,$$

Bell states and quantum correlations

- Bell states are four specific *maximally entangled* quantum states of two qubits.
- Consider for instance the state $|\Phi^+\rangle$. If the pointer associated to A measures its qubit in the standard basis the outcome would be perfectly random, either possibility having probability 1/2. But if the pointer associated to B then measures its qubit, the outcome, although random for it alone, is the same as the one A gets. There is *quantum correlation*.

Bell inequality and its violation

- It is straightforward to transpose into the present setting the 1964 analysis and result presented by Bell [10] in his discussion about the EPR paper [11] and about the subsequent Bohm's approaches [12] based on the assumption of hidden variables.
- Let us replace the spin one-half particles considered by Bell as examples with the parallel (i.e., $+1$) and perpendicular (i.e., -1) quantum orientations in the plane as the only possible issues of the observable

$$\sigma_\phi = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix}, \quad \sigma_\phi|\phi/2\rangle = |\phi/2\rangle\,, \quad \sigma_\phi|\phi/2+\pi/2\rangle = -|\phi/2+\pi/2\rangle$$

supposing that there exists a pointer device designed for measuring such orientations with outcomes ± 1 only.

- Let us consider a bipartite system of two quantum orientations described respectivement by $\vec{\boldsymbol{\sigma}}^A$ and $\vec{\boldsymbol{\sigma}}^B$ in the so-called *singlet* state Ψ^-.
- In such a state, if measurement of the component $\sigma^A_{\phi_a} := \vec{\boldsymbol{\sigma}}^A \cdot \widehat{\mathbf{u}}_{\phi_a}$ ($\widehat{\mathbf{u}}_{\phi_a}$ is unit vector with polar angle ϕ_a) yields the value $+1$, then measurement of $\sigma^B_{\phi_b}$ when $\phi_b = \phi_a$ must yield the value -1, and vice-versa.

10. J. S. Bell, On the Einstein Podolski Rosen Paradox, *Physics* **1** 195-200 (1964).
11. A. Einstein, N. Rosen and B. Podolsky, *Phys. Rev.* **47** 777-780 (1935)
12. D. Bohm, A suggested interpretation of the Quantum Theory in Terms of "Hidden" Variables, I, *Phys. Rev.* **85** 166-179 (1952) ; idem, II, *Phys. Rev.* **85** 180-193 (1952)

Bell inequality and its violation (continued I)

- From a classical perspective, the explanation of such a correlation needs a predetermination by means of the existence of *hidden* parameters $\lambda \in \Lambda$
- The result $\varepsilon^A \in \{-1, +1\}$ (resp. $\varepsilon^B \in \{-1, +1\}$) of measuring $\sigma^A_{\phi_a}$ (resp. $\sigma^B_{\phi_b}$) is then determined by ϕ_a and λ only, not by ϕ_b also, i.e. $\varepsilon^A = \varepsilon^A(\phi_a, \lambda)$ (resp. $\varepsilon^B = \varepsilon^B(\phi_b, \lambda)$).
- Given a probability distribution $\rho(\lambda)$ on Λ, the **classical** expectation value of the product of the two components $\sigma^A_{\phi_a}$ and $\sigma^B_{\phi_b}$ is given by

$$\mathsf{P}(\phi_a, \phi_b) = \int_\Lambda \mathrm{d}\lambda\, \rho(\lambda)\, \varepsilon^A(\phi_a, \lambda)\, \varepsilon^B(\phi_b, \lambda)\,.$$

- Since $\int_\Lambda \mathrm{d}\lambda\, \rho(\lambda) = 1$ and $\varepsilon^{A,B} = \pm 1$, we have $-1 \leq \mathsf{P}(\phi_a, \phi_b) \leq 1$.
- Equivalent predictions **with the quantum setting** then imposes the equality between the classical and quantum expectation values :

$$\mathsf{P}(\phi_a, \phi_b) = \left\langle \Psi^- \middle| \sigma^A_{\phi_a} \otimes \sigma^B_{\phi_b} \middle| \Psi^- \right\rangle = -\widehat{\mathbf{u}}_{\phi_a} \cdot \widehat{\mathbf{u}}_{\phi_b} = -\cos(\phi_a - \phi_b)\,.$$

- In the above equation, the value -1 is reached at $\phi_a = \phi_b$. This is possible for $\mathsf{P}(\phi_a, \phi_a)$ only if $\varepsilon^A(\phi_a, \lambda) = -\varepsilon^B(\phi_a, \lambda)$. Hence, we can write $\mathsf{P}(\phi_a, \phi_b)$ as

$$\mathsf{P}(\phi_a, \phi_b) = -\int_\Lambda \mathrm{d}\lambda\, \rho(\lambda)\, \varepsilon(\phi_a, \lambda)\, \varepsilon(\phi_b, \lambda)\,, \quad \varepsilon(\phi, \lambda) \equiv \varepsilon^A(\phi, \lambda) = \pm 1\,.$$

Bell inequality and its violation (continued II)

- Let us now introduce a third unit vector $\widehat{\mathbf{u}}_{\phi_c}$. Due to $\varepsilon^2 = 1$, we have

$$\mathsf{P}(\phi_a,\phi_b) - \mathsf{P}(\phi_a,\phi_c) = \int_\Lambda \mathrm{d}\lambda\,\rho(\lambda)\,\varepsilon(\phi_a,\lambda)\,\varepsilon(\phi_b,\lambda)\,\left[\varepsilon(\phi_b,\lambda)\,\varepsilon(\phi_c,\lambda) - 1\right]\,.$$

- It results the (baby) Bell inequality :

$$|\mathsf{P}(\phi_a,\phi_b) - \mathsf{P}(\phi_a,\phi_c)| \leq \int_\Lambda \mathrm{d}\lambda\,\rho(\lambda)\,\left[1 - \varepsilon(\phi_b,\lambda)\,\varepsilon(\phi_c,\lambda)\right] = 1 + \mathsf{P}(\phi_b,\phi_c)\,.$$

- Hence, the validity of the existence of hidden variable(s) for justifying the quantum correlation in the singlet state Ψ^-, and which is encapsulated by the above equation, has the following consequence on the arbitrary triple (ϕ_a,ϕ_b,ϕ_c) :

$$1 - \cos(\phi_b - \phi_c) \geq |\cos(\phi_b - \phi_a) - \cos(\phi_c - \phi_a)|\ ,$$

equivalently, in terms of the two independent angles $\zeta = \dfrac{\phi_a - \phi_b}{2}$ and $\eta = \dfrac{\phi_b - \phi_c}{2}$,

$$\left|\sin^2\zeta - \sin^2(\eta + \zeta)\right| \leq \sin^2\eta\,.$$

Bell inequality and its violation (continued III)

- It is easy to find pairs (ζ, η) for which the inequality

$$\left|\sin^2\zeta - \sin^2(\eta+\zeta)\right| \leq \sin^2\eta\,.$$

does not hold true. For instance with $\eta = \zeta \neq 0$, i.e., $\phi_b = \dfrac{\phi_a+\phi_c}{2}$,

$$|4\sin^2\eta - 3| \leq 1\,,$$

and this does not hold true for all $|\eta| < \pi/4$, i.e., for $|\phi_a - \phi_b| = |\phi_b - \phi_c| < \pi/2$.

- Actually, we did not follow here the proof given by Bell, which is a lot more elaborate. Also, Bell considered unit vectors in 3-space. Restricting his proof to vectors in the plane does not make any difference, as it is actually the case in many works devoted to the foundations of quantum mechanics

Complex two-dimensional Hilbert space $\mathbb{C}^2$

- As a complex vector space, $\mathbb{C}^2$, with canonical basis $\mathbf{e}_1$, $\mathbf{e}_2$, has a real structure, i.e. is isomorphic to a real vector space which makes it isomorphic to $\mathbb{R}^4$, itself isomorphic to $\mathbb{R}^2 \otimes \mathbb{R}^2$,
- A **real** structure is obtained by considering the vector expansion

$$\mathbb{C}^2 \in \mathbf{v} = z_1\mathbf{e}_1 + z_2\mathbf{e}_2 = x_1\mathbf{e}_1 + y_1\,(\mathrm{i}\mathbf{e}_1) + x_2\mathbf{e}_2 + y_2\,(\mathrm{i}\mathbf{e}_2)\ ,$$

- i.e., by writing $z_1 = x_1 + \mathrm{i}y_1$, $z_2 = x_2 + \mathrm{i}y_2$, and considering the set of vectors

$$\{\mathbf{e}_1, \mathbf{e}_2, (\mathrm{i}\mathbf{e}_1)\,, (\mathrm{i}\mathbf{e}_2)\}$$

as forming a basis of $\mathbb{R}^4$.

Map from $\mathbb{C}^2$ to $\mathbb{R}^4$ to $\mathbb{R}^2 \otimes \mathbb{R}^2$

- Forgetting about the superfluous subscripts A and B, the map Euclidean plane $\mathbb{R}^2 \mapsto$ complex "plane" $\mathbb{C}$ can be chosen as

$$|0\rangle \mapsto 1\,, \qquad \left|\frac{\pi}{2}\right\rangle \mapsto \mathrm{i}\,.$$

- Of course, it could be instead chosen as :

$$|0\rangle \mapsto 1\,, \qquad \left|\frac{\pi}{2}\right\rangle \mapsto -\mathrm{i}\,.$$

- With the first choice, we write the correspondence between bases as

$$|0\rangle \otimes |0\rangle = \mathbf{e}_1\,, \quad \left|\frac{\pi}{2}\right\rangle \otimes \left|\frac{\pi}{2}\right\rangle = -\mathbf{e}_2\,, \quad |0\rangle \otimes \left|\frac{\pi}{2}\right\rangle = (\mathrm{i}\mathbf{e}_1)\,, \quad \left|\frac{\pi}{2}\right\rangle \otimes |0\rangle = (\mathrm{i}\mathbf{e}_2)$$

“Up” and “down” complex objects.

- Real composite nature to “up” and “down” complex objects :
$$\mathbf{e}_1 \equiv |\uparrow\rangle \equiv \begin{pmatrix}1\\0\end{pmatrix}, \qquad \mathbf{e}_2 \equiv |\downarrow\rangle \equiv \begin{pmatrix}0\\1\end{pmatrix},$$
- Unitary map from the Bell basis to the basis of real structure of $\mathbb{C}^2$
$$\left(|\Phi^+\rangle \quad |\Phi^-\rangle \quad |\Psi^+\rangle \quad |\Psi^-\rangle\right) = \left(\mathbf{e}_1 \quad \mathbf{e}_2 \quad (\mathrm{i}\mathbf{e}_1) \quad (\mathrm{i}\mathbf{e}_2)\right) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$
- In terms of respective components of vectors in their respective spaces,
$$\begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x^+ \\ x^- \\ y^+ \\ y^- \end{pmatrix}.$$
- Equivalently, in complex notations, with $z^\pm = x^\pm + \mathrm{i}y^\pm$,
$$\begin{pmatrix} z^+ \\ z^- \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -\mathsf{C} \\ \mathsf{C} & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \equiv \mathcal{C}_@ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix},$$
- Here we have introduced the conjugaison operator $\mathsf{C}z = \bar{z}$, i.e. the mirror symmetry with respect to the real axis, $-\mathsf{C}$ being the mirror symmetry with respect to the imaginary axis.

Real Bell state nature of Shrödinger cat states

- Operator $\mathcal{C}_@$ can be expressed as

$$\mathcal{C}_@ = \frac{1}{\sqrt{2}}(I + \mathsf{F}) \,, \quad \mathsf{F} := C\hat{\jmath} = \begin{pmatrix} 0 & -C \\ C & 0 \end{pmatrix} \,, \quad \hat{\jmath} \equiv -\mathrm{i}\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \tau_2 \,.$$

- Therefore, with the above choice of isomorphisms, Bell entanglement in $\mathbb{R}^2 \otimes \mathbb{R}^2$ is not represented by a linear superposition in $\mathbb{C}^2$. It involves also the two mirror symmetries $\pm$C.

- Operator F is a kind of "flip" whereas "cat" operator $\mathcal{C}_@$ builds from the *up* and *down* basic states the two elementary Schrödinger cats

$$\mathsf{F}\,|\uparrow\rangle = |\downarrow\rangle\,, \quad \mathsf{F}\,|\downarrow\rangle = -|\uparrow\rangle\,,$$

$$\mathcal{C}_@\,|\uparrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)\,, \quad \mathcal{C}_@\,|\downarrow\rangle = \frac{1}{\sqrt{2}}(-|\uparrow\rangle + |\downarrow\rangle)\,.$$

Flip F in the construction of the spin one-half coherent states

- Interesting too is the appearance of the flip F in the construction of the spin one-half coherent state defined in terms of spherical coordinates (θ, ϕ) by

$$\mathbb{S}^2 \ni \hat{\mathbf{n}}(\theta,\varphi) \mapsto |\theta,\varphi\rangle = \left(\cos\frac{\theta}{2}\,|\uparrow\rangle + e^{\mathrm{i}\phi}\sin\frac{\theta}{2}\,|\downarrow\rangle\right) \equiv \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{\mathrm{i}\phi}\sin\frac{\theta}{2} \end{pmatrix}$$

$$= D^{\frac{1}{2}}\left(\xi_{\hat{\mathbf{n}}}^{-1}\right)|\uparrow\rangle \equiv \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2}e^{-\mathrm{i}\phi} \\ \sin\frac{\theta}{2}e^{\mathrm{i}\phi} & \cos\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- Here, $\xi_{\hat{\mathbf{n}}}$ corresponds, through homomorphism SO$(3) \mapsto$ SU(2), to the specific rotation $\mathcal{R}_{\hat{\mathbf{n}}}$ mapping the unit vector pointing to the north pole, $\hat{\boldsymbol{k}} = (0,0,1)$, to $\hat{\mathbf{n}}$

- UIR operator $D^{\frac{1}{2}}\left(\xi_{\hat{\mathbf{n}}}^{-1}\right)$ represents the element $\xi_{\hat{\mathbf{n}}}^{-1}$ in SU(2)

- Second column of $D^{\frac{1}{2}}\left(\xi_{\hat{\mathbf{n}}}^{-1}\right)$ is precisely the flip of the first one,

$$D^{\frac{1}{2}}\left(\xi_{\hat{\mathbf{n}}}^{-1}\right) = \left(|\theta,\phi\rangle \quad \mathsf{F}|\theta,\varphi\rangle\right)\,.$$

Flip and matrix representations of quaternions

- This is the key for grasping the isomorphisms $\mathbb{C}^2 \cong \mathbb{H} \cong \mathbb{R}_+ \times \mathrm{SU}(2)$. Using quaternionic algebra, e.g., $\hat{\boldsymbol{\imath}} = \hat{\boldsymbol{\jmath}}\hat{\boldsymbol{k}}$ + even permutations,

$$\mathbb{H} \ni q = q_0 + q_1\hat{\boldsymbol{\imath}} + q_2\hat{\boldsymbol{\jmath}} + q_3\hat{\boldsymbol{k}} = q_0 + q_3\hat{\boldsymbol{k}} + \hat{\boldsymbol{\jmath}}\left(q_1\hat{\boldsymbol{k}} + q_2\right) \equiv \begin{pmatrix} q_0 + \mathrm{i}q_3 \\ q_2 + \mathrm{i}q_1 \end{pmatrix} \equiv Z_q \in \mathbb{C}^2\,,$$

  after identifying $\hat{\boldsymbol{k}} \equiv \mathrm{i}$ as both are roots of -1.

- Then the flip appears naturally in the final identification $\mathbb{H} \cong \mathbb{R}_+ \times \mathrm{SU}(2)$ as

$$q \equiv \begin{pmatrix} q_0 + \mathrm{i}q_3 & -q_2 + \mathrm{i}q_1 \\ q_2 + \mathrm{i}q_1 & q_0 - \mathrm{i}q_3 \end{pmatrix} = \begin{pmatrix} Z_q & \mathsf{F}Z_q \end{pmatrix}\,. \tag{12}$$

Spin $1/2$ coherent states as vectors in $\mathbb{R}^2_A \otimes \mathbb{R}^2_B$

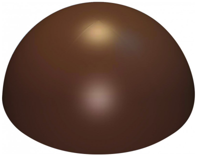
- As "cat states" in $\mathbb{C}^2$

$$\mathbb{C}^2 \ni |\theta, \varphi\rangle = \left(\cos\frac{\theta}{2}\,|\uparrow\rangle + e^{i\phi}\sin\frac{\theta}{2}\,|\downarrow\rangle\right)$$

- As entangled states in $\mathbb{R}^2_A \otimes \mathbb{R}^2_B$, $|\theta, \varphi\rangle \mapsto \begin{pmatrix} \cos\frac{\theta}{2} \\ -\sin\frac{\theta}{2}\cos\phi \\ \sin\frac{\theta}{2}\sin\phi \\ 0 \end{pmatrix}$

$$|\theta, \varphi\rangle = \cos\frac{\theta}{2}|0\rangle_A \otimes |0\rangle_B - \sin\frac{\theta}{2}\cos\phi\left|\frac{\pi}{2}\right\rangle_A \otimes \left|\frac{\pi}{2}\right\rangle_B + \sin\frac{\theta}{2}\sin\phi|0\rangle_A \otimes \left|\frac{\pi}{2}\right\rangle_B + 0\left|\frac{\pi}{2}\right\rangle_A \otimes |0\rangle_B$$

- i.e., set of orientations in space $\mathbb{R}^3$ as spin $1/2$ coherent states ~ 2 entangled orientations in plane



$|\uparrow\rangle \sim |0\rangle_A \otimes |0\rangle_B$ and $|\downarrow\rangle \sim \left|\frac{\pi}{2}\right\rangle_A \otimes \left|\frac{\pi}{2}\right\rangle_B$