Introduction to k-contact geometry and applications

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Geometry and Applications. Modern Mathematical Approaches.

Scheme of the talk

Main claim

Jet geometry and its main structures can be described via a recently developed geometric structure: k-contact manifolds.

- Introduce k-contact geometry as a generalisation of contact geometry.
- Provide a distributional approach to k-contact geometry.
- Recover jet manifolds as N^r_π-contact distribution with a certain type of polarisation.
- Explain how this gives a general framework for studying differential equations and related techniques with certain advantages: it fits well when jet geometry does not completely.

Main aim

k-contact geometry contains jet geometry as a special case, but it is open to see how far this can give practical advantages.

Contact geometry

Definition

A **contact distribution** on a $(2n+1)$-dimensional manifold M is a **maximally non-integrable distribution** D of corank one, i.e. any η on U such that $D = \ker \eta$ satisfies

$$\eta \wedge (\mathrm{d}\eta)^n \neq 0 \,.$$

Then, (M, D) is called a **contact manifold** and every η for D is called a **(local) contact form**. If $D = \ker \eta$ for a globally defined one-form η, then (M, η) is called a **co-oriented contact manifold**.

Being a contact form on U amounts to

$$\mathrm{T}U = \ker \eta \oplus \ker \mathrm{d}\eta \,.$$

Contact geometry

Proposition

Given a co-oriented contact manifold (M, η), there exists a unique vector field $R \in \mathfrak{X}(M)$, called **Reeb vector field**, such that

$$\iota_R \, \mathrm{d}\eta = 0 \,, \quad \iota_R \eta = 1 \,.$$

The Reeb vector field R generates the distribution $\ker \mathrm{d}\eta$, called the **Reeb distribution**. It is related to studying periodic orbits of Hamiltonian systems (Weinstein conjecture).

Darboux theorem for contact manifolds

Consider a contact manifold (M, η) of dimension $2n + 1$. Then, around every point $p \in M$, there exists a local chart (U, q^i, p_i, s), $i = 1, \ldots, n$, such that

$$\eta|_U = \mathrm{d}s - \sum_{i=1}^{n} p_i \, \mathrm{d}q^i \,.$$

These coordinates are called **Darboux** or **canonical coordinates** of (M, η).

In Darboux coordinates, the Reeb vector field is $R = \partial/\partial s$.

$J^1\pi$ as a contact manifold for $\pi : E \to Q$ of rank one

Let $\pi : E \to Q$ be a fibred manifold of rank one and $\dim Q = n$. On $J^1\pi$, take adapted coordinates (x^i, u, u_i), with $i = 1, \ldots, n$. The Cartan distribution is, locally, the kernel of the one-form

$$\theta = \mathrm{d}u - \sum_{i=1}^{n} u_i \, \mathrm{d}x^i .$$

This is a contact form as $\mathrm{d}\theta = \sum_{i=1}^n \mathrm{d}x^i \wedge \mathrm{d}u_i$ and $\ker \mathrm{d}\theta = \langle \partial/\partial u \rangle$. Hence, the Cartan distribution is

$$\mathcal{C}^1_\pi = \ker\theta = \left\langle D_i = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u}, \frac{\partial}{\partial u_i} \right\rangle_{i=1,\ldots,n} .$$

Since $\dim J^1\pi = 2n+1$ and $\theta \wedge (\mathrm{d}\theta)^n \neq 0$, the pair $(J^1\pi, \mathcal{C}^1_\pi)$ is an ordinary contact manifold. A local Reeb vector field is $R = \frac{\partial}{\partial u}$.

If the rank of $\pi : E \to Q$ is $k > 1$, the so-called contact forms $\theta^\alpha = du^\alpha - \sum_{i=1}^n u_i^\alpha \, \mathrm{d}x^i$, with $\alpha = 1, \ldots, k$ are NOT contact.

k-Contact Manifolds

Contact geometry appears in thermodynamics, e.g. $dU - TdS$, in dissipative systems, etc. A generalisation for dissipative field theories appear around five years ago.

X. Rivas Guijarro, *Geometrical Aspects of Contact Mechanical Systems and Field Theories*, PhD thesis, Universitat Politècnica de Catalunya, 2021.

Definition

A *k-contact form* on M is a differential one-form $\boldsymbol{\eta} \in \Omega^1(U, \mathbb{R}^k)$ such that:

1. $\ker \boldsymbol{\eta} \subset \mathrm{T}M$ is a regular distribution of corank k,
2. $\ker \boldsymbol{\eta} \oplus \ker \mathrm{d}\boldsymbol{\eta} = TU$.

If $\eta \in \Omega^1(M, \mathbb{R}^k)$, the pair $(M, \boldsymbol{\eta})$ is called a *co-oriented k-contact manifold*. If, in addition, $\dim M = n + nk + k$ for some $n \in \mathbb{N}$ and M is endowed with an integrable distribution $\mathcal{V}$ contained in $\ker \boldsymbol{\eta}$ with $\operatorname{rank} \mathcal{V} = nk$, we say that $(M, \boldsymbol{\eta}, \mathcal{V})$ is a *polarised co-oriented k-contact manifold* and we call $\mathcal{V}$ a *first-order polarisation*.

Reeb Vector Fields and distributions

Theorem

Let $(M, \boldsymbol{\eta} = \sum_\alpha \eta^\alpha \otimes e_\alpha)$ be a co-oriented k-contact manifold. There exists a unique family of vector fields $R_1, \ldots, R_k \in \mathfrak{X}(M)$, called the Reeb vector fields, such that

$$\begin{cases} \iota_{R_\alpha}\eta^\beta = \delta_\alpha^\beta, \\ \iota_{R_\alpha}\mathrm{d}\eta^\beta = 0, \end{cases}$$

for $\alpha, \beta = 1, \ldots, k$. Moreover, $[R_\alpha, R_\beta] = 0$ and $R_1 \wedge \ldots \wedge R_k \neq 0$.

Note that $\mathcal{L}_{R_\alpha}\boldsymbol{\eta} = 0$ for $\alpha = 1, \ldots, k$ and the flow of each Reeb vector field leaves invariant the vector fields taking values in $\ker \boldsymbol{\eta}$ for $\alpha = 1, \ldots, k$.

k-Contact distributions

Definition

A k-**contact distribution** is a distribution D on a manifold M such that, for each point $x \in M$, admits an open $U \ni x$ and a k-contact form $\boldsymbol{\eta}$ on U such that $D|_U = \ker \boldsymbol{\eta}$.

Definition

A regular distribution D on M is **maximally non-integrable** when the vector bundle mapping $\rho : D \times_M D \to \mathrm{T}M/D$ given by

$$\rho(v, v') = \pi([X, X']_x)\,, \qquad \forall v, v' \in D_x\,,$$

where $\pi : \mathrm{T}M \to \mathrm{T}M/D$ is the natural vector bundle projection on M and X, X' are vector fields taking values in D such that $X(x) = v$ and $X'(x) = v'$, is non-degenerate.

If $D = \ker \boldsymbol{\zeta} \in \Omega^1(U, \mathbb{R}^k)$, then one has an isomorphism $\boldsymbol{\eta} : \mathrm{T}_x M/D_x \simeq \mathbb{R}^k$ and one can write

$$\rho(v, v') = -\,\mathrm{d}\boldsymbol{\eta}(v, v').$$

k-Contact distributions

Due to the existence of Reeb vector fields, one can write.

Theorem

If (M, η) is a k-contact manifold, then $\mathrm{d}\eta$ is non-degenerate in $\ker \eta$. *In other words,* $\ker \eta$ *is maximally non-integrable.*

Definition

Given a distribution D on M, a Lie symmetry of D is a vector field X on M such that $[X, Y]$ takes values in D for every vector field Y taking values in D.

Theorem

A distribution D on M is a k-contact distribution if and only if it is maximally non-integrable and admits k-commuting Lie symmetries $S_1, \ldots, S_k$ of D spanning a complement.

Adapted coordinates on $J^1\pi$

Let $\pi : E \to Q$ be a fibre bundle with $\dim Q = n$ and $\operatorname{rank}\pi = k$. In coordinates adapted to the fibration, we write (x^i, u^α), with $i = 1, \ldots, n$ and $\alpha = 1, \ldots, k$. The induced coordinates on $U \subset J^1\pi$ are $(x^i, u^\alpha, u_i^\alpha)$. The first-order k-contact form associated with these coordinates is

$$\boldsymbol{\eta} = \sum_{\alpha=1}^{k} \eta^\alpha \otimes e_\alpha = \sum_{\alpha=1}^{k} (du^\alpha - \sum_{i=1}^{n} u_i^\alpha \, dx^i) \otimes e_\alpha, \qquad e_1, \ldots, e_k \text{ base of } \mathbb{R}^k.$$

The Cartan distribution is

$$\mathcal{C}_\pi^1 = \ker \boldsymbol{\eta} = \bigcap_{\alpha=1}^{k} \ker \eta^\alpha = \left\langle D_i = \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{k} u_i^\alpha \frac{\partial}{\partial u^\alpha}, \frac{\partial}{\partial u_i^\alpha} \right\rangle_{i=1,\ldots,n,\alpha=1,\ldots,k}.$$

and the adapted Reeb fields are $R_\alpha = \partial/\partial u^\alpha$. A first-order polarisation is $\mathcal{V} = \langle \partial/\partial u_i^\alpha \rangle$.

Isotropic, coisotropic and Legendrian submanifolds

Definition

For a subspace $E_x \subset \mathcal{D}_x$ of a k-contact manifold $(M, \mathcal{D})$ with $\mathcal{D}_x = \ker \boldsymbol{\eta}_x$, define its k-contact orthogonal by

$$E_x^{\perp_{\mathcal{D}}} := \{v_x \in \mathcal{D}_x : \mathrm{d}\boldsymbol{\eta}(v_x, w_x) = 0 \text{ for all } w_x \in E_x\}.$$

A submanifold $N \subset M$ with $TN \subset \mathcal{D}$ is $\mathcal{D}$-isotropic if $T_x N \subset (T_x N)^{\perp_{\mathcal{D}}}$, $\mathcal{D}$-coisotropic if $(T_x N \cap \mathcal{D}_x)^{\perp_{\mathcal{D}}} \subset T_x N$, and Legendrian if it is maximal isotropic with a complement in D_x that is isotropic. A k'-contact submanifold is instead a submanifold whose induced distribution $TN \cap \mathcal{D}$ is again a k'-contact distribution.

For instance

$$\mathcal{V} = \langle \partial / \partial u_i^\alpha \rangle, \qquad H_\pi^1 = \langle D_1, \ldots, D_n \rangle$$

are isotropic and Legendrian. Indeed. $\mathcal{V}$ is a polarisation of rank nk.

k-contact Hamiltonian vector fields

Definition

A k-contact vector field is a Lie symmetry of $\mathcal{D}$, namely $[X, \Gamma(\mathcal{D})] \subset \Gamma(\mathcal{D})$. Its adapted Hamiltonian on U where $\mathcal{D} = \ker \boldsymbol{\eta}$ is the $\mathbb{R}^k$-valued function

$$\boldsymbol{h}_X = -\iota_X \boldsymbol{\eta} = -(\eta^1(X), \ldots, \eta^k(X)).$$

Although $\boldsymbol{h}_X$ depends on the adapted presentation, its geometric role is clear: the zero locus $\boldsymbol{h}_X^{-1}(0)$ is the locus where X becomes tangent to $\mathcal{D}$.

For Cartan distributions on $J^1\pi$, a k-contact Hamiltonian vector field X is a Lie symmetry and its local Hamiltonian $\boldsymbol{h}_X$ is, when X is a prolonged vector field, its characteristic,
$\boldsymbol{Q} = \sum_{\alpha=1}^k (\phi^\alpha - \sum_{i=1}^n \xi^i u_i^\alpha) \otimes e_\alpha$.
There is a one-to-one relation between k-contact Hamiltonian vector fields and their adapted Hamiltonians, and there is a Lie bracket also

$$\{\boldsymbol{h}_1, \boldsymbol{h}_2\} = -\boldsymbol{\eta}([X_1, X_2]).$$

Classical finite jets

Let $\pi : E \to Q$ be a fibred manifold, $\dim Q = n$, fibre rank m. Consider multi-indexes $J = (j_1, \ldots, j_n)$ with $j_1, \ldots, j_n$ non-negative integers and $|J| = \sum_{i=1}^n j_i$. On $J^r\pi$ use adapted coordinates (x^i, u_I^α), $|I| \leq r$. We write $\pi_{r,s} : J^r\pi \to J^s\pi$ for the natural projections.

Cartan distribution

$$\mathcal{C}_\pi^r = \left\langle D_i, \frac{\partial}{\partial u_K^\alpha} : |K| = r \right\rangle, \qquad D_i = \frac{\partial}{\partial x^i} + \sum_{\alpha,|I|\leq r-1} u^\alpha_{I+\widehat{e}_i} \frac{\partial}{\partial u_I^\alpha}.$$

- $\mathcal{C}_\pi^r$ encodes holonomicity: $T(j^r\sigma)(TQ) \subset \mathcal{C}_\pi^r$.
- Its corank is $N_\pi^r = m\binom{n+r-1}{r-1}$.

The canonical structure on $J^r\pi$

Besides the Cartan distribution, $J^r\pi$ has a canonical vector-valued one-form

$$S^r_\pi \in \Omega^1\left(J^r\pi, \pi^*_{r,r-1} V\pi_{r-1}\right), \quad S^r_\pi = \sum_{\alpha=1}^{m} \sum_{|I|\leq r-1} \theta^\alpha_I \otimes \frac{\partial}{\partial u^\alpha_I}.$$

- $\ker S^r_\pi = \mathcal{C}^r_\pi$.
- S^r_π measures the failure of a tangent vector to project to the tangent vector on $J^{r-1}\pi$ prescribed by the r-jet.
- Equivalently, S^r_π is the Spencer operator written as a canonical structure form.

Why the standard higher-order coframe is not enough

The standard contact forms are

$$\theta^\alpha_I = du^\alpha_I - \sum_i u^\alpha_{I+\widehat{e_i}} dx^i, \qquad |I| \leq r-1, \qquad \alpha = 1, \ldots, m.$$

They span the annihilator of $\mathcal{C}^r_\pi$, but for $r > 1$ they do not define an adapted N^r_π-contact form.

Key obstruction

If $\Theta = (\theta^\alpha_I)$, then $\ker \Theta = \mathcal{C}^r_\pi$, but typically

$$\ker \Theta \oplus \ker d\Theta \neq TJ^r\pi.$$

Thus the usual contact coframe detects $\mathcal{C}^r_\pi$ but misses the Reeb vector fields needed for k-contact geometry.

Illustrative example I: the model

Take $\pi : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$, with base coordinates (x^1, x^2) and fields (u, v). On $J^2\pi$ use coordinates

$$(x^1, x^2, q^u, q^v, q^u_1, q^u_2, q^v_1, q^v_2, q^u_{11}, q^u_{12}, q^u_{22}, q^v_{11}, q^v_{12}, q^v_{22}).$$

Then

$$\mathcal{C}^2_\pi = \left\langle D_1, D_2, \frac{\partial}{\partial q^\alpha_{11}}, \frac{\partial}{\partial q^\alpha_{12}}, \frac{\partial}{\partial q^\alpha_{22}} \right\rangle_{\alpha=u,v}.$$

- $\dim J^2\pi = 14$, $\operatorname{rank} \mathcal{C}^2_\pi = 8$.
- The expected k is $N^2_\pi = 2\binom{3}{1} = 6$.

Illustrative example II: the naive coframe fails

The six standard contact forms are

$$\theta_0^\alpha = dq^\alpha - q_1^\alpha dx^1 - q_2^\alpha dx^2, \quad \theta_1^\alpha = dq_1^\alpha - q_{11}^\alpha dx^1 - q_{12}^\alpha dx^2,$$

$$\theta_2^\alpha = dq_2^\alpha - q_{12}^\alpha dx^1 - q_{22}^\alpha dx^2, \quad \alpha \in \{u, v\}.$$

They satisfy $\ker \Theta = \mathcal{C}_\pi^2$. However,

$$\ker d\Theta = \left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right\rangle, \quad \dim(\ker \Theta \oplus \ker d\Theta) = 8 + 2 < 14.$$

Conclusion

The standard contact coframe is not a six-contact form, even though its kernel is the correct Cartan distribution.

Illustrative example III: the adapted six-contact form

A Reeb frame transverse to $\mathcal{C}^2_\pi$ is

$$R^0_\alpha = \frac{\partial}{\partial q^\alpha}, \quad R^i_\alpha = x^i \frac{\partial}{\partial q^\alpha} + \frac{\partial}{\partial q^\alpha_i}, \quad \alpha \in \{u, v\}, \ i = 1, 2.$$

These vector fields commute, preserve $\mathcal{C}^2_\pi$, and span a complement of it. The dual adapted form is

$$\eta^2_\pi = \sum_{\alpha \in \{u,v\}} \left((\theta^\alpha_0 - x^1\theta^\alpha_1 - x^2\theta^\alpha_2) \otimes e^0_\alpha + \theta^\alpha_1 \otimes e^1_\alpha + \theta^\alpha_2 \otimes e^2_\alpha \right).$$

Then, $\ker \eta^2_\pi = \mathcal{C}^2_\pi$ and $\ker \mathrm{d}\eta^2_\pi = \langle R^0_\alpha, R^i_\alpha \rangle$.

Illustrative example IV: Spencer already appears

Let $R : \mathbb{R}^6 \to \mathfrak{X}(J^2\pi)$ be the Reeb lift, $R(e^i_\alpha) = R^i_\alpha$. Then

$$T\pi_{2,1} \circ R \circ \eta^2_\pi = S^2_\pi.$$

Indeed, the triangular terms cancel:

$$(\theta^\alpha_0 - x^1\theta^\alpha_1 - x^2\theta^\alpha_2) \otimes \frac{\partial}{\partial q^\alpha} + \theta^\alpha_1 \otimes \left(x^1 \frac{\partial}{\partial q^\alpha} + \frac{\partial}{\partial q^\alpha_1}\right)$$

$$+\theta^\alpha_2 \otimes \left(x^2 \frac{\partial}{\partial q^\alpha} + \frac{\partial}{\partial q^\alpha_2}\right) = \theta^\alpha_0 \otimes \frac{\partial}{\partial q^\alpha} + \theta^\alpha_1 \otimes \frac{\partial}{\partial q^\alpha_1} + \theta^\alpha_2 \otimes \frac{\partial}{\partial q^\alpha_2}.$$

What the example proves

The adapted k-contact form does not merely recover $\mathcal{C}^2_\pi$; after Reeb lifting, it recovers the canonical jet structure.

Main theorem I: finite jets are N^r_π-contact

Theorem

The Cartan distribution $\mathcal{C}^r_\pi$ on $J^r\pi$ is an N^r_π-contact distribution, where

$$N^r_\pi = m\binom{n+r-1}{r-1}.$$

- $\mathcal{C}^r_\pi$ is maximally non-integrable.
- In every adapted chart there is a natural local Reeb frame transverse to $\mathcal{C}^r_\pi$.
- The associated adapted contact form has kernel $\mathcal{C}^r_\pi$.

The local Reeb frame

In adapted coordinates (x^i, u^α_I), define for $|J| \leq r-1$:

$$R^J_\alpha = \sum_{I \leq J} \frac{x^{J-I}}{(J-I)!} \frac{\partial}{\partial u^\alpha_I} = \mathrm{pr}^{(r)} \left(\frac{x^J}{J!} \frac{\partial}{\partial u^\alpha} \right).$$

- The fields R^J_α commute pairwise.
- They preserve $\mathcal{C}^r_\pi$, being prolongations of vertical polynomial fields on E.
- They span a complement: $TJ^r\pi = \mathcal{C}^r_\pi \oplus \left\langle R^J_\alpha \right\rangle_{|J|<r}$ locally.

This is the nontrivial ingredient behind higher-order k-contactity.

Adapted N^r_π-contact forms

The adapted coframe dual to the Reeb frame is

$$\eta^\alpha_J = \sum_{L\geq J,\ |L|\leq r-1} (-1)^{|L-J|}\frac{x^{L-J}}{(L-J)!}\theta^\alpha_L, \qquad |J|\leq r-1.$$

Then

$$\boldsymbol{\eta}^r_\pi = \sum_{\alpha=1}^m \sum_{|J|\leq r-1} \eta^\alpha_J \otimes e^J_\alpha, \quad \ker \boldsymbol{\eta}^r_\pi = \mathcal{C}^r_\pi,$$

and

$$TJ^r\pi = \ker \boldsymbol{\eta}^r_\pi \oplus \ker \mathrm{d}\boldsymbol{\eta}^r_\pi.$$

Important distinction

$\boldsymbol{\eta}^r_\pi$ is local and chart-dependent; $\mathcal{C}^r_\pi$ is global and canonical.

The Spencer operator reappears

Let $R:\mathbb{R}^{N^r_\pi}\to\mathfrak{X}(U)$ be the Reeb lift, $R(e^J_\alpha)=R^J_\alpha$. If $\pi_{r,s}:J^r\pi\to J^s\pi$, then

Spencer identity

$$\mathrm{T}\pi_{r,r-1}\circ R\circ\boldsymbol{\eta}^r_\pi=S^r_\pi.$$

In adapted coordinates, this is exactly

$$S^r_\pi=\sum_{\alpha=1}^{m}\sum_{|I|\leq r-1}\theta^\alpha_I\otimes\frac{\partial}{\partial u^\alpha_I}.$$

Consequently, the adapted k-contact form does not merely have the right kernel: it reconstructs the canonical jet structure / Spencer operator.

Spencer contractions from $\mathrm{d}\eta$

Use the local splitting

$$\mathcal{C}^r_\pi = H^r_\pi \oplus \mathcal{G}^r_\pi, \quad H^r_\pi = \langle D_1, \dots, D_n \rangle, \quad \mathcal{G}^r_\pi = \ker T\pi_{r,r-1} = \left\langle \frac{\partial}{\partial u^\alpha_J} \right\rangle_{|J|=r}.$$

For $h = \sum_i h^i D_i$ and $A = \sum_{|K|=r} A^\alpha_K \partial_{u^\alpha_K}$,

$$R \circ \mathrm{d}\boldsymbol{\eta}^r_\pi(h, A) = \sum_{\alpha=1}^{m} \sum_{|I|=r-1} \left(\sum_i h^i A^\alpha_{I+\widehat{e}_i} \right) \frac{\partial}{\partial u^\alpha_I}.$$

Under $\mathcal{G}^r_\pi \simeq S^r T^*Q \otimes V\pi$, this is precisely the algebraic Spencer contraction of totally symmetric tensors A with vectors h

$$(h, A) \in TQ \times S^r T^*Q \otimes V\pi \mapsto \iota_h A \in S^{r-1} T^*Q \otimes V\pi.$$

What is already recovered

At this point, the polarised k-contact viewpoint has recovered the canonical finite-jet data:

- the Cartan distribution $\mathcal{C}^r_\pi$ as a genuine N^r_π-contact distribution;
- the canonical structure form / Spencer operator via $\mathrm{T}\pi_{r,r-1} \circ R \circ \boldsymbol{\eta}^r_\pi = S^r_\pi$;
- the algebraic Spencer contraction from the mixed part of $R \circ d\boldsymbol{\eta}^r_\pi$;
- the highest-order vertical direction $\mathcal{G}^r_\pi$ as the natural candidate polarisation.

Transition

The next question is the converse one: which polarised k-contact manifolds actually come from finite-order jet bundles?

Higher-order polarisation

Definition

A **polarisation** in a k-contact manifold $(M, \mathcal{D})$ is an integrable Legendrian subbundle $P \subset \mathcal{D}$ of maximal rank among integrable Legendrian subbundles.

For jet bundles the relevant polarisation is not the full vertical bundle over Q, but

$$\mathcal{G}^r_\pi = \ker T\pi_{r,r-1} \subset \mathcal{C}^r_\pi.$$

Theorem

$\mathcal{G}^r_\pi$ is a polarisation of $(J^r\pi, \mathcal{C}^r_\pi)$.

The vertical tower

On $J^r\pi$ one has the vertical tower

$$\mathcal{V}_0 = \mathcal{G}^r_\pi, \quad \mathcal{V}_s = \ker T\pi_{r,r-s-1} \text{ for } s = 1, \dots, r-1, \quad \mathcal{V}_r = \ker T\pi_r.$$

The graded quotients satisfy, for $\mathcal{V}_{s-1} = 0$, that

$$\mathcal{V}_s/\mathcal{V}_{s-1} \simeq \pi^*_{r,0}(S^{r-s}T^*Q \otimes V\pi) \quad \text{for } s = 0, \dots, r.$$

The action of horizontal total derivatives on this tower is the Spencer contraction:

$$[D_i, \partial_{u^\alpha_K}] = -\partial_{u^\alpha_{\widehat{K-e_i}}}, \qquad K_i > 0$$

and zero otherwise.

Split and Spencer type

A polarisation $P \subset \mathcal{D}$ is split if locally there is an integrable isotropic complement $H \subset \mathcal{D}$ with

$$\mathcal{D} = H \oplus P.$$

It is of Spencer type if the derived quotients of $\mathcal{D}$ have the model at every point

$$P \simeq S^r H^* \otimes W, \qquad \mathcal{D}^{(s)}/\mathcal{D}^{(s-1)} \simeq S^{r-s} H^* \otimes W,$$

and the bracket action of H is the Spencer contraction.
In jet manifolds $J^r\pi$, the polarisation $\mathcal{G}^r_\pi$ is split and of Spencer type, with

$$\mathcal{P} = \mathcal{G}^r_\pi = \left\langle \frac{\partial}{\partial u^\alpha_J} \right\rangle_{|J|=r}, \quad H = H^r_\pi = \langle D_1, \dots, D_n \rangle.$$

Jet type polarisation

A polarised k-contact manifold $(M, \mathcal{D}, P)$ is of jet type and order r if, locally, there exist:

- an integrable isotropic complement H of P in $\mathcal{D}$,
- an integrable complement $\mathcal{V}_r$ of H in TM,
- regular involutive distributions

$$\mathcal{V}_s = \mathcal{D}^{(s)} \cap \mathcal{V}_r, \quad s = 0, \ldots, r-1,$$

- $\mathcal{D}^{(r)} = TM$ and there exist Spencer-type identifications on the quotients $D^{(s)}/D^{(s-1)}$ and the action of $H \times D^{(s)}/D^{(s-1)} \to D^{(s+1)}/D^{(s)}$ by Lie brackets.

Main theorem II: recognition of finite jets

Local recognition theorem

A polarised k-contact manifold $(M, \mathcal{D}, P)$ is of jet type and order r if and only if, locally around every point there exist coordinates $x^i, u^\alpha, u^\alpha_J$ such that

$$\mathcal{D} = \left\langle D_i = \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{m} \sum_{|J|\leq r-1} u^\alpha_{J+\widehat{e}_i} \frac{\partial}{\partial u^\alpha_J}, \frac{\partial}{\partial u^\alpha_J} : |J| = r \right\rangle, \qquad \mathcal{P} = \left\langle \frac{\partial}{\partial u^\alpha_J} \right\rangle_{|J|=r}.$$

Moreover, if $n = \operatorname{rank} H$ and $m = \operatorname{rank}(\mathcal{V}_r/\mathcal{V}_{r-1})$, then

$$k = m\binom{n+r-1}{r-1}.$$

Recovering the jet bundle itself

The theorem reconstructs the ordinary jet bundle from intrinsic polarised k-contact data.

$$
\begin{gathered}
(M, \mathcal{D}, P, H, \mathcal{V}_r) \\
\Downarrow \\
Q = M/\mathcal{V}_r, \quad E = M/\mathcal{V}_{r-1}, \quad \pi : E \to Q \\
\Downarrow \\
\text{coordinates } x^i,\ u^\alpha,\ u^\alpha_I = D_I u^\alpha \\
\Downarrow \\
\text{local polarised transcontactomorphism } M \simeq J^r\pi.
\end{gathered}
$$

Point

Jets are not assumed as background coordinates; they are recovered from the polarisation, the derived flag and the Spencer bracket action.

What we have?

The recognition theorem is a kind of Darboux theorem: it gives coordinates to put the manifold as a jet manifold. In particular, the following are recovered:

- The Cartan distribution is recovered as the k-contact distribution $\mathcal{D}$.
- The highest-order symbol is recovered as the polarisation P.
- The vertical tower is reconstructed from $\mathcal{D}^{(s)} \cap \mathcal{V}_r$.
- The classical Spencer action is encoded by brackets with H.

Conclusion

We can use k-contact geometry to study jet manifolds, even when we perform techniques that are not intrinsic to the jet bundle. The local recognition theorem shows when these techniques give jet manifolds again.

Order raising as polarised Legendrian prolongation

The next jet space is recovered from Legendrian complements:

$$J^{r+1}\pi \simeq \mathrm{Leg}_{G_\pi^r}(\mathcal{C}_\pi^r).$$

A point of $J^{r+1}\pi$ determines an n-plane in $\mathcal{C}_\pi^r$ complementary to G_π^r; isotropy encodes symmetry of the $(r+1)$-st derivatives.

Tautological statement

Under this identification, the tautological distribution is $\mathcal{C}_\pi^{r+1}$.

Prolonging an equation

For a PDE $\mathcal{E} \subset J^r\pi$, its first prolongation can be read as a polarised Legendrian prolongation with a tangency condition.

Geometric formulation

$\mathcal{E}^{(1)} \subset J^{r+1}\pi \simeq \mathrm{Leg}_{G^r_\pi}(\mathcal{C}^r_\pi)$ consists of those polarised Legendrian n-planes over points of $\mathcal{E}$ whose projection is tangent to $\mathcal{E}$.

This says: prolongation is the Legendrian extension compatible with the equation.

Holonomic sections become polarised Legendrian submanifolds

A connected submanifold $S \subset J^r\pi$ is locally the image of a holonomic section if and only if

$$TS \subset \mathcal{C}^r_\pi, \qquad \mathcal{C}^r_\pi|_S = TS \oplus G^r_\pi|_S.$$

Meaning

Holonomicity is not only contact tangency; it is contact tangency plus transversality to the highest-order polarisation.

Systems of PDEs as polarised k-contact data

Let $\mathcal{E} \subset J^r\pi$ be a regular PDE. Define

$$\mathcal{C}_{\mathcal{E}} = T\mathcal{E} \cap \mathcal{C}^r_\pi, \qquad \mathcal{G}^r_{\mathcal{E}} = T\mathcal{E} \cap G^r_\pi.$$

- $\mathcal{C}_{\mathcal{E}}$ is the induced distribution.
- $\mathcal{G}^r_{\mathcal{E}}$ is the geometric symbol of the equation.
- A classical solution is an n-dimensional polarised Legendrian submanifold contained in $\mathcal{E}$.

Caution

$\mathcal{C}_{\mathcal{E}}$ need not itself be a k-contact distribution; it is the induced contact-compatible distribution of the equation.

k-contact Hamiltonian vector fields

Cartan locus

Let $(M, \mathcal{D})$ be a k-contact manifold and let X be a k-contact vector field. The tangency locus of X with respect to a distribution $\mathcal{D}$ is

$$L_{\mathcal{D}}(X) = \{p : X_p \in \mathcal{D}_p\}.$$

If locally $\mathcal{D}|_U = \ker \boldsymbol{\eta}$, then $L_{\mathcal{D}}(X) \cap U = \{p : \boldsymbol{h}_X(p) = 0\}$.

The Cartan locus is invariant under the flow of X. If X is polarised, the polarised locus $L_P(X) = \{p : X_p \in P_p\}$ is invariant under its flow.
In jet manifolds, if X is the prolongation of a vector field on E, then $L_{\mathcal{C}^r_\pi}(X)$ is the classical characteristic locus of X where its characteristic and their total derivatives vanish.

Lie characteristics from k-contact Hamiltonians

Let X be an N^r_π-contact vector field on $(J^r\pi, \mathcal{C}^r_\pi)$ that is projectable onto E, namely it is a prolongation of a vector field $\sum_{i=1}^n \xi^i \partial_{x^i} + \sum_{\alpha=1}^k \phi^\alpha \partial_{u^\alpha}$ on E. Its characteristic is

$$Q^\alpha = \phi^\alpha - \sum_{i=1}^n \xi^i u_i^\alpha, \qquad \alpha = 1, \dots, m.$$

For any adapted contact form $\boldsymbol{\eta}^r_\pi$, the Hamiltonian $h_X = -\iota_X \boldsymbol{\eta}^r_\pi$ has components

$$h_J^\alpha = -\sum_{L \geq J,\ |L| \leq r-1} (-1)^{|L-J|} \frac{x^{L-J}}{(L-J)!} D_L Q^\alpha, \qquad \alpha = 1, \dots, m.$$

Conversely,

$$Q^\alpha = -\sum_{|L| \leq r-1} \frac{x^L}{L!} h_L^\alpha, \qquad \alpha = 1, \dots, m.$$

Thus, classical Lie characteristics are a distinguished part of the local k-contact Hamiltonian theory. The Lie bracket of Hamiltonian vector fields induces the Lie bracket of characteristics. Indeed, it is only a particular case.

Why reduction is subtle for finite jets

Let $\pi : E \to Q$ and let $(J^r\pi, \mathcal{C}^r_\pi, \mathcal{G}^r_\pi)$ be the Cartan distribution with its highest-order polarisation. Classical symmetry reduction usually requires the symmetry to preserve $\mathcal{C}^r_\pi$. This is too restrictive for many useful reductions, for instance λ-symmetries, coverings, hodograph-type transformations, or reductions that change the effective jet presentation.

Key idea: reduce the enlarged data, not necessarily $\mathcal{C}^r_\pi$ itself.

$$\mathcal{C}^r_\pi \quad \rightsquigarrow \quad \mathcal{C}^r_\pi + \mathcal{D}_G, \qquad \mathcal{G}^r_\pi \quad \rightsquigarrow \quad \mathcal{G}^r_\pi + \mathcal{D}_G.$$

Thus the quotient is allowed to forget the original fibration and reconstruct a new jet model only afterwards.

Projective k-contact reduction distribution

Let $\mathcal{D}_G \subset TJ^r\pi$ be the distribution generated by the infinitesimal action. We do not assume that $\mathcal{D}_G$ preserves $\mathcal{C}^r_\pi$. Instead, we require projectability modulo the group directions:

$$[\Gamma(\mathcal{D}_G), \Gamma(\mathcal{C}^r_\pi + \mathcal{D}_G)] \subset \Gamma(\mathcal{C}^r_\pi + \mathcal{D}_G).$$

Compatibility with the highest-order polarisation means

$$[\Gamma(\mathcal{D}_G), \Gamma(\mathcal{G}^r_\pi + \mathcal{D}_G)] \subset \Gamma(\mathcal{G}^r_\pi + \mathcal{D}_G).$$

If $N \subset J^r\pi$ is an ambient constraint, we also impose (although generally unncessary)

$$[\Gamma(\mathcal{D}_G|_N), \Gamma(TN + \mathcal{D}_G|_N)] \subset \Gamma(TN + \mathcal{D}_G|_N).$$

So $\mathcal{C}^r_\pi$, $\mathcal{G}^r_\pi$, and N become projectable only after enlarging them by $\mathcal{D}_G$.

The reduced polarised distribution

Assume that $q : N \to \overline{N} = N/G$ is a smooth quotient. The reduced Cartan-type and polarisation-type distributions are obtained from

$$\mathcal{F}_N := (TN + \mathcal{D}_G) \cap (\mathcal{C}^r_\pi + \mathcal{D}_G), \qquad \mathcal{P}_N := (TN + \mathcal{D}_G) \cap (\mathcal{G}^r_\pi + \mathcal{D}_G),$$

by pushing them to the quotient:

$$\overline{\mathcal{C}} := Tq(\mathcal{F}_N), \qquad \overline{\mathcal{G}} := Tq(\mathcal{P}_N).$$

$$(N, \mathcal{F}_N, \mathcal{P}_N) \xrightarrow{\quad q \quad} (\overline{N}, \overline{\mathcal{C}}, \overline{\mathcal{G}})$$

The quotient need not be a jet bundle a priori. It is only a reduced polarised distribution. The decisive question is whether $(\overline{N}, \overline{\mathcal{C}}, \overline{\mathcal{G}})$ satisfies the intrinsic jet-type recognition conditions.

Recognition after reduction

The reduced space becomes a genuine finite-order jet model precisely when the reduced polarisation is of jet type. In that case the recognition theorem gives local coordinates and a fibred manifold $\rho : \overline{E} \to \overline{Q}$ such that

$$(\overline{N}, \overline{\mathcal{C}}, \overline{\mathcal{G}}) \simeq (J^s\rho, \mathcal{C}^s_\rho, \mathcal{G}^s_\rho)$$

as a polarised k-contact jet distribution.

If $\mathcal{E} \subset J^r\pi$ is a differential equation and the quotient is regular, then

$$\overline{\mathcal{E}} := (\mathcal{E} \cap N)/G \subset \overline{N}$$

is interpreted as a differential equation inside the reconstructed jet bundle $J^s\rho$.

$\boxed{\text{Reduction first, jet recognition second.}}$

What is gained? The case of λ-symmetries

For λ-symmetries of ODEs, the λ-prolonged field $X_\lambda^{(r)}$ is generally not a Cartan symmetry. Nevertheless its line distribution

$$\mathcal{L} = \langle X_\lambda^{(r)} \rangle$$

satisfies the projective conditions

$$[\Gamma(\mathcal{L}), \Gamma(\mathcal{C}_\pi^r + \mathcal{L})] \subset \Gamma(\mathcal{C}_\pi^r + \mathcal{L}), \qquad [\Gamma(\mathcal{L}), \Gamma(\mathcal{G}_\pi^r + \mathcal{L})] \subset \Gamma(\mathcal{G}_\pi^r + \mathcal{L}).$$

Thus λ-reduction fits the same scheme:

$$(J^r\pi, \mathcal{C}_\pi^r, \mathcal{G}_\pi^r) \quad \rightsquigarrow \quad (\overline{N}, \overline{\mathcal{C}}, \overline{\mathcal{G}}) \quad \overset{\text{jet type}}{\simeq} \quad (J^s\rho, \mathcal{C}_\rho^s, \mathcal{G}_\rho^s).$$

The point is that the reduced system need not live in the original jet presentation; the polarised k-contact structure tells us when it is again a jet equation.

Bäcklund transformations and Lax pairs

Coverings, Bäcklund transformations and Lax representations usually introduce auxiliary variables or relations between different systems.

- The enlarged space carries an induced or enlarged Cartan distribution.
- Compatibility is expressed as Cartan compatibility or as a curvature-defect condition.

Conservative statement

The framework supplies a common geometric language; it does not claim that every auxiliary system is automatically a Hamiltonian zero locus without the needed k-contact hypotheses.

Hodograph transformations

A hodograph transformation can exchange dependent and independent variables:

$$(x^i, u^\alpha) \longmapsto (X^i(x,u), U^\alpha(x,u)).$$

It is usually not projectable with respect to the original fibration, but it may preserve the Cartan distribution.

Interpretation

Hodograph transformations are non-projectable k-contact transformations between different jet presentations.

Other non-projectable operations

The same language also accommodates transformations that are awkward in a fixed fibration.

- Hodograph transformations may change independent variables.
- Sundman-type transformations may mix the independent variable with auxiliary differential data.
- Miura-type maps and coverings can be treated as correspondences between enlarged Cartan spaces.

Unifying principle

Preserve the Cartan geometry first; impose the relevant transversality or polarisation condition only when a specific jet presentation is required.

Open directions

- Infinite-order limit: can be described in terms of finite models in certain cases.
- Formal integrability, Spencer cohomology, characteristic varieties and Vessiot theory in terms of the vertical tower $\mathcal{V}_0 \subset \cdots \subset \mathcal{V}_r$.
- Variational theory on polarised k-contact manifolds.
- Relation with multisymplectic, multicontact, polysymplectic and nonconservative field-theoretic formalisms.
- Concrete applications to Riemann invariants, reductions, solitons and coverings.

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Thank you