When is the Drinfeld Double Hopf algebra part of a Hopf brace structure?

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Introduction and Preliminaries about Hopf braces

Hopf braces and crossed products

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Introduction

L. Guarnieri and L. Vendramin,
Skew braces and the Yang-Baxter operators,
Math. Comput. **86**(307) (2017) 2519–2534.

A **skew brace** $(G, \cdot, \star)$ consists in a pair of groups $(G, \cdot)$ and $(G, \star)$ such that

$$g \star (h \cdot k) = (g \star h) \cdot g^{-1} \cdot (g \star k) \quad \text{for all } g, h, k \in G.$$

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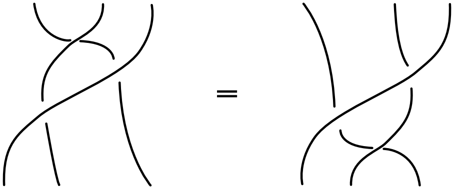
Skew braces induce

- **non-degenerate**
- **non-necessarily involutive**

solutions of the **set-theoretical Yang-Baxter Equation**.



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I. Angiono, C. Galindo and L. Vendramin,
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$\mathsf{C} = {}_{\mathbb{K}}\mathsf{Vect}$

Hopf brace $\longleftrightarrow$ *quantum version* of a skew brace

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Skew brace $\xrightarrow{\textbf{LINEARISATION PROCESS}}$ **Hopf brace**

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$\mathsf{C} = {}_{\mathbb{K}}\mathsf{Vect}$

Hopf brace $\longleftrightarrow$ *quantum version* of a skew brace

Notation: $\mathsf{C} = (\mathsf{C}, \otimes, K, c)$ is a **strict symmetric monoidal category**:

- $\otimes \colon \mathsf{C} \times \mathsf{C} \longrightarrow \mathsf{C}$ *tensor functor*,
- K, the *unit object*,
- A natural isomorphism called the *braiding*:

$$\{c_{X,Y} \colon X \otimes Y \to Y \otimes X\}_{X,Y \in \mathsf{C}}, \text{ with } \boldsymbol{c^{-1}_{X,Y} = c_{Y,X}}.$$

The notion of a Hopf brace

- $(H, \varepsilon_H, \delta_H)$ a coalgebra in C,
- (H, η_H^1, μ_H^1) and (H, η_H^2, μ_H^2) algebras in C.

The notion of a Hopf brace

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A **Hopf brace** $\mathbb{H} = (H_1, H_2)$ in C is a 9-tuple

$$(H, \eta_H^1, \mu_H^1, \eta_H^2, \mu_H^2, \varepsilon_H, \delta_H, \lambda_H^1, \lambda_H^2)$$

satisfying the following conditions:

(i) $H_1 = (H, \eta_H^1, \mu_H^1, \varepsilon_H, \delta_H, \lambda_H^1)$ is a **Hopf algebra** in C,

(ii) $H_2 = (H, \eta_H^2, \mu_H^2, \varepsilon_H, \delta_H, \lambda_H^2)$ is a **Hopf algebra** in C,

(iii) $\boldsymbol{\mu_H^2 \circ (H \otimes \mu_H^1) = \mu_H^1 \circ (\mu_H^2 \otimes \Gamma_{H_1}) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes H \otimes H)}$, where

$$\boldsymbol{\Gamma_{H_1} := \mu_H^1 \circ (\lambda_H^1 \otimes \mu_H^2) \circ (\delta_H \otimes H)}.$$

The category of Hopf braces

HBr

- $\mathbb{H} = (H_1, H_2)$
- $f : \mathbb{H} = (H_1, H_2) \to \mathbb{B} = (B_1, B_2)$

$$:\iff \begin{array}{l} f : H_1 \to B_1 \\ f : H_2 \to B_2 \end{array} \quad \text{are } \textbf{Hopf algebra morphisms} \text{ in C}$$

The category of Hopf braces

$$\mathrm{HBr} \xleftarrow{\ i\ } \mathrm{cocHBr}$$

- $\mathbb{H} = (H_1, H_2)$
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$$:\Longleftrightarrow \quad \begin{array}{l} f : H_1 \to B_1 \\ f : H_2 \to B_2 \end{array}$$ are **Hopf algebra morphisms** in C

$\mathbb{H} = (H_1, H_2)$ is ***cocommutative*** $:\Longleftrightarrow$ $(H, \varepsilon_H, \delta_H)$ is cocommutative

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Some remarks...

1) $\mathsf{C} = \mathsf{Set} \implies$ [**Hopf brace** $=$ **Skew brace**].

2) $H \in \mathsf{Hopf} \iff \mathbb{H}_{\mathrm{triv}} = (H, H)$.

$\mathbb{H} = (H_1, H_2)$ is ***cocommutative*** $:\Longleftrightarrow$ $(H, \varepsilon_H, \delta_H)$ is cocommutative

Algebraic properties of Hopf braces

Let $\mathbb{H} = (H_1, H_2) \in$ HBr.

Algebraic properties of Hopf braces

Let $\mathbb{H} = (H_1, H_2) \in \mathsf{HBr}$.

$$\eta_H^1 = \eta_H^2$$

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Let $\mathbb{H} = (H_1, H_2) \in$ HBr.

$$\boxed{\eta_H^1 = \eta_H^2} \longrightarrow \eta_H \equiv \eta_H^i, \; i = 1, 2$$

Algebraic properties of Hopf braces

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(H_1, Γ_{H_1}) is a **left H_2-module algebra**

Algebraic properties of Hopf braces

Let $\mathbb{H} = (H_1, H_2) \in \mathsf{HBr}$.

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(H_1, Γ_{H_1}) is a **left H_2-module algebra**

If $\mathbb{H}$ is **cocommutative**,

then Γ_{H_1} is a **coalgebra morphism**

Introduction and Preliminaries about Hopf braces

Hopf braces and crossed products

Problem

S. Majid,
Physics for algebrists: noncommutative and cocommutative Hopf algebras by a bicrossproduct construction,
J. Algebra **130** (1990) 17–64.

Theorem

If $(A, H, \varphi_A, \phi_H)$ is a **matched pair** of Hopf algebras in C, then

$$\boldsymbol{A \bowtie H} = (A \otimes H, \eta_A \otimes \eta_H, \mu_{A\bowtie H}, \varepsilon_A \otimes \varepsilon_H, \delta_{A\otimes H}, \lambda_{A\bowtie H})$$

is a **Hopf algebra** in C, where

$$\mu_{A\bowtie H} := (\mu_A \otimes \mu_H) \circ (A \otimes \Psi \otimes H),$$
$$\lambda_{A\bowtie H} := \Psi \circ (\lambda_H \otimes \lambda_A) \circ c_{A,H}$$

and $\Psi := (\varphi_A \otimes \phi_H) \circ \delta_{H\otimes A}$. $\boldsymbol{A \bowtie H}$ is the **bicrossed product Hopf algebra** of A and H.

Problem (P)

- $\mathbb{A} = (A_1, A_2)$
- $\mathbb{H} = (H_1, H_2)$

such that

- $(A_1, H_1, \varphi_A^1, \phi_H^1)$
- $(A_2, H_2, \varphi_A^2, \phi_H^2)$

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- $(A_1, H_1, \varphi_A^1, \phi_H^1) \rightsquigarrow \boldsymbol{A_1 \bowtie H_1} = (A \otimes H, \eta_A \otimes \eta_H, \mu^1_{A\bowtie H}, \varepsilon_A \otimes \varepsilon_H, \delta_{A\otimes H}, \lambda^1_{A\bowtie H})$
- $(A_2, H_2, \varphi_A^2, \phi_H^2) \rightsquigarrow \boldsymbol{A_2 \bowtie H_2} = (A \otimes H, \eta_A \otimes \eta_H, \mu^2_{A\bowtie H}, \varepsilon_A \otimes \varepsilon_H, \delta_{A\otimes H}, \lambda^2_{A\bowtie H})$

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- $(A_2, H_2, \varphi_A^2, \phi_H^2) \rightsquigarrow \boldsymbol{A_2 \bowtie H_2} = (A \otimes H, \eta_A \otimes \eta_H, \mu^2_{A\bowtie H}, \varepsilon_A \otimes \varepsilon_H, \delta_{A\otimes H}, \lambda^2_{A\bowtie H})$

$$(\boldsymbol{A_1 \bowtie H_1}, \boldsymbol{A_2 \bowtie H_2}) \in \mathrm{HBr?}$$

Problem (P)

$$\bullet\ \mathbb{A} = (A_1, A_2) \qquad \bullet\ \mathbb{H} = (H_1, H_2)$$

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- $(A_1, H_1, \varphi_A^1, \phi_H^1) \rightsquigarrow \boldsymbol{A_1 \bowtie H_1} = (A \otimes H, \eta_A \otimes \eta_H, \mu^1_{A\bowtie H}, \varepsilon_A \otimes \varepsilon_H, \delta_{A\otimes H}, \lambda^1_{A\bowtie H})$
- $(A_2, H_2, \varphi_A^2, \phi_H^2) \rightsquigarrow \boldsymbol{A_2 \bowtie H_2} = (A \otimes H, \eta_A \otimes \eta_H, \mu^2_{A\bowtie H}, \varepsilon_A \otimes \varepsilon_H, \delta_{A\otimes H}, \lambda^2_{A\bowtie H})$

$$(\boldsymbol{A_1 \bowtie H_1}, \boldsymbol{A_2 \bowtie H_2}) \in \mathrm{HBr}?$$

- **We have solved it in two particular cases**:
  - $\to$ $(A_1 \otimes H_1, A_2 \bowtie H_2)$ $(\varphi_A^1 = \varepsilon_H \otimes A,\ \phi_H^1 = H \otimes \varepsilon_A)$,
  - $\to$ $(A_1 \bowtie H_1, A_2 \sharp H_2)$ $(\phi_H^2 = H \otimes \varepsilon_A)$.

R. González Rodríguez, B. Ramos Pérez, **About Hopf braces and crossed products**, *São Paulo J. Math. Sci.* **20** (2026).

Main result 1

Theorem

In the situation **(P)**, if it is assumed that

- $\mathbb{H}$ is **cocommutative**,
- the actions φ_A^1 and ϕ_H^1 are **trivial** ($\Longrightarrow$ $A_1 \bowtie H_1 = A_1 \otimes H_1$),

then

$$(A_1 \otimes H_1, A_2 \bowtie H_2) \in \mathbf{HBr}$$

if and only if

(E1) $\mu_A^1 \circ (\varphi_A^2 \otimes \varphi_A^2) \circ (H \otimes c_{H,A} \otimes A) \circ (\delta_H \otimes A \otimes A) = \varphi_A^2 \circ (H \otimes \mu_A^1)$,

(E2) $\mu_H^1 \circ (\mu_H^2 \otimes \Omega) \circ (H \otimes c_{H,H} \otimes A \otimes H) \circ (((\phi_H^2 \otimes H) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A)) \otimes H \otimes A \otimes H)$
$= \mu_H^2 \circ ((\phi_H^2 \circ (H \otimes \mu_A^1)) \otimes \mu_H^1) \circ (H \otimes A \otimes c_{H,A} \otimes H)$,

where

$$\Omega := \mu_H^1 \circ (\lambda_H^1 \otimes (\mu_H^2 \circ (\phi_H^2 \otimes H))) \circ (\delta_H \otimes A \otimes H).$$

If it is assumed in the previous theorem that:

- $A \in$ **Hopf** $\longleftrightarrow \mathbb{A} = \mathbb{A}_{\text{triv}} = (A, A) \in$ **HBr**,
- $H \in$ **cocHopf** $\longleftrightarrow \mathbb{H} = \mathbb{H}_{\text{triv}} = (H, H) \in$ **cocHBr**,
- ϕ_H^2 is **trivial** ($\Longrightarrow A \bowtie H = A\sharp H$).

If it is assumed in the previous theorem that:

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- ϕ_H^2 is **trivial** ($\Longrightarrow$ $A \bowtie H = A\sharp H$).

Corollary

If A is a Hopf algebra and H is a **cocommutative** Hopf algebra in C such that (A, φ_A^2) is a **left H-module algebra**, then

$$(A \otimes H, A\sharp H) \in \mathbf{HBr}.$$

Y. Doi and M. Takeuchi,
Multiplication alteration by two-cocycles - the quantum version,
Comm. Algebra **22**(14) (1994) 5715–5732.

$\boldsymbol{H}$ a **finite** Hopf algebra in C

→ $\widehat{\boldsymbol{H}} := (\boldsymbol{H}^{\mathrm{cop}})^*$.

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H a **finite** Hopf algebra in C
→ $\widehat{\boldsymbol{H}} := (\boldsymbol{H}^{\mathrm{cop}})^*$.

Proposition

$(\widehat{\boldsymbol{H}}, \boldsymbol{H}, \varphi_{\widehat{H}}, \phi_{\boldsymbol{H}})$ is a **matched pair** of Hopf algebras, where

$$\varphi_{\widehat{H}} := (H^* \otimes b_H(K)) \circ (H^* \otimes R \otimes H^*) \circ (a_H(K) \otimes H \otimes H^*),$$
$$\phi_H := (H \otimes b_H(K)) \circ (\overline{R} \otimes H^*),$$

and

$$R := \mu_H \circ c_{H,H} \circ (\mu_H \otimes \lambda_H^{-1}) \circ (H \otimes \delta_H),$$
$$\overline{R} := (H \otimes (\mu_H \circ (\lambda_H^{-1} \otimes H))) \circ (\delta_H \otimes H) \circ c_{H,H} \circ \delta_H.$$

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Proposition

$(\widehat{H}, H, \varphi_{\widehat{H}}, \phi_H)$ is a **matched pair** of Hopf algebras.

Theorem

$$(\widehat{H}, H, \varphi_{\widehat{H}}, \phi_H) \xrightarrow{\bowtie} \boxed{D(H) = \widehat{H} \bowtie H}$$

Corollary

If A is a Hopf algebra and H is a **cocommutative** Hopf algebra in C such that $(\boldsymbol{A}, \varphi_{\boldsymbol{A}}^{2})$ is a **left H-module algebra**, then

$$(\boldsymbol{A} \otimes \boldsymbol{H}, \boldsymbol{A} \sharp \boldsymbol{H}) \in \mathbf{HBr}.$$

If $\boldsymbol{H}$ is a **finite** and **cocommutative** Hopf algebra in C, then

$$\overline{R} = H \otimes \eta_H \implies \phi_H = H \otimes \varepsilon_{\widehat{H}} \implies \boldsymbol{D(H) = \widehat{H} \sharp H}.$$

Corollary

If A is a Hopf algebra and H is a **cocommutative** Hopf algebra in C such that (A, φ_A^2) is a **left H-module algebra**, then

$$(A \otimes H, A \sharp H) \in \mathbf{HBr}.$$

If H is a **finite** and **cocommutative** Hopf algebra in C, then

$$\overline{R} = H \otimes \eta_H \implies \phi_H = H \otimes \varepsilon_{\widehat{H}} \implies D(H) = \widehat{H} \sharp H.$$

Theorem

If H is a **finite** and **cocommutative** Hopf algebra in C, then

$$(T(H), D(H)) \in \mathbf{HBr}$$

where $T(H) = \widehat{H} \otimes H$ is the tensor product Hopf algebra of $\widehat{H}$ with H.

Main result 2

Theorem

In the situation **(P)**, if it is assumed that

- ϕ_H^2 is **trivial** ($\Longrightarrow$ $A_2 \bowtie H_2 = A_2 \sharp H_2$),

then

$$(A_1 \bowtie H_1, A_2 \sharp H_2) \in \mathbf{HBr}$$

if and only if

(C1) $(A \otimes \mu_H^1) \circ (\Psi^1 \otimes \mu_H^2) \circ (H \otimes \Psi^2 \otimes H) \circ (((\Gamma'_{H_1} \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H)) \otimes A \otimes H)$
$= (A \otimes \mu_H^2) \circ (\Psi^2 \otimes \mu_H^1) \circ (H \otimes \Psi^1 \otimes H)$,

(C2) $(\Gamma_{A_1} \otimes H) \circ (A \otimes \Psi^1) = \Psi^1 \circ (H \otimes \Gamma_{A_1}) \circ (c_{A,H} \otimes A)$,

(C3) $\Psi^2 \circ (H \otimes \mu_A^1) = (\mu_A^1 \otimes H) \circ (A \otimes \Psi^2) \circ (\Psi^2 \otimes A)$.

If it is assumed in the previous theorem that:

- $A \in \textbf{Hopf} \longleftrightarrow \mathbb{A} = \mathbb{A}_{\mathrm{triv}} = (A, A) \in \textbf{HBr}$,
- $H \in \textbf{Hopf} \longleftrightarrow \mathbb{H} = \mathbb{H}_{\mathrm{triv}} = (H, H) \in \textbf{HBr}$,
- φ_A^2 is **trivial** ($\Longrightarrow A \sharp H = A \otimes H$).

If it is assumed in the previous theorem that:

- $A \in \mathbf{Hopf} \longleftrightarrow \mathbb{A} = \mathbb{A}_{\text{triv}} = (A, A) \in \mathbf{HBr}$,
- $H \in \mathbf{Hopf} \longleftrightarrow \mathbb{H} = \mathbb{H}_{\text{triv}} = (H, H) \in \mathbf{HBr}$,
- φ_A^2 is **trivial** ($\Longrightarrow$ $A \sharp H = A \otimes H$).

Corollary

Let A and H be Hopf algebras in C such that $(A, H, \varphi_A^1, \phi_H^1)$ is a **matched pair**.
If H is **commutative**, then

$$(A \bowtie H, A \otimes H) \in \mathbf{HBr}.$$

Corollary

Let A and H be Hopf algebras in C such that $(\boldsymbol{A}, \boldsymbol{H}, \varphi_{\boldsymbol{A}}^{\boldsymbol{1}}, \phi_{\boldsymbol{H}}^{\boldsymbol{1}})$ is a **matched pair**.
If H is **commutative**, then

$$(\boldsymbol{A} \bowtie \boldsymbol{H}, \boldsymbol{A} \otimes \boldsymbol{H}) \in \mathbf{HBr}.$$

→ Applying **Corollary above** to the matched pair $(\widehat{\boldsymbol{H}}, \boldsymbol{H}, \varphi_{\widehat{\boldsymbol{H}}}, \phi_{\boldsymbol{H}})$...

Theorem

If $\boldsymbol{H}$ is a **finite** and **commutative** Hopf algebra in C, then

$$(\boldsymbol{D}(\boldsymbol{H}), \boldsymbol{T}(\boldsymbol{H})) \in \mathbf{HBr}.$$