Quantization of web geometry

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- Triality symmetry of classical quasigroups
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- Quantum quasigroups
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Classical quasigroups and homotopies

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$(Q, \cdot, /, \backslash)$ [binary **multiplication**, **right** and **left division**] with

$$(x \cdot y)/y \overset{\text{(IR)}}{=} x \overset{\text{(IL)}}{=} y\backslash(y \cdot x)$$

and

$$(x/y) \cdot y \overset{\text{(SR)}}{=} x \overset{\text{(SL)}}{=} y \cdot (y\backslash x).$$

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Quasigroups are **isotopic**, $Q \simeq Q'$, if related by an isotopy.

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Cayley diagram of S_3 with respect to left multiplication by
(1 2) — single-shafted double-headed arrows $\leftrightarrow$ and
(2 3) — double-shafted double-headed arrows $\Leftrightarrow$:

$$\begin{tikzcd}
\boxed{\begin{array}{c}(Q,\cdot)^{(1)} = \\ (Q,\cdot,/,\backslash)\end{array}} \arrow[r, leftrightarrow] \arrow[d, Leftrightarrow] & \boxed{\begin{array}{c}(Q,\cdot)^{(1\ 2)} = \\ (Q,\circ,\backslash\!\backslash,/\!/)\end{array}} \arrow[r, Leftrightarrow] & \boxed{\begin{array}{c}(Q,\cdot)^{(1\ 2\ 3)} = \\ (Q,/\!/,\backslash,\circ)\end{array}} \arrow[d, leftrightarrow] \\
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$$

. . . permuting $x_1 \cdot x_2 = x_3$ to $x_{1\pi} * x_{2\pi} = x_{3\pi}$.

Web geometry

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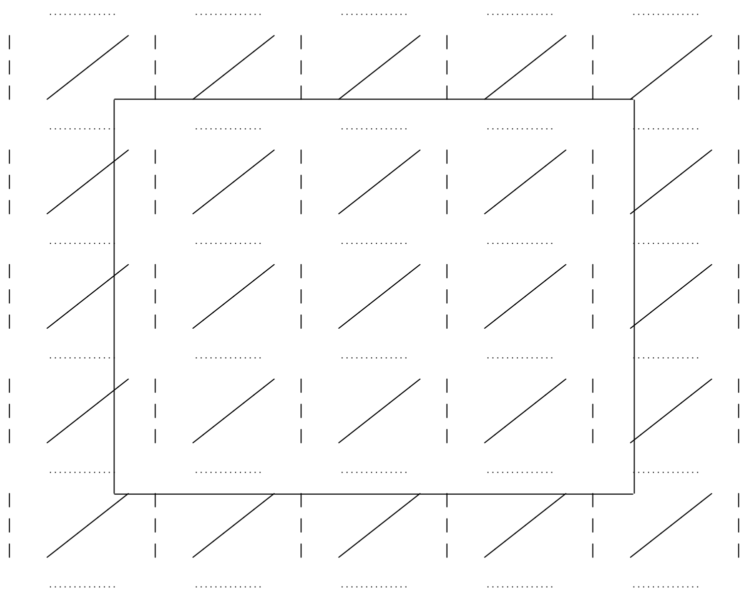
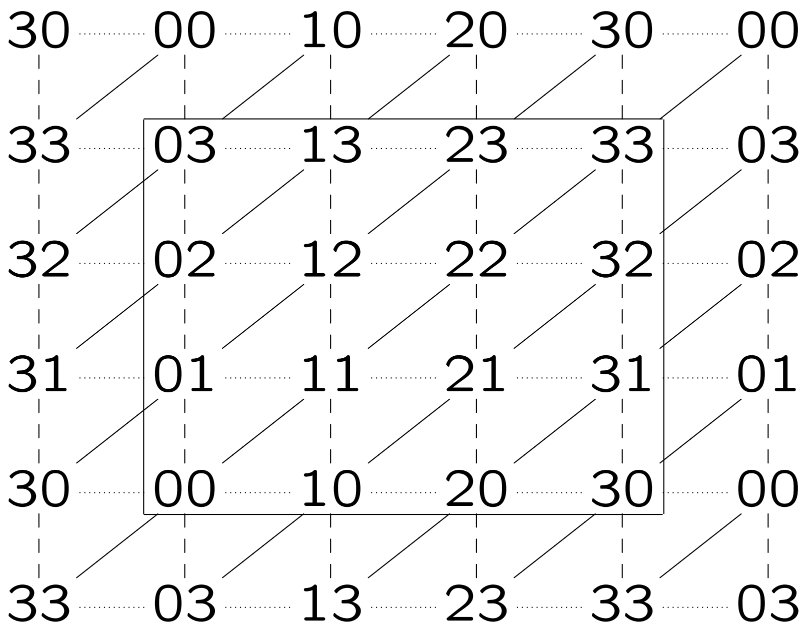
Three **pencils** of disjoint lines on $W = H \times V = V \times D = D \times H$



Horizontal: dotted
Vertical: dashed
Diagonal: solid

Quasigroups coordinatize webs

Three **pencils** of disjoint lines on $Q \times Q$



Horizontal: common second component y
Vertical: common first component x
Diagonal: common product $x \cdot y$;

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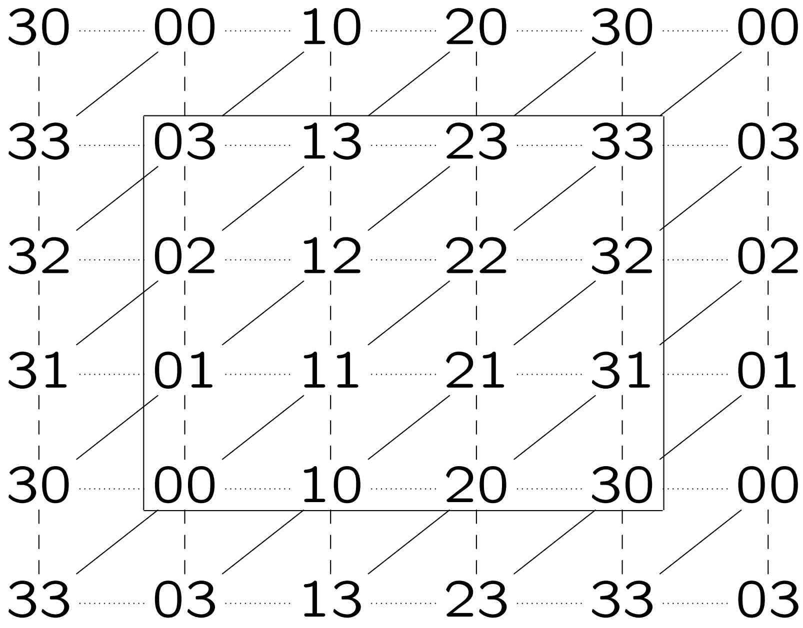

Horizontal: common second component y
Vertical: common first component x
Diagonal: common product $x \cdot y$;   here $x - y$ on $\mathbb{Z}/_4$.

Classical semisymmetrization

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On $Q \times Q \times Q$, using $\otimes$ as separator for tuples, define

$$[(x_1 \quad \otimes \quad x_2 \quad \otimes \quad x_3) \quad \bullet$$

$$(y_1 \quad \otimes \quad y_2 \quad \otimes \quad y_3)] =$$

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- Semisymmetrization on Q^3 is **semisymmetric**: $x \circ y = x/y$, or $(Q^3, \bullet)^{(2\ 1)} = (Q^3, \bullet)^{(1\ 3)} = (Q^3, \bullet)^{(3\ 2)}$ in triality.

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  from quasigroup homotopies to homomorphisms.
- The web of a quasigroup Q is the set of **points**
  $\top = \top^3 \to Q^3; v \otimes h \otimes d \mapsto x_1 \otimes x_2 \otimes x_3$
  in the category of quasigroup homotopies.

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$$\begin{tikzcd}
a \otimes b \arrow[r, mapsto, "\nabla"] \arrow[d, mapsto, "\Delta\otimes\Delta"'] & a \cdot b \arrow[r, mapsto, "\Delta"] & (a \cdot b)^L \otimes (a \cdot b)^R \\
a^L \otimes a^R \otimes b^L \otimes b^R \arrow[rr, mapsto, "1_A\otimes\tau\otimes 1_A"'] & & a^L \otimes b^L \otimes a^R \otimes b^R \arrow[u, mapsto, "\nabla\otimes\nabla"']
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\end{tikzcd}$$

Says that $\boldsymbol{\Delta}$ is a magma homomorphism,

or that ∇ is a comagma homomorphism.

Quantum quasigroups

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Bimagma (A, ∇, Δ) in a symmetric monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$.

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Left composite $\quad \mathsf{G}\colon A \otimes A \xrightarrow{\Delta \otimes 1_A} A \otimes A \otimes A \xrightarrow{1_A \otimes \nabla} A \otimes A\,;$

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Bimagma (A, ∇, Δ) is a **quantum quasigroup**
if its left and right composites are invertible morphisms
in the symmetric monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$.

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commuting for $i \in \{0, 1, 2\}$.

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$$\begin{tikzcd}
Q \otimes Q \arrow[r, "\rotatebox[origin=c]{180}{G}_i"] \arrow[d, leftrightarrow, "\tau"'] & Q \otimes Q \arrow[d, leftrightarrow, "\tau"] \\
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So each bimagma (Q, ∇_i, Δ_i) is a quantum quasigroup.

Triality symmetry of quantum T-quasigroups

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$$
\begin{tikzcd}
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For $x\otimes y \overset{\nabla_0}{\mapsto} x\circ y$, $x\otimes y \overset{\nabla_1}{\mapsto} x/y$, $x\otimes y \overset{\nabla_2}{\mapsto} x\backslash y$, $x \overset{\Delta_i}{\mapsto} x\otimes x$, from quasigroup $(Q,\circ,/,\backslash)$ in $(\mathbf{Set},\times,\mathbf{1})$, recovers quasigroup triality.

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or equivalently, the diagram
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[The similarity to
$$\begin{tikzcd} Q\otimes Q \arrow[rr, "\Game_i"] \arrow[d, leftrightarrow, "\tau"'] & & Q\otimes Q \arrow[d, leftrightarrow, "\tau"] \\ Q\otimes Q & & Q\otimes Q \arrow[ll, "\mathsf{G}_{i+1}"] \end{tikzcd}$$
is apparent.]

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Write $Q(\rho, \lambda, L, R)$ for the linear quantum quasigroup structure.

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Semisymmetrization

$$\boxed{\begin{array}{c} [(x_1 \quad \oplus \quad x_2 \quad \oplus \quad x_3) \quad \oplus \\ (y_1 \quad \oplus \quad y_2 \quad \oplus \quad y_3)] \\ (\nabla_1 \quad \oplus \quad \nabla_2 \quad \oplus \quad \nabla_3) \quad = \\ (x_2 \oplus y_3)\nabla_1 \oplus (x_3 \oplus y_1)\nabla_2 \oplus (x_1 \oplus y_2)\nabla_3 \\ = (x_2^{\rho_1} + y_3^{\lambda_1}) \oplus (x_3^{\rho_2} + y_1^{\lambda_2}) \oplus (x_1^{\rho_3} + y_2^{\lambda_3}) \end{array}}$$

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gives $Q^3(\mathsf{P}, \mathsf{P}^{-1}, 1_{Q^3}, 1_{Q^3})$ with **Rho-matrix** $\mathsf{P} = \begin{bmatrix} 0 & 0 & \rho_3 \\ \rho_1 & 0 & 0 \\ 0 & \rho_2 & 0 \end{bmatrix}$.

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Monomial solutions: $L = C(\alpha, 0, 0), C(0, \beta, 0)$ or $C(0, 0, \gamma)$
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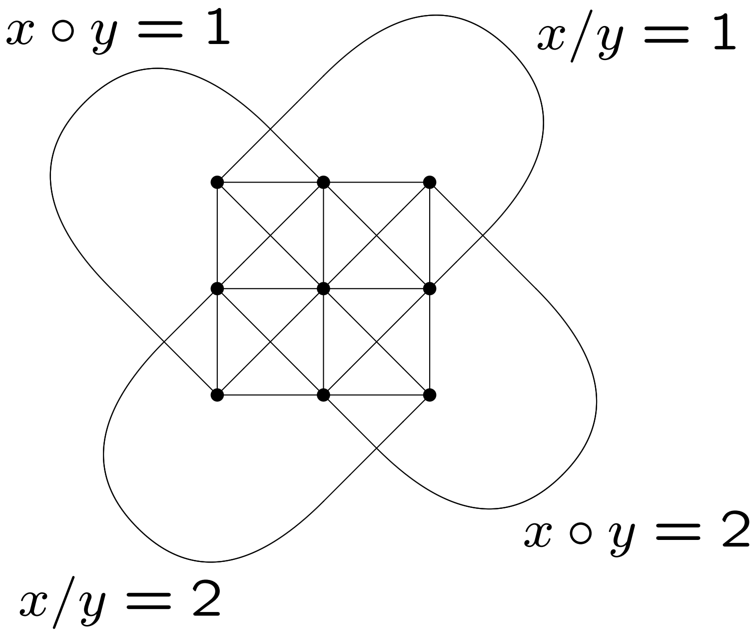

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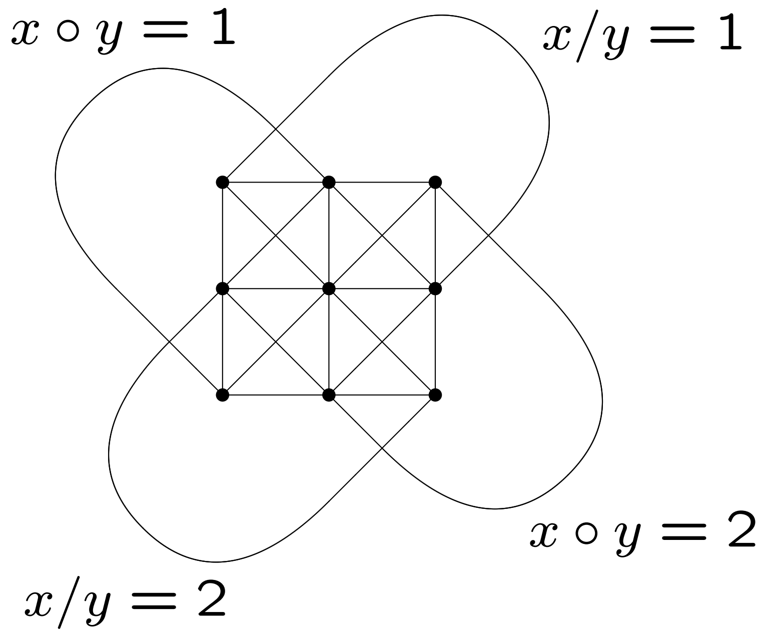

Verticals: constant x; and **horizontals:** constant y. Note $\mathsf{P}^3 = -1$.

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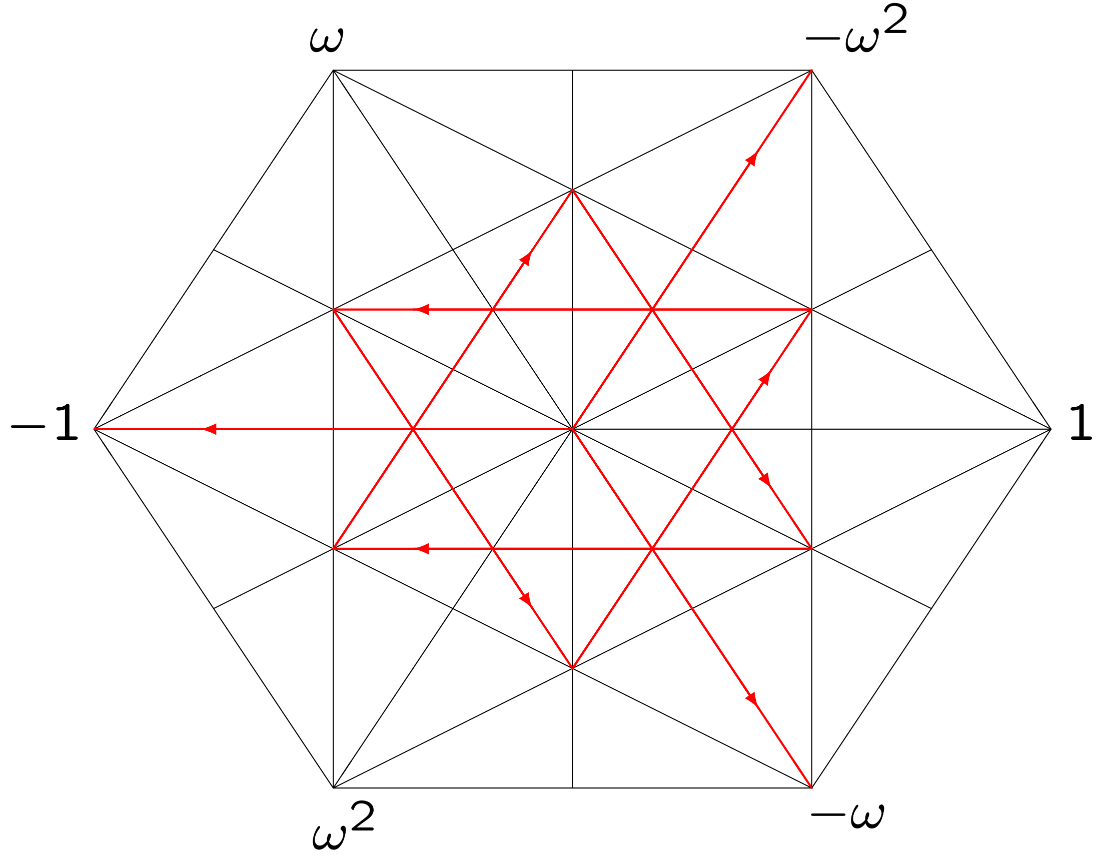
Over $\mathbb{R}$, only have the 3 monomial solutions.

Monomial and non-monomial complex solutions



Here, a red vector from u to v means a solution $(x, y, z) = (v, u, u)$.
Also, have cyclic rotations $(y, z, x), (z, x, y)$ of these solutions.

Geometry of non-monomial comultiplications

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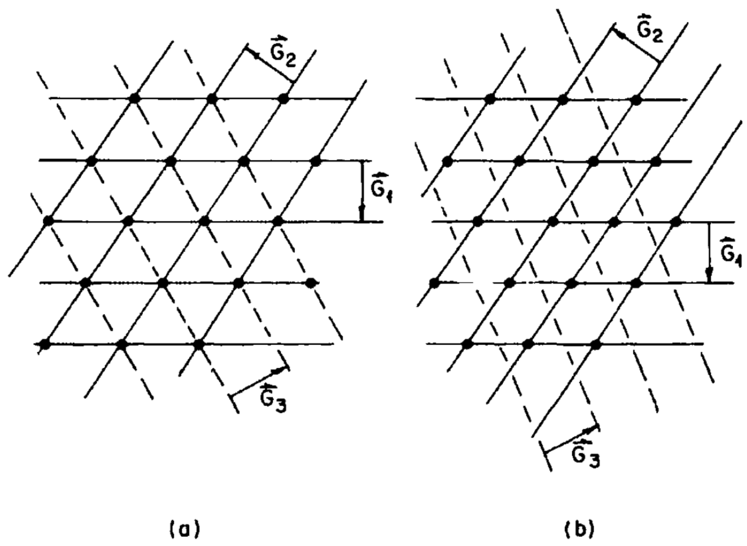

Figure 2-5 Three density waves form a two-dimensional lattice.

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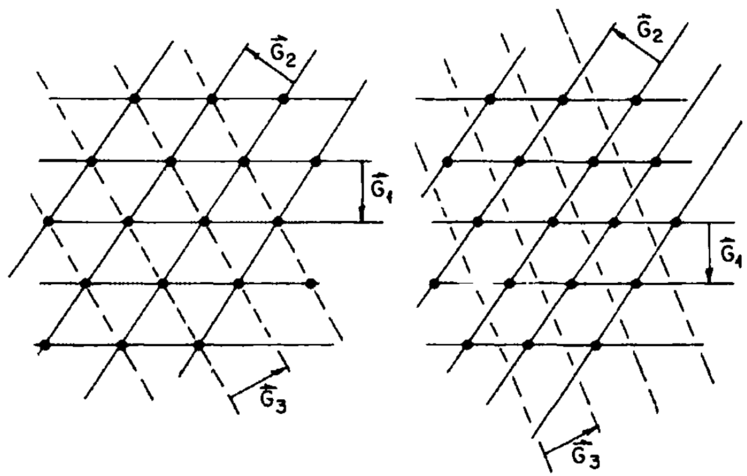

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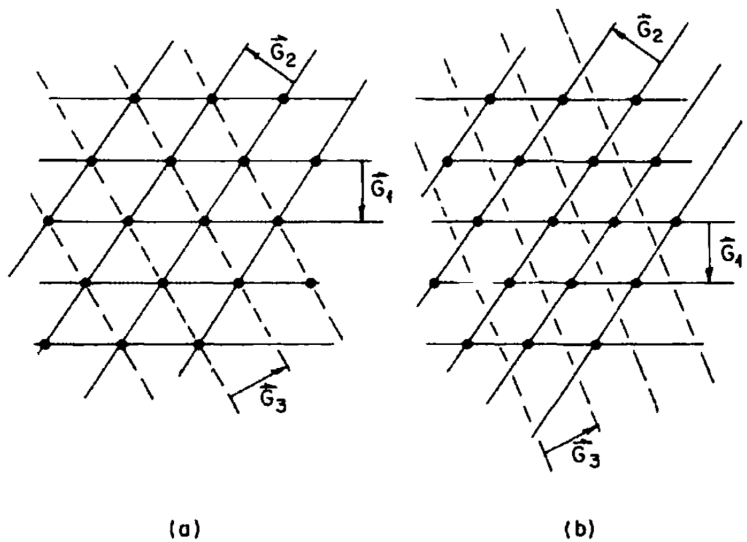

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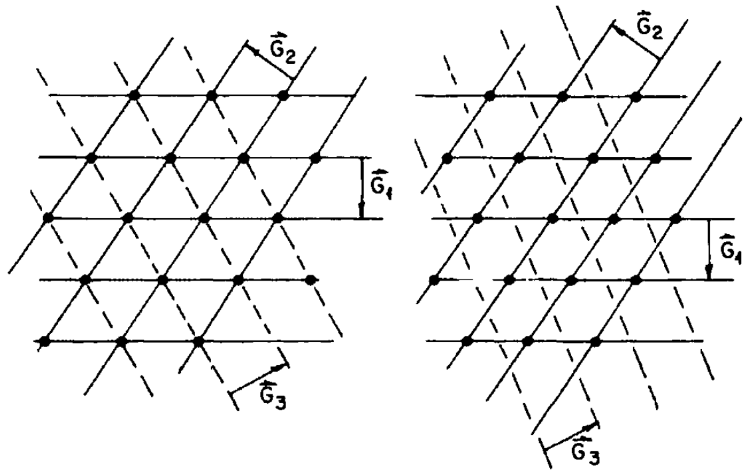

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Thank you for your attention!