The Riemann sphere of $B(H)$

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Notation

- H denotes a Hilbert space and $\mathcal{A} = B(H)$ will denote the C*-algebra of bounded linear operators from H to H.
- $M_2(\mathcal{A})$ denotes the C*-algebra of 2×2 matrices with entries in $\mathcal{A}$, and $\mathcal{U}_2(\mathcal{A})$ its unitary group.
- Matrices $\tilde{a} \in M_2(\mathcal{A})$ will carry a $\tilde{}$.
- $\mathcal{U}_2(\mathcal{A})$ acts on the space $\mathcal{P} \subset M_2(\mathcal{A})$ of orthogonal projections via $L_{\tilde{u}}(\tilde{p}) = \tilde{u}\tilde{p}\tilde{u}^{-1}$ for $\tilde{u} \in \mathcal{U}_2(\mathcal{A})$.
- The **Riemann sphere of $\mathcal{A}$** is the unitary orbit of the projection $\tilde{p}_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in M_2(\mathcal{A})$, that is
$$\mathcal{R} = \{\tilde{u}\tilde{p}_0\tilde{u}^{-1} : \tilde{u} \in \mathcal{U}_2(\mathcal{A})\}.$$
- The **unit sphere** of the C*-module $\mathcal{A}^2$ is
$$\mathcal{K} = \{\tilde{u}e_1 \in \mathcal{A}^2 : \tilde{u} \in \mathcal{U}_2(\mathcal{A}), e_1 = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\}.$$
- The **Hopf fibration** is $\mathfrak{h} : \mathcal{K} \to \mathcal{R}$, defined by
$$\mathfrak{h}(\mathrm{x}) = \mathrm{x}\mathrm{x}^* = \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right)\left(\begin{smallmatrix} x_1^* & x_2^* \end{smallmatrix}\right) = \begin{pmatrix} x_1x_1^* & x_1x_2^* \\ x_2x_1^* & x_2x_2^* \end{pmatrix} = \tilde{u}e_1e_1^*\tilde{u}^{-1} = \tilde{u}\tilde{p}_0\tilde{u}^{-1} = \tilde{p}_{\mathrm{x}}$$
(it is a principal bundle with group $\mathcal{U}$ acting on the right).

Structure

- $\mathcal{R}$ is a reductive homogeneous space of the group $\mathcal{U}_2$.

- This differential geometric structure contains an affine connection on $\mathcal{R}$ and its geodesics.

- It also contains an invariant Finsler metric (defined using spectral norm of $B(H)$ on the tangent spaces) which makes $\mathcal{R}$ into a metric space where geodesics are minimal curves.

- There are 2 charts for $\mathcal{R}$, one of which we will mention right away, and another given by the inverse of the exponential map.

Examples

- **In finite dimensions**
  - If $\mathcal{A} = B(\mathbb{C})$,
    then $\mathcal{R}$ is the one-dimensional complex projective line $\mathbb{P}(\mathbb{C})$ (homeomorphic to S^2) and $\mathcal{K}$ is the unit sphere in $\mathbb{C}^2$ (homeomorphic to S^3).
  - If $\mathcal{A} = B(\mathbb{C}^n)$
    The case $n > 1$ involves the noncommutative C*-algebra $M_n(\mathbb{C})$ of operators on $H = \mathbb{C}^n$. Here $M_2(\mathcal{A})$ is naturally identified with $M_{2n}(\mathbb{C})$ acting on $\mathbb{C}^n \times \mathbb{C}^n$, which is naturally identified with $\mathbb{C}^{2n}$.

Atlas on $\mathcal{R}$

The main chart $(\mathcal{V}_0, \varphi_0, \mathcal{A})$ of $\mathcal{R}$ consists of the neighborhood

$$\begin{aligned}\mathcal{V}_0 &= \{\tilde{p} \in \mathcal{R} : \|\tilde{p} - \tilde{p}_0\| < 1\} \\ &= \mathfrak{h}(\{\mathrm{x}\mathrm{x}^* : \mathrm{x} \in \mathcal{K}, x_1 \text{ is invertible}\}) \subset \mathcal{R}\end{aligned}$$

and the bijection

$$\begin{aligned}\mathcal{V}_0 &\xrightarrow{\varphi_0} \mathcal{A} \\ \tilde{p}_{\mathrm{x}} &\mapsto x_2 x_1^{-1}\end{aligned}$$

for $\tilde{p}_{\mathrm{x}} = \mathrm{x}\mathrm{x}^*$, with $\mathrm{x} = \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right)$, x_1 invertible.

And for each $\tilde{u} \in \mathcal{U}_2(\mathcal{A})$ the chart $(\mathcal{V}_{\tilde{u}}, \varphi_{\tilde{u}})$ is constructed as follows.

$$\mathcal{V}_{\tilde{u}} = L_{\tilde{u}}(\mathcal{V}_0) \;\text{ and }\; \varphi_{\tilde{u}} : \mathcal{V}_{\tilde{u}} \to \mathcal{A} \text{ for } \varphi_{\tilde{u}} = \varphi_0 \circ L_{\tilde{u}^{-1}}.$$

Möbius transformations

Given $\mathrm{x} = \tilde{u}\mathrm{e}_1 \;\; \wedge \;\; \mathrm{y} = \tilde{v}\mathrm{e}_1$, for $a \in \mathcal{V}_{\tilde{u}} \cap \mathcal{V}_{\tilde{v}}$ and writing $(\tilde{u}\tilde{v})^* = \left(\begin{smallmatrix} c & d \\ e & f \end{smallmatrix}\right) \in \mathcal{U}_2(\mathcal{A})$, we obtain the Möbius transformation

$$\left(\varphi_{\tilde{p}_{\mathrm{x}}} \circ \varphi_{\tilde{p}_{\mathrm{y}}}^{-1}\right)(a) = (e + fa)(c + da)^{-1}.$$

As a consequence this atlas defines a C^∞ structure on $\mathcal{R}$.

Example (chart). For $\mathcal{A} = B(\mathbb{C}^n) = M_n(\mathbb{C}^n)$, the domain $\mathcal{V}_0$ of the main chart consists of the orthogonal projections $P_{\mathrm{Gr}(A)}$ for $A \in M_n(\mathbb{C}^n)$, and $\varphi_0(P_{\mathrm{Gr}(A)}) = A$ holds.

Theorem

With the previous notation for the main chart $(\mathcal{V}_0, \varphi_0)$, if $\tilde{p} \in \mathcal{R}$, the following are equivalent

1. $\tilde{p} \in \mathcal{V}_0$, *(that is,* $\tilde{p} = \mathfrak{h}(\mathrm{x}) = \mathrm{x}\mathrm{x}^*$, $\mathrm{x} \in \mathcal{K}$, *with* x_1 *invertible)*
2. $\|\tilde{p} - \tilde{p}_0\| < 1$
3. *if* $\tilde{p} = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$ *in the* $\tilde{p}_0 \oplus (\tilde{1} - \tilde{p}_0)$ *decomposition, then* p_{11} *is invertible*
4. $\tilde{p}$ *is the projection* $P_{Gr(a)} \in B(H \oplus H)$ *onto the graph of the operator* $a = \varphi_0(\tilde{p})$

Example

- ***In infinite dimensions***
  If $\mathcal{A} = B(H)$, then consider the densely defined closed operators T (with closed graph and possible unbounded) on a Hilbert space, then ***the orthogonal projections*** $P_{Gr(T)}$ ***onto the graphs***

  $$Gr(T) = \{(h, T(h)) : h \in Dom(T)\} \subset H \times H$$

  belong to $\mathcal{R}$ and can be written as:

  $$P_{Gr(T)} = \begin{pmatrix} (1 + T^*T)^{-1} & (1 + T^*T)^{-1}T^* \\ T(1 + T^*T)^{-1} & T(1 + T^*T)^{-1}T^* \end{pmatrix}$$

(in this formula the entries are bounded operators).

Definition

*A **bounded deformation** of an unbounded closed operator T on H is a family $\{T_t\}_{t\in[0,\alpha)}$ of bounded operators T_t such that*

- *$t \mapsto T_t$ is continuous in the norm topology*
- *$\lim_{t\to\alpha^-} P_{Gr(T_t)} = P_{Gr(T)}$ where $P_{Gr(T_t)}$ and $P_{Gr(T)}$ lie in the Riemann sphere $\mathcal{R}$ of the algebra $B(H)$ and the limit is taken in the Finsler metric of $\mathcal{R}$.*

In particular, if the bounded deformation $\{T_t\}_{t\in[0,\alpha)}$ of the unbounded operator T satisfies the condition

$$dist(P_{Gr(T_{t_0})}, P_{Gr(T_\alpha)}) = length\ P_{Gr(T_t)}|_{t_0}^{\alpha}, \forall t_0 \in [0,\alpha)$$

*we will call it an **optimal bounded deformation**. Here $dist(P_{Gr(T_{t_0})}, P_{Gr(T_\alpha)})$ denotes the Finsler distance in $\mathcal{R}$ and $length\ P_{Gr(T_t)}|_{t_0}^{\alpha}$ is the Finsler length of the curve where $P_{Gr(T_\alpha)} = P_{Gr(T)}$.*

Example (The differentiation operator $-i\frac{d}{dx}$ and 0)

Consider the operator $$\begin{aligned} -i\frac{d}{dx} : \mathcal{D} &\to L^2[0,1] \\ f &\mapsto -if' \end{aligned}$$ *with domain*

$$\mathcal{D} = \{f \in L^2[0,1] : f \textit{ is abs. cont., } f' \in L^2[0,1] \textit{ and } f(0) = f(1)\}.$$

This operator is unbounded, self-adjoint, densely defined on $L^2[0,1]$*, and closed.*

Using general properties of geodesics of projections, since

$$\begin{aligned} \dim(\overbrace{\operatorname{ran}(P_{Gr(0)}) \cap \ker(P_{Gr(-i\frac{d}{dx})})}^{\mathcal{H}_{10}}) &= \dim(\overbrace{\ker(P_{Gr(0)}) \cap \operatorname{ran}(P_{Gr(-i\frac{d}{dx})})}^{\mathcal{H}_{01}}) \\ &= 0 \end{aligned}$$

there exists a unique geodesic between $P_{Gr(0)}$ *and* $P_{Gr(-i\frac{d}{dx})}$.

As usual, geodesics are studied via the following decomposition (here $\mathcal{H}_{10} = \mathcal{H}_{01} = 0$ holds), making use of the fact that these spaces reduce $P_{\mathrm{Gr}(0)}$ and $P_{\mathrm{Gr}(-i\frac{d}{dx})}$ simultaneously,

$$H \times H = L^2[0,1] \times L^2[0,1] = \mathcal{H}_{11} \oplus \mathcal{H}_{00} \oplus \mathcal{H}_0,$$

for

$$\begin{aligned}
&\mathcal{H}_{11} := \operatorname{ran}(P_{\mathrm{Gr}(0)}) \cap \operatorname{ran}(P_{\mathrm{Gr}(-i\frac{d}{dx})}) = [1] \times \{0\},\\
&\mathcal{H}_{00} := \ker(P_{\mathrm{Gr}(0)}) \cap \ker(P_{\mathrm{Gr}(-i\frac{d}{dx})}) = \{0\} \times [1]\\
\text{and}\ \ &\mathcal{H}_0 := (\mathcal{H}_{11} \oplus \mathcal{H}_{00})^\perp = [1]^\perp \times [1]^\perp
\end{aligned}$$

where $[1] = \{\lambda 1 : \lambda \in \mathbb{C}\}$ denotes the constant functions in $L^2[0,1]$. In this case we have

$$P_{\mathrm{Gr}(0)}|_{\mathcal{H}_{11}\oplus\mathcal{H}_{00}} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) = P_{\mathrm{Gr}(-i\frac{d}{dx})}|_{\mathcal{H}_{11}\oplus\mathcal{H}_{00}}.$$

We now have to study the restriction to $\mathcal{H}_0 = [1]^\perp \times [1]^\perp$.
We will use block matrices in the Fourier basis $\{\xi_n\}_{n\in\mathbb{Z}\setminus\{0\}}$ of $[1]^\perp$ (with $\xi_n(x) = e^{i2\pi nx}$), making use of the fact that the ξ_n are eigenvectors of $-i\frac{d}{dx}$.
For $(h,k) \in [1]^\perp \times [1]^\perp$ one can compute

$$P_{\mathrm{Gr}(-i\frac{d}{dx})}\Big|_{[1]^\perp\times[1]^\perp} \begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} D_1 h & D_2 k \\ D_2 h & D_3 k \end{pmatrix}$$

for the following diagonal operators in the basis $\{\xi_n\}_{n\in\mathbb{Z}\setminus\{0\}}$

$$\begin{aligned} D_1 &= \operatorname{diag}\left(\left\{1/\left(1+(2\pi n)^2\right)\right\}_{n\in\mathbb{Z}\setminus\{0\}}\right) \\ D_2 &= \operatorname{diag}\left(\left\{2\pi n/\left(1+(2\pi n)^2\right)\right\}_{n\in\mathbb{Z}\setminus\{0\}}\right) \\ D_3 &= \operatorname{diag}\left(\left\{(2\pi n)^2/\left(1+(2\pi n)^2\right)\right\}_{n\in\mathbb{Z}\setminus\{0\}}\right). \end{aligned}$$

Note that D_1 and D_2 are diagonal, compact, positive semidefinite operators, and D_3 is positive definite (invertible), all bounded on $[1]^\perp$.

Following ideas of Halmos and Andruchow, one constructs Z_0 such that

$$\delta(t) = e^{itZ_0}\, P_{\mathrm{Gr}(0)}\big|_{[1]^\perp \oplus [1]^\perp}\, e^{-itZ_0}, \qquad t \in [0,1],$$

is a minimal geodesic joining $P_{\mathrm{Gr}(0)}\big|_{[1]^\perp \oplus [1]^\perp}$ to $P_{\mathrm{Gr}(-i\frac{d}{dx})}\big|_{[1]^\perp \oplus [1]^\perp}$, with respect to the Finsler metric defined by the operator norm. For $\binom{f}{g} \in L^2[0,1] \times L^2[0,1] = \mathcal{H}_{11} \oplus \mathcal{H}_{00} \oplus \mathcal{H}_0$, we can write

$$\gamma(t)\binom{f}{g} = \begin{pmatrix} \int_0^1 f \\ 0 \end{pmatrix} + \begin{pmatrix} A(t)^2 & A(t)B(t) \\ B(t)A(t) & B(t)^2 \end{pmatrix}\binom{h}{k} = \begin{pmatrix} \hat{A}_d(t)^2 & \hat{A}(t)\hat{B}(t) \\ \hat{B}(t)\hat{A}(t) & \hat{B}(t)^2 \end{pmatrix}\binom{f}{g},$$

where $\hat{A}_d(t)$, $\hat{A}(t)$ and $\hat{B}(t)$ are self-adjoint diagonal operators, and $\hat{A}_d(t) : L^2[0,1] \to L^2[0,1]$ is invertible for $0 < t < 1$. Then, one can prove that

$$\gamma(t) = P_{\mathrm{Gr}(\hat{B}(t)\hat{A}_d(t)^{-1})}.$$

Theorem (Properties of $-i\frac{d}{dx}$)

The unique geodesic $\gamma : [0,1] \to \mathcal{P}(L^2[0,1] \oplus L^2[0,1])$ joining the orthogonal projection $\gamma(0) = P_{Gr(0)} = P_{L^2[0,1]\oplus\{0\}}$ with $\gamma(1) = P_{Gr(-i\frac{d}{dx})}$ satisfies the following properties.

1. $\gamma(t) = P_{\mathrm{Gr}_{T(t)}}$, for all $t \in (0,1)$, where $T(t)$ are the self-adjoint bounded diagonal operators (in the Fourier basis) $T(t) : L^2[0,1] \to L^2[0,1]$, which can be written as

$$T(t) = \begin{pmatrix} \mathrm{diag}_{n\in\mathbb{Z}_{<0}}\{\tan\big(t\tan^{-1}(2\pi n)\big)\} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathrm{diag}_{p\in\mathbb{Z}_{>0}}\{\tan\big(t\tan^{-1}(2\pi p)\big)\} \end{pmatrix}$$

   in terms of the blocks determined by subspaces of $L^2[0,1]$ generated from the Fourier basis corresponding to $\{\xi_j\}_{j\in\mathbb{Z}_{<0}}$, $\{\xi_0\}$ and $\{\xi_j\}_{j\in\mathbb{Z}_{>0}}$.

2. $\| T(t)\| = \tan(t\pi/2)$, for $0 < t < 1$, and therefore

$$\lim_{t\to 0} \| T(t)\| = 0 \quad \text{and} \quad \lim_{t\to 1} \| T(t)\| = +\infty.$$

3. Every projection $\gamma(t)$ for $0 \leqslant t < 1$ belongs to the chart $(\mathcal{V}_0, \varphi_0, \mathcal{A})$.

4. $\gamma(1) = P_{\mathrm{Gr}(-i\frac{d}{dx})}$ does not belong to the chart $(\mathcal{V}_0, \varphi_0, \mathcal{A})$; however, $P_{\mathrm{Gr}(-i\frac{d}{dx})}$ is on the boundary of $(\mathcal{V}_0, \varphi_0, \mathcal{A})$.

5. $\{T(t)\}_{t\in[0,1)}$ is an optimal bounded deformation between the operator 0 and the unbounded operator $-i\frac{d}{dx}$.

Theorem (Case with infinitely many geodesics)

Let $P_{Gr(0)}$ and Q be orthogonal projections in $B(H \oplus H)$ such that $\dim(\operatorname{ran}(P_{Gr(0)}) \cap \ker(Q)) = \dim(\ker(P_{Gr(0)}) \cap \operatorname{ran}(Q)) > 0$. *Using the decomposition $H \times H = \mathcal{H}_{11} \oplus \mathcal{H}_{00} \oplus \mathcal{H}' \oplus \mathcal{H}_0$, with*

$$\mathcal{H}' = (\operatorname{ran}(P_{Gr(0)}) \cap \ker(Q)) \oplus (\ker(P_{Gr(0)}) \cap \operatorname{ran}(Q)),$$

***all geodesics** $\gamma : [0,1] \to \mathcal{R}$ joining $P_{Gr(0)}$ with Q and with length$(\gamma) \leqslant \pi/2$ are of the form*

$$\gamma_u(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \begin{pmatrix} \cos^2(t\pi/2) & \cos(t\pi/2)(\sin(t\pi/2))u \\ (\sin(t\pi/2))(\cos(t\pi/2))u^* & \sin^2(t\pi/2) \end{pmatrix} & 0 \\ 0 & 0 & 0 & \delta_0(t) \end{pmatrix}$$

- where u is any isomorphism between
$$u : \operatorname{ran}(P_{\mathrm{Gr}(0)}) \cap \ker(Q) \to \ker(P_{\mathrm{Gr}(0)}) \cap \operatorname{ran}(Q),$$
- $\dot{\gamma}|_{\mathcal{H}'}(0) = X = \begin{pmatrix} 0 & \frac{\pi}{2}u \\ \frac{\pi}{2}u^* & 0 \end{pmatrix}$,
- δ_0 is the unique geodesic between $P_{\mathrm{Gr}(0)}\big|_{\mathcal{H}_0}$ and $Q\big|_{\mathcal{H}_0}$,
- and γ_u has minimal length $\pi/2$.

Definition

Given a geodesic $\gamma(t)$, $t \in [0,1]$, *a parameter value* $t_0 \in [0,1]$ *is called* ***conjugate to*** 0 ***along*** γ *if there exists a nontrivial Jacobi field vanishing at* 0 *and at* t_0. *In this case the* ***index of*** t_0 *is the dimension of the space of Jacobi fields vanishing at* 0 *and at* t_0. *The parameter* t_0 *is called* ***conjugate*** *if this index is greater than* 0.

The following is an example of conjugate values in $\mathcal{R}$ involving Fredholm operators.

Theorem

Let F *be a Fredholm operator of index zero and*
$n = \dim(\ker(F)) = \dim(\mathrm{ran}(F)^{\perp}) > 0$.
Consider the infinite geodesics obtained in the previous theorem that join the projection $P_{Gr(0)}$ *with* $P_{\mathrm{invGr}(F)}$ *(for* $\mathrm{invGr}(F) = \{(Fx, x) : x \in H\}$*).*
Then, 1 *is a parameter conjugate to* 0 *with* $t \in [0,1]$, *and the index of this conjugate parameter has dimension* n^2.

Some general results

Proposition (Geodesics between bounded operators)

*Given S and T **bounded** operators on H, then*

there exists a (minimal) geodesic in $\mathcal{R}$ joining $P_{Gr(S)}$ with $P_{Gr(T)}$

$$\Leftrightarrow \quad \dim\left(\ker(1+S^*T)\right) = \dim\left(\ker(1+T^*S)\right)$$

(and it is unique if these subspaces are trivial).

Corollary

Starting from $P_{Gr(0)}$ there always exist (unique) geodesics to other $P_{Gr(T)}$ for bounded T.

Corollary

If both S and T are bounded, and one of them is compact, then there exists a geodesic between $Gr(S)$ and $Gr(T)$.

Theorem

Let $T : D(T) \subset H \to H$ be a densely defined closed operator. Then there exists a unique minimal geodesic γ between $\gamma(0) = P_{Gr(0)}$ and $\gamma(1) = P_{Gr(T)}$. It is given by

$$\gamma(t) = e^{itZ} P_{Gr(0)} e^{-itZ},$$

$$\text{for } Z = \begin{pmatrix} 0 & i \arctan(|T|) V^* \\ -iV \arctan(|T|) & 0 \end{pmatrix},$$

where $T = V|T|$ is the polar decomposition.
Moreover, the intermediate projections $\gamma(t)$, $t \in (0,1)$ correspond to graphs of bounded operators: $\gamma(t) = P_{Gr(B(t))}$, $B(t) \in B(H)$.

Remark

If T is bounded, $\|Z\| = \|\arctan(|T|)\| = \arctan(\|T\|) < \pi/2$, while if it is not, $\|Z\| = \|\arctan(|T|)\| = \pi/2$.

Any pair of projections $P_{Gr(A)}$ and $P_{Gr(B)}$ of **self-adjoint** operators A, B can be joined by a geodesic of $\mathcal{P}(H \times H)$:

Theorem

Let $A : D(A) \subset H \to H$ and $B : D(B) \subset H \to H$ be self-adjoint operators. Then there exists a minimal geodesic δ of $\mathcal{P}(H \times H)$ such that $\delta(0) = P_{Gr(A)}$ and $\delta(1) = P_{Gr(B)}$. Moreover, the intermediate projections $\delta(t)$, $t \in (0,1)$ are the graphs of bounded operators: $\delta(t) = P_{Gr(B(t))}$, where $B(t) \in B(H)$.

Lemma

Suppose that S and T are bounded. There exists a (minimal) geodesic between $P_{Gr(S)}$ and $P_{Gr(T)}$ for $S, T \in B(H)$ if and only if $\dim\ker(I + S^*T) = \dim\ker(I + T^*S)$.

Corollary

If both S and T are bounded, and one of them is compact, then (using the Fredholm alternative) there exists a geodesic between $P_{Gr(S)}$ and $P_{Gr(T)}$.

If S or T are **non self-adjoint**, there may not exist geodesics joining their graphs, as can be seen in the following simple example:

Example

Consider $\mathcal{S}$ the (unilateral) shift operator in ℓ^2 defined by $\mathcal{S}(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$. Put $S_2 = -2\mathcal{S}$ and $T = I$ (the identity). Then $\ker(1 + T^* S_2) = \ker(1 - 2\mathcal{S}) = \ker(\frac{1}{2} - \mathcal{S}) = \{0\}$ (the shift has no eigenvalues).
On the other hand $\ker(1 + S_2^* T) = \ker(1 - 2\mathcal{S}^*) = \ker(\frac{1}{2} - \mathcal{S}^*)$ which has dimension 1.
Therefore, by the previous lemma, since

$$0 = \dim\left(\ker(I + T^* S_2)\right) \neq \dim\left(\ker(I + S_2^* T)\right) = 1,$$

the orthogonal projections of $Gr(I) = \{(x, x) : x \in \ell^2\}$ and $Gr(-2\mathrm{S}) = \{(y, -2\mathrm{S}y) : y \in \ell^2\}$ cannot be joined by a geodesic of $\mathcal{R}$.

Theorem (Geodesics from $P_{\mathrm{Gr}(0)}$ with initial conditions)

Let T be a densely defined, closed operator with domain $\mathcal{D}(T)$. The unique minimal geodesic $\gamma : [0,1] \to \mathcal{P}(H \oplus H)$ such that $\gamma(0) = P_{Gr(0)}$, $\gamma(1) = P_{Gr(T)}$ and $\dot\gamma(0) = \left(\begin{smallmatrix} 0 & a \\ a^* & 0 \end{smallmatrix}\right)$, with $\|a\| \leqslant \pi/2$, consists of orthogonal projections onto the graphs $\gamma(t) = P_{Gr(A(t))}$ of the operators

$$A(t) = ta^*(\operatorname{sinc}|ta^*|)(\cos|ta^*|)^{-1} = v\tan|ta^*| \ \in B(H),$$

for $t \in [0,1)$ and v the partial isometry of the polar decomposition of $a^* = v|a^*|$, with $\|a\| \leqslant \pi/2$.
Moreover,

$$\gamma(t) = (\cos^2|t\tilde X|)\tilde p_0 + (\sin^2|t\tilde X|)(1-\tilde p_0) + \operatorname{sinc}(2t|\tilde X|)\tilde X\tilde\rho_0$$

for $\tilde X = \left(\begin{smallmatrix} 0 & -a \\ a^* & 0 \end{smallmatrix}\right)$ and $\tilde\rho_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

Corollary

The deformation
$t \mapsto \{T_t\}_{t\in[0,1]} = \{\gamma(t)\}_{t\in[0,1]} = \{P_{Gr(A(t))}\}_{t\in[0,1]}$, *with*

$$A(t) = ta^*(\operatorname{sinc}|ta^*|)(\cos|ta^*|)^{-1} = v\tan|ta^*| \in B(H)$$

is an optimal bounded deformation.
That is

$$length|_{t_0}^{1}(\tilde{p}_{t_0}, \tilde{p}_1) = dist(\tilde{p}_{t_0}, \tilde{p}_1)$$

for $t_0 \in [0,1)$, *where length and dist (distance in the Riemann sphere* $\mathcal{R}$*) are defined by the Finsler metric on* $\mathcal{R}$.

Remark

Then, for any densely defined closed unbounded operator T on H, its orthogonal projection onto the graph $P_{Gr(T)}$ is in the boundary of the domain of the image of the chart φ_0 when we identify the operators $A \in B(H)$ with their orthogonal projections onto their graphs $P_{Gr(A)}$.

THANK YOU VERY MUCH!

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