Kähler and Poisson Geometry of affine coadjoint orbits of infinite-dimensional unitary groups

Alice Barbora Tumpach

Institut CNRS Pauli, Vienna, Austria
& Laboratoire Painlevé, Lille, France

WGMP 2026, 29.06.2026

Supported by FWF grant I 5015-N
“Banach Poisson Lie Groups and Integrable systems”
and
FWF AI Austria grant PAT1179524
"Geometric Green Learning on Groups and Quotient Spaces”

Based on joined works with T. Goliński, G. Larotonda and A. Tahiri:

1. KAEHLER GEOMETRY
   - A. Tahiri, A.B.Tumpach, *The restricted p-Schatten-class Grassmannian as affine coadjoint orbit*, ArXiv.
   - T. Goliński, G. Larotonda, A.B. Tumpach, *On the Kähler geometry of infinite dimensional affine coadjoint orbits of unitary groups*, in preparation.
2. POISSON GEOMETRY
   - A.B.Tumpach, *Bruhat-Poisson structure of the restricted Grassmannian and the KdV hierarchy*, Communications of Mathematical Physics.
   - T. Goliński, A.B. Tumpach, *Banach Poisson-Lie groups, Lax equations and the AKS Theorem in infinite dimensions*, Differential Geometry and its Applications.
   - A. Tahiri, A.B.Tumpach, *Banach Poisson-Lie groups in duality*, in preparation.
3. HYPERKAEHLER GEOMETRY
   - A.B.Tumpach, *Hyperkähler structures and infinite-dimensional Grassmannians*, Journal of Functional Analysis.
   - A.B.Tumpach, *Infinite-dimensional hyperkähler manifolds associated with Hermitian-symmetric affine coadjoint orbits*, Annales de l'Institut Fourier.

Let $\mathscr{H}$ be a complex separable infinite-dimensional Hilbert space endowed with a decomposition $\mathscr{H} = \mathscr{H}_+ \oplus \mathscr{H}_-$ onto the orthogonal sum of two closed infinite-dimensional subspaces.

Definition

The **restricted Schatten class Grassmannian** $\mathrm{Gr}_{\mathrm{res},p}(\mathscr{H})$ is defined as the set of all closed subspaces $V \subseteq \mathscr{H}$ such that

1. the orthogonal projection $\mathrm{pr}_+|_V : V \to \mathscr{H}_+$ is Fredholm (i.e. has finite index);
2. the orthogonal projection $\mathrm{pr}_-|_V : V \to \mathscr{H}_-$ is of class L_p.

For two Hilbert spaces $\mathscr{H}_1, \mathscr{H}_2$ and $1 \leq p < +\infty$, the Schatten class ideal $L_p(\mathscr{H}_1, \mathscr{H}_2)$ is the Banach space of bounded linear operators A such that

$$\mathrm{Tr}\,(A^*A)^{\frac{p}{2}} < +\infty$$

endowed with the norm $\|A\|_p := \left(\mathrm{Tr}\,(A^*A)^{\frac{p}{2}}\right)^{\frac{1}{p}}$.

$\mathscr{H} = \mathscr{H}_+ \oplus \mathscr{H}_-$, $d = i(p_+ - p_-)$.

$$L_{res,p}(\mathscr{H}) = \{A \in B(\mathscr{H}), A_{+-}, A_{-+} \in L_p\}.$$

$$L_{1,q}(\mathscr{H}) = \{A \in B(\mathscr{H}) : A_{++}, A_{--} \in L_1, \text{ and } A_{+-}, A_{-+} \in L_q\}.$$

Definition

The restricted unitary group $\mathrm{U}_{\mathrm{res},p}(\mathscr{H})$ is defined as follows:

$$\mathsf{U}_{\mathrm{res},p}(\mathscr{H}) = \{u \in \mathsf{U}(\mathscr{H}) \mid [d, u] \in L_p(\mathscr{H})\} = \mathsf{U}(\mathscr{H}) \cap L_{\mathrm{res},p}(\mathscr{H}).$$

Proposition (Tahiri-T.)

The Banach Lie group $\mathsf{U}_{\mathrm{res},p}(\mathscr{H})$ acts transitively on the restricted p-Schatten class Grassmannian, and the stabilizer of $\mathscr{H}_+ \in \mathrm{Gr}_{\mathrm{res},p}(\mathscr{H})$ is $\mathsf{U}(\mathscr{H}_+) \times \mathsf{U}(\mathscr{H}_-)$.

$$\mathscr{H} = \mathscr{H}_+ \oplus \mathscr{H}_-, \; d = i(p_+ - p_-).$$

Definition

For $1 < p \leq 2$, and $A, B \in L_{\mathrm{res},p}(\mathscr{H})$, define the Schwinger 2-cocycle by

$$s(A, B) := \mathrm{Tr}_{\mathrm{res}}(A[d, B]), \tag{1}$$

Definition

For $1 < p \leq 2$, we define the Banach Lie algebra $\tilde{\mathfrak{u}}_{\mathrm{res},p}$ as the central extension of $\mathfrak{u}_{\mathrm{res},p}$ with continuous two-cocycle s given by

$$s(A, B) := \mathrm{Tr}_{\mathrm{res}}(A[d, B]), \tag{2}$$

for all $A, B \in \mathfrak{u}_{\mathrm{res},p}$. That is, $\tilde{\mathfrak{u}}_{\mathrm{res},p}$ is the Banach Lie algebra $\mathfrak{u}_{\mathrm{res},p} \oplus \mathbb{R}$ endowed with the bracket $[\cdot, \cdot]_d$ defined by

$$[(A, a), (B, b)]_d = ([A, B], s(A, B)) \,. \tag{3}$$

Symplectic structures

Symplectic form = smoothly varying skew-symmetric bilinear form

$$\begin{array}{cccc} \omega_x & : & T_xM \times T_xM & \to & \mathbb{R} \\ & & (U, V) & \mapsto & \omega_x(U, V) \end{array}$$

with $dw = 0$ and $(T_xM)^{\perp_w} = \{0\}$

strong symplectic form = for every $x \in M$, $\omega_x \ : T_xM \to (T_xM)^*$ is an isomorphism

weak symplectic form = for every $x \in M$, $\omega_x \ : T_xM \to (T_xM)^*$ is just injective

Darboux Theorem does not hold for a weak symplectic form

Theorem (Tahiri-T.)

For $1 < p \leq 2$, the homogeneous space

$$\mathrm{Gr}_{\mathrm{res},p}(\mathscr{H}) = \mathsf{U}_{\mathrm{res},p}(\mathscr{H}) / \left(\mathsf{U}(\mathscr{H}_+) \times \mathsf{U}(\mathscr{H}_-)\right)$$

admits a natural $\mathsf{U}_{\mathrm{res},p}(\mathscr{H})$-invariant weak symplectic structure, whose expression at the class of the identity operator is given by

$$\Omega_{[\mathrm{id}]}\left([A],[B]\right) = 2\Im\mathrm{Tr}\left(A_{-+}^* B_{-+}\right) = -\frac{1}{2}\mathrm{Tr}_{\mathrm{res}} A[d,B] = -\frac{1}{2}s(A,B),$$

where

$$[A] = \left[\begin{pmatrix} A_{++} & -A_{-+}^* \\ A_{-+} & A_{--} \end{pmatrix}\right] = \left[\begin{pmatrix} 0 & -A_{-+}^* \\ A_{-+} & 0 \end{pmatrix}\right]$$

denotes the class of A modulo the isotropy Lie algebra.

Poisson Structures

Poisson bracket = family of bilinear maps
$\{\cdot,\cdot\}_U : \mathscr{C}^\infty(U) \times \mathscr{C}^\infty(U) \to \mathscr{C}^\infty(U)$, U open in M with

- skew-symmetry $\{f,g\}_U = -\{g,f\}_U$
- Jacobi identity $\{f,\{g,h\}_U\}_U + \{g,\{h,f\}_U\}_U + \{h,\{f,g\}_U\}_U = 0$
- Leibniz rule $\{f,gh\}_U = \{f,g\}_U h + g\{f,h\}_U$

A strong symplectic form defines a Poisson bracket by
$\{f,g\} = \omega(X_f, X_g)$ where $df = \omega(X_f,\cdot)$ and $dg = \omega(X_g,\cdot)$

A Poisson bracket may not be given by a bivector field
References: D. Beltita, T. Goliński, A.B.Tumpach, *Queer Poisson brackets*,
T. Goliński, P. Rahangdale and A.B. Tumpach, *Poisson structures in the Banach setting: comparison of different approaches.*

Definition of a Banach Poisson manifold

Definition of a Poisson tensor: [T.]
M Banach manifold, $\mathbb{F}$ a subbundle of T^*M in duality with TM.
π smooth section of $\Lambda^2\mathbb{F}^*(\mathbb{F})$ is called a Poisson tensor on M with respect to $\mathbb{F}$ if :

1. for any closed local sections α, β of $\mathbb{F}$, the differential $d\left(\pi(\alpha,\beta)\right)$ is a local section of $\mathbb{F}$;
2. (Jacobi) for any closed local sections α, β, γ of $\mathbb{F}$,

$$\pi\left(\alpha, d\left(\pi(\beta,\gamma)\right)\right) + \pi\left(\beta, d\left(\pi(\gamma,\alpha)\right)\right) + \pi\left(\gamma, d\left(\pi(\alpha,\beta)\right)\right) = 0.$$

Definition of a Poisson Manifold [T.] A Banach Poisson manifold is a triple $(M, \mathbb{F}, \pi)$ consisting of a smooth Banach manifold M, a subbundle $\mathbb{F}$ of the cotangent bundle T^*M in duality with TM, and a Poisson tensor π on M with respect to $\mathbb{F}$.

Theorem (Tahiri-T.)

Set $1 < p \leq 2$ and q such that $\frac{1}{p} + \frac{1}{q} = 1$, and $q = \infty$ for $p = 1$. The Banach space $\tilde{\mathfrak{u}}_{1,q} := \mathfrak{u}_{1,q} \oplus \mathbb{R}$ is a Banach Lie-Poisson space with respect to $\tilde{\mathfrak{u}}_{\mathrm{res},p}$ for the Poisson bracket

$$\{f, g\}_d(\mu, \gamma) := \langle \mu, [D_\mu f(\mu, \gamma), D_\mu g(\mu, \gamma)] \rangle + \gamma s(D_\mu f(\mu, \gamma), D_\mu g(\mu, \gamma)) \tag{4}$$

where f, g are smooth real functions on $\tilde{\mathfrak{u}}_{1,q}$, (μ, γ) is an arbitrary element in $\tilde{\mathfrak{u}}_{1,q}$, and D_μ denotes the partial Fréchet derivative with respect to $\mu \in \tilde{\mathfrak{u}}_{1,q}$.

Theorem (Tahiri-T.)

The symplectic structure on the restricted Grassmannian $\mathrm{Gr}_{\mathrm{res},p}(\mathscr{H}_+)$, $1 \leq p \leq 2$, coincide (modulo constant) with the Kirillov-Kostant-Souriou symplectic forms on the affine coadjoint orbit $\mathscr{O}_{(0,\gamma)}$ of $(0,\gamma) \in \tilde{\mathfrak{u}}_{1,q}$, $\gamma \neq 0$ after the identification

$$\begin{array}{ccl} \mathrm{Gr}_{\mathrm{res},p}(\mathscr{H}) & \longrightarrow & \mathscr{O}_{(0,\gamma)} \\ W & \longmapsto & \gamma\left(i(\mathrm{pr}_W - \mathrm{pr}_{W^\perp}) - i(\mathrm{pr}_+ - \mathrm{pr}_-)\right). \end{array}$$

where pr_W (resp. $\mathrm{pr}_{W^\perp}$) is the orthogonal projection onto W (resp. $W^\perp$).

Theorem (Goliński-Larotonda-T.)

The restricted Grassmannian $\mathrm{Gr}_{\mathrm{res},p}(\mathscr{H}_+)$, $1 < p \leq 2$, is a Kähler manifold.

For $p = 1$, the complex structure is only densely defined
For $p > 2$, the symplectic form is only densely defined

Banach Poisson-Lie groups

A **Banach Poisson-Lie group** B is a Banach Lie group equipped with a Banach Poisson manifold structure compatible with the multiplication

Proposition (T.)

Let B be a Banach Lie group and $(B, \mathbb{B}, \pi)$ a Banach Poisson structure on B. Then B is a Banach Poisson-Lie group if and only if

1. *$\mathbb{B}$ is invariant under left and right multiplications by elements in B,*
2. *the subspace $\mathfrak{u} := \mathbb{B}_e \subset \mathfrak{b}^*$, where e is the unit element of B, is invariant under the coadjoint action of B on $\mathfrak{b}^*$ and the map*

$$\begin{array}{rcccl} \Pi_r^B & : & B & \to & \Lambda^2\mathfrak{u}^* \\ & & g & \mapsto & R^{**}_{g^{-1}}\pi_g, \end{array}$$

   is a 1-cocycle on B with respect to the coadjoint representation of B in $\Lambda^2\mathfrak{u}^*$.

Banach Manin triple

A **Banach Manin triple** is a triple of Banach Lie algebras $(\mathfrak{d}, \mathfrak{g}_+, \mathfrak{g}_-)$ and a non-degenerate continuous symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{d}$ such that

- $\mathfrak{d} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$
- $\langle \cdot, \cdot \rangle$ is $\mathfrak{d}$-invariant
- $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are Banach Lie subalgebras of $\mathfrak{d}$
- $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are isotropic for $\langle \cdot, \cdot \rangle$

Banach Lie bialgebras

Definition : Let $\mathfrak{b}$ be a Banach Lie algebra, and a duality pairing $\langle \cdot, \cdot \rangle_{\mathfrak{b},\mathfrak{u}}$ between $\mathfrak{b}$ and a Banach Lie algebra $\mathfrak{u}$. One says that $\mathfrak{b}$ is a Banach Lie bialgebra with respect to $\mathfrak{u}$ if

(1) $\mathfrak{b}$ acts continuously by coadjoint action on $\mathfrak{u}$.

(2) there is a 1-cocycle $\theta \; : \mathfrak{b} \rightarrow \Lambda^2 \mathfrak{u}^*$ with respect to the adjoint representation of $\mathfrak{b}$ on $\Lambda^2 \mathfrak{u}^*$, i.e. satisfying

$$\begin{aligned} \theta\left([x, y]\right)(\alpha, \beta) = \; & \theta(y)(\mathrm{ad}_x^* \alpha, \beta) + \theta(y)(\alpha, \mathrm{ad}_x^* \beta) \\ & - \theta(x)(\mathrm{ad}_y^* \alpha, \beta) - \theta(x)(\alpha, \mathrm{ad}_y^* \beta) \end{aligned}$$

where $x, y \in \mathfrak{b}$ and $\alpha, \beta \in \mathfrak{u}$.

Definition

We will say that $\mathfrak{b}$ is a Banach Lie-Poisson space with respect to $\mathfrak{u}$ if $\mathfrak{u}$ is in duality with $\mathfrak{b}$ and is a Banach Lie algebra $(\mathfrak{u}, [\cdot, \cdot]_{\mathfrak{u}})$ which acts continuously on $\mathfrak{b}$ by coadjoint action.

Poisson–Lie groups in the infinite-dimensional case

Manin triple

$\Updownarrow$

Banach Lie-bialgebra + Banach Lie-Poisson space

$\Uparrow$

Banach Poisson–Lie group G + $\pi^\sharp(\alpha) := \pi(\alpha, \cdot)$ takes values in TG

A.B.Tumpach, *Bruhat-Poisson structure of the restricted Grassmannian and the KdV hierarchy*, Communications of Mathematical Physics.

Poisson–Lie groups in the infinite-dimensional case

Manin triple

$\Updownarrow$

Banach Lie-bialgebra + Banach Lie-Poisson space

$\Uparrow$ $\Downarrow$

Banach Poisson–Lie group G + $\pi^\sharp(\alpha) := \pi(\alpha, \cdot)$ takes values in TG

A.B.Tumpach, *Bruhat-Poisson structure of the restricted Grassmannian and the KdV hierarchy*, Communications of Mathematical Physics.
P. Rahangdale, *Drinfeld Correspondence in Infinite Dimensions*, ArXiv.

Theorem (Tahiri-T.)

Let $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and set $r = \min\{p, q\}$. Consider the Banach Lie group $U_{\mathrm{res},p}(\mathscr{H})$, and

1. $\mathfrak{g}_- = L_{1,r}(\mathscr{H})/\mathfrak{u}_{1,r}(\mathscr{H}) \subset \mathfrak{u}_{\mathrm{res},p}(\mathscr{H})^*$;
2. $\mathbb{U} \subset T^* U_{\mathrm{res},p}(\mathscr{H})$, *given by* $\mathbb{U}_u = r^*_{u^{-1}}\mathfrak{g}_-$;
3. $\Pi\colon U_{\mathrm{res},p}(\mathscr{H}) \to L^2_{skew}(\mathfrak{g}_-)$, *defined by*
$$\Pi(u)([x_1], [x_2]) = \Im \mathrm{Tr}\left(u^{-1}\mathrm{pr}_{\mathfrak{b}_r}(x_1) u \mathrm{pr}_{\mathfrak{u}_r}(u^{-1}\mathrm{pr}_{\mathfrak{b}_r}(x_2) u)\right);$$
4. $\pi_u := r^{**}_{u^{-1}}\Pi(u)$.

Then the triple $(U_{\mathrm{res},p}(\mathscr{H}), \mathbb{U}, \pi)$ is a Banach Poisson-Lie group.

Theorem (Tahiri-T.)

For $1 < p < \infty$ and $r = \min\{p, q\}$, the restricted p-Schatten class Grassmannian, $\mathrm{Gr}_{\mathrm{res},p}(\mathscr{H})$, carries a Poisson structure $(\mathrm{Gr}_{\mathrm{res},p}(\mathscr{H}), \mathbb{G}, \pi^{\mathrm{res},p})$ such that:

1. *the canonical projection $\mathrm{pr}\colon U_{\mathrm{res},p}(\mathscr{H}) \to \mathrm{Gr}_{\mathrm{res},p}(\mathscr{H})$ is a Poisson morphism,*
2. *the natural left action $U_{\mathrm{res},p}(\mathscr{H}) \times \mathrm{Gr}_{\mathrm{res},p}(\mathscr{H}) \to \mathrm{Gr}_{\mathrm{res},p}(\mathscr{H})$ is a Poisson action.*

Definition

A hyperkähler manifold can be defined as a smooth manifold $\mathscr{M}$ endowed with

- one Riemannian metric g,
- three (formally integrable) complex structures I_1, I_2, I_3 satisfying the same identities as the quaternions, in particular $I_1 I_2 I_3 = -1$,
- three symplectic forms ω_1, ω_2, ω_3 linked to the Riemannian metric g and the complex structures I_i by $\omega_i(X, Y) = g(X, I_i Y)$.

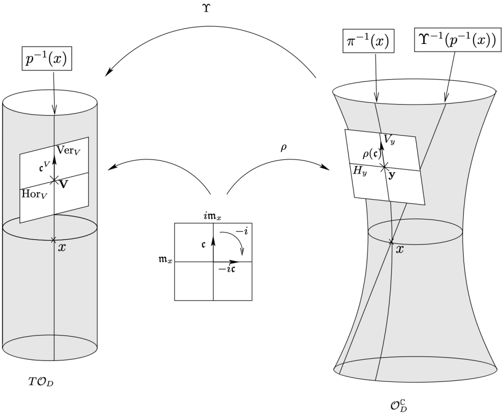
- $\mathscr{M}$ endowed with I_1 and $\omega_{\mathbb{C}} := \omega_2 + i\omega_3$ is a complex symplectic manifold.
- for any $(a, b, c) \in \mathbb{S}^2$, $I_{(a,b,c)} = aI_1 + bI_2 + cI_3$ is a complex structure on $\mathscr{M}$.
- $\Rightarrow$ any hyperkähler manifold admits a family of complex symplectic structures indexed by the two-sphere of complex structures defined by I_1, I_2, I_3.

Theorem (T.)

For $p = 2$, the cotangent space $T^*\mathrm{Gr}_{\mathrm{res},2}(\mathscr{H}_+)$ of the restricted Grassmannian is a hyperkähler manifold isomorphic to the complexification $\mathrm{Gr}^{\mathbb{C}}_{\mathrm{res},2}(\mathscr{H}_+)$ of the Grassmannian defined as

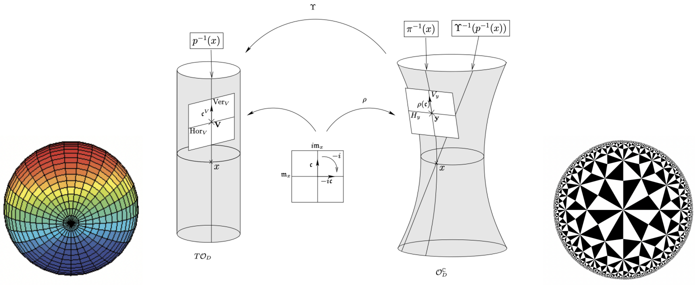
$$\mathrm{Gr}^{\mathbb{C}}_{\mathrm{res},2}(\mathscr{H}_+) = \{(P, Q) \in \mathrm{Gr}_{\mathrm{res},2}(\mathscr{H}_+) \times \mathrm{Gr}_{\mathrm{res},2}(\mathscr{H}_-), P \oplus Q = \mathscr{H}\}$$

A. B. Tumpach, *Hyperkähler structures and infinite-dimensional Grassmannians,* Journal of Functional Analysis
A.B.Tumpach, *Infinite-dimensional hyperkähler manifolds associated with Hermitian-symmetric affine coadjoint orbits*, Annales de l'Institut Fourier.



[2] A. B. Tumpach, *Hyperkähler structures and infinite-dimensional Grassmannians,* Journal of Functional Analysis, https://doi.org/10.1016/j.jfa.2006.05.019

The 2-sphere and the hyperbolic disc are living in the same coadjoint orbit of $SL(2, \mathbb{C})$



[2] A. B. Tumpach, *Hyperkähler structures and infinite-dimensional Grassmannians*, Journal of Functional Analysis, https://doi.org/10.1016/j.jfa.2006.05.019

The 2-sphere and the hyperbolic disc are living in the same coadjoint orbit of $SL(2,\mathbb{C})$

- we will identify the 2-sphere with the complex projective line $\mathbb{CP}(1)$ endowed with its Kähler metric known as the Fubini-Study metric.
- The corresponding Riemannian metric is $\frac{1}{4}$ times the Riemannian metric of the round sphere of radius 1,
- the complex structure is the one of the Riemannian sphere $\mathbb{S}^2 \simeq \mathbb{C} \cup \{\infty\}$.
- The hyperbolic Poincaré disc will be denoted by $\mathbb{H}^2 = \{z \in \mathbb{C}, |z| < 1\}$.

$$G = SU(2) := \left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \mid |a|^2 + |b|^2 = 1, a, b \in \mathbb{C} \right\},$$

$$G^{\mathbb{C}} = SL(2,\mathbb{C}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1, a, b, c, d \in \mathbb{C} \right\},$$

$$x := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in \mathfrak{su}(2).$$

Let G and $G^{\mathbb{C}}$ act by adjoint action on their Lie algebras, and denote by $\mathscr{O}_x$ the compact adjoint orbit of x under G, and $\mathscr{O}_x^{\mathbb{C}}$ the complex adjoint orbit of x under $G^{\mathbb{C}}$. One has:

$$\begin{array}{ccc} \mathscr{O}_x & \simeq & \mathbb{CP}(1) \\ \left(\begin{smallmatrix} a & -\bar{c} \\ c & \bar{a} \end{smallmatrix}\right)\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)\left(\begin{smallmatrix} a & -\bar{c} \\ c & \bar{a} \end{smallmatrix}\right)^{-1} & \mapsto & \mathrm{Eig}(i) = \mathbb{C}\begin{pmatrix} a \\ c \end{pmatrix}, \end{array}$$

where $\mathrm{Eig}(i)$ is the eigenspace for the eigenvalue i, and similarly

$$\begin{array}{ccc} \mathscr{O}_x^{\mathbb{C}} & \simeq & \left\{(\ell_1, \ell_2) \in \mathbb{CP}(1) \times \mathbb{CP}(1), \ell_1 \oplus \ell_2 = \mathbb{C}^2\right\} \\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^{-1} & \mapsto & \left(\mathrm{Eig}(i) = \mathbb{C}\begin{pmatrix} a \\ c \end{pmatrix}, \mathrm{Eig}(-i) = \mathbb{C}\begin{pmatrix} b \\ d \end{pmatrix}\right). \end{array}$$

The stabilizers of x for the adjoint actions of G and $G^{\mathbb{C}}$ are respectively:

$$\mathfrak{t} := \mathrm{Stab}_{SU(2)}(x) = U(1) = \left\{\left(\begin{smallmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{smallmatrix}\right), \theta \in \mathbb{R}\right\},$$
$$\mathfrak{t}^{\mathbb{C}} := \mathrm{Stab}_{SL(2,\mathbb{C})}(x) = \mathbb{C}^* = \left\{\left(\begin{smallmatrix} a & 0 \\ 0 & \frac{1}{a} \end{smallmatrix}\right), a \in \mathbb{C}^*\right\}.$$

- the orthogonal $\mathfrak{m}$ of $\mathfrak{t}$ in $\mathfrak{su}(2)$ with respect to the Killing form $\kappa(A, B) := \mathrm{Tr}\,(\mathrm{ad}(A)\mathrm{ad}(B)) = 4\mathrm{Tr}\, AB$:

$$\mathfrak{m} := \left\{ \begin{pmatrix} 0 & -\bar{c} \\ c & 0 \end{pmatrix}, c \in \mathbb{C} \right\}.$$

- $\mathfrak{g} = \mathfrak{su}(2) = \mathfrak{t} \oplus \mathfrak{m}$
- $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}(2, \mathbb{C}) = \mathfrak{t} \oplus \mathfrak{m} \oplus i\mathfrak{t} \oplus i\mathfrak{m}$, with the commutation relations $[\mathfrak{t}, \mathfrak{m}] \subset \mathfrak{m}$ and $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{t}$.

$$\mathfrak{g}^{n.c.} := \mathfrak{su}(1, 1) = \mathfrak{t} \oplus i\mathfrak{m} = \left\{ \begin{pmatrix} i\theta & \bar{\beta} \\ \beta & -i\theta \end{pmatrix}, \theta \in \mathbb{R}, \beta \in \mathbb{C} \right\}$$

$$SU(1, 1) := \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, a, b \in \mathbb{C}, |a|^2 - |b|^2 = 1 \right\}.$$

Proposition

The non-compact orbit $\mathscr{O}_x^{n.c.}$ is diffeomorphic to the quotient space $SU(1,1)/U(1)$ hence to $\mathbb{H}^2$,

$$\mathscr{O}_x^{n.c.} \simeq SU(1,1)/U(1) \simeq \mathbb{H}^2$$

and can be parameterized by $\beta \in \mathbb{C}$ as follows

$$\begin{array}{ccccc} \mathscr{O}_x^{n.c.} & \longrightarrow & SU(1,1)/U(1) & \longrightarrow & \mathbb{H}^2 \\ Ad\begin{pmatrix} \cosh|\beta| & \frac{\beta}{|\beta|}\sinh|\beta| \\ \frac{\bar{\beta}}{|\beta|}\sinh|\beta| & \cosh|\beta| \end{pmatrix}(x) & \longmapsto & \left[\begin{pmatrix} \cosh|\beta| & \frac{\beta}{|\beta|}\sinh|\beta| \\ \frac{\bar{\beta}}{|\beta|}\sinh|\beta| & \cosh|\beta| \end{pmatrix}\right]_{U(1)} & \longmapsto & \frac{\beta}{|\beta|}\tanh|\beta|. \end{array}$$

Proposition

Mostow decomposition $SL(2,\mathbb{C}) = SU(2) \times \exp i\mathfrak{m} \times \exp i\mathfrak{t}$, *maps a matrix* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$ *to the product* $k \cdot f \cdot e$*, where*

$$e := \begin{pmatrix} \alpha & 0 \\ 0 & 1/\alpha \end{pmatrix} \in \exp i\mathfrak{t}, \quad \textit{with } \alpha := \sqrt{\tfrac{|a|^2+|c|^2}{|b|^2+|d|^2}},$$

$$f := \begin{pmatrix} \cosh|\beta| & \frac{\beta}{|\beta|}\sinh|\beta| \\ \frac{\bar{\beta}}{|\beta|}\sinh|\beta| & \cosh|\beta| \end{pmatrix} \in \exp i\mathfrak{m},$$

with

$$\beta := \frac{\bar{a}b+\bar{c}d}{|\bar{a}b+\bar{c}d|}\,\frac{\ln\left(\sqrt{(|a|^2+|c|^2)(|b|^2+|d|^2)}+|\bar{a}b+\bar{c}d|\right)}{2}$$

$$k := \begin{pmatrix} a & b \\ c & d \end{pmatrix} e^{-1} f^{-1}.$$

Theorem

There is a $SU(2)$-equivariant fibration $\pi : \mathscr{O}_x^{\mathbb{C}} \to \mathscr{O}_x$ of the complex (co-)adjoint orbit

$$\mathscr{O}_x^{\mathbb{C}} \simeq \left\{(\ell_1, \ell_2) \in \mathbb{CP}(1) \times \mathbb{CP}(1), \ell_1 \oplus \ell_2 = \mathbb{C}^2\right\}$$

over the compact (co-)adjoint orbit $\mathscr{O}_x \simeq \mathbb{CP}(1) \simeq \mathbb{S}^2$ given by

$$\begin{array}{cccc} \pi : & \mathscr{O}_x^{\mathbb{C}} & \longrightarrow & \mathscr{O}_x \\ & Ad(k \cdot f \cdot e)(x) & \longmapsto & Ad(k)(x), \end{array}$$

where $(k, f, e) \in SU(2) \times \exp i\mathfrak{m} \times \exp i\mathfrak{t}$. In particular, the fiber $\pi^{-1}(x)$ over $x \in \mathscr{O}_x^{\mathbb{C}}$ is $Ad(e^{i\mathfrak{m}})(x) = \mathscr{O}_x^{n.c.} \simeq \mathbb{H}^2$.

Theorem

Moreover the complex (co-)adjoint orbit is isomorphic to the (co-)tangent bundle of $\mathbb{CP}(1)$ via the following $SU(2)$-equivariant diffeomorphism

$$\begin{array}{rccc} \Phi : & T\mathbb{CP}(1) = T\mathcal{O}_x & \longrightarrow & \mathcal{O}_x^{\mathbb{C}} \\ & \left(z = g\ x\ g^{-1}, v = g\ \mathfrak{a}\ g^{-1}\right) & \longmapsto & g\ e^{i\mathfrak{a}}\ x\ e^{-i\mathfrak{a}}\ g^{-1} \end{array}$$

with $g \in SU(2)$ and $\mathfrak{a} \in \mathfrak{m}$.

In other words, $\mathcal{O}_x^{\mathbb{C}}$ is the complexification of the compact orbit $\mathcal{O}_x \simeq \mathbb{CP}(1)$ and is isomorphic to the tangent bundle of $\mathbb{CP}(1)$. The hyperbolic space $\mathbb{H}^2$ sits inside $\mathcal{O}_x^{\mathbb{C}}$ as the fiber over $x \in \mathcal{O}_x \simeq \mathbb{CP}(1)$ and is the orbit of x under the adjoint action of $SU(1,1) \subset SL(2,\mathbb{C})$.

Theorem (Biquard-Gauduchon)

1. *The complex adjoint orbit $\mathcal{O}_x^{\mathbb{C}}$ admits a G-invariant hyperkähler structure compatible with the complex symplectic form $\omega_{\mathbb{C}}$ of Kirillov-Kostant-Souriau and extending the natural Kähler structure of the Hermitian-symmetric orbit of compact type $\mathcal{O}_x$.*
2. *The Kähler form ω_1 associated with the natural complex structure $I_1 := i$ of the complex orbit $\mathcal{O}_x^{\mathbb{C}}$ is given by $\omega_1 = dd^c K$, where the potential K has the following expression*

$$K(y) = \alpha \operatorname{Re}\langle y, \pi(y)\rangle, \textit{ where } y \in \mathcal{O}_x^{\mathbb{C}}, \tag{5}$$

where the constant α is defined by $ad_x^2 = -\alpha^2$ on $\mathfrak{m}_x$.

Theorem (T.)

For $p = 2$, the cotangent space $T^*\mathrm{Gr}_{\mathrm{res},2}(\mathscr{H}_+)$ of the restricted Grassmannian is a hyperkähler manifold isomorphic to the complexification $\mathrm{Gr}^{\mathbb{C}}_{\mathrm{res},2}(\mathscr{H}_+)$ of the Grassmannian defined as

$$\mathrm{Gr}^{\mathbb{C}}_{\mathrm{res},2}(\mathscr{H}_+) = \{(P, Q) \in \mathrm{Gr}_{\mathrm{res},2}(\mathscr{H}_+) \times \mathrm{Gr}_{\mathrm{res},2}(\mathscr{H}_-), P \oplus Q = \mathscr{H}\}$$

A. B. Tumpach, *Hyperkähler structures and infinite-dimensional Grassmannians,* Journal of Functional Analysis
A.B.Tumpach, *Infinite-dimensional hyperkähler manifolds associated with Hermitian-symmetric affine coadjoint orbits*, Annales de l'Institut Fourier.

- A.Tahiri, A.B.Tumpach, *The restricted p-Schatten-class Grassmannian as affine coadjoint orbit*, ArXiv.
- T.Goliński, G. Larotonda, A.B. Tumpach, *On the Kähler geometry of infinite dimensional affine coadjoint orbits of unitary groups*, in preparation.
- A.B.Tumpach, *Bruhat-Poisson structure of the restricted Grassmannian and the KdV hierarchy*, ArXiv.
- D.Beltita, T. Golinski, A.B.Tumpach, *Queer Poisson Brackets*, Journal of Geometry and Physics.
- T.Goliński, A.B. Tumpach, *Banach Poisson-Lie groups, Lax equations and the AKS Theorem in infinite dimensions*, Differential Geometry and its Applications.
- A.Tahiri, A.B.Tumpach, *Banach Poisson-Lie groups in duality*, in preparation.
- D. Beltita, T. Ratiu, A.B. Tumpach, *The restricted Grassmannian, Banach Lie-Poisson spaces, and coadjoint orbits*, Journal of Functional Analysis.
- A.B.Tumpach, *Hyperkähler structures and infinite-dimensional Grassmannians*, Journal of Functional Analysis.
- A.B.Tumpach, *Infinite-dimensional hyperkähler manifolds associated with Hermitian-symmetric affine coadjoint orbits*, Annales de l'Institut Fourier.