Realizations of sl(3)

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Abstract

Although not honoured as one of the "fundamental group of physics", despite the Lie group $\mathrm{SL}(3,\mathbb{R})$ and the associated Lie algebra play an important role in the study of symmetries, namely of the second order ordinary differential equations. It is therefore worthwhile to have realizations of this algebra in reserve in the case they are needed. In this contribution, we will be interested in how the realizations of sl(3) calculated by Shirokov's method look like when using the classification of sl(3)-subalgebras up to inner automorphism, which is known thanks to Douglas and Repka.

1 Subalgebras and realizations

The problem of the construction of realizations (representations of Lie algebras by vector fields) is at least as old as Sophus Lie. Nevertheless, people are still involved, because various open problems regarding realizations still exist and realizations have wide mathematical and physical applications; for a long list of where realizations can be used to help to solve various physical and mathematical problems, see [6] and references therein.

Despite intensive study, some sorts of classification of all possible realizations is finished for a relatively small number of low dimensional Lie algebras (in addition to the well known original works of S. Lie, one can mention the pioneering article on the classification [6]).

The problem of equivalence of two realizations, which is fundamentally related to the classification problem, is not quite that simple. For a comprehensive comments on the problem of equivalence see [5].

In 2015, Shirokov et al. developed a breakthrough method [3] to construct an explicit realization of a finite-dimensional Lie algebra by left- and right-invariant vector fields on the associated local Lie group using only the structure constants of the algebra. This very-easily-applicable method can be used to list inequivalent realizations of given Lie algebra by vector fields depending on various numbers of independent variables. This can be done using the classification of inequivalent subalgebras of the initial Lie algebra and by the projection of the left-invariant vector fields on the corresponding spaces. To compute the list of realizations of given Lie algebra, one needs first a list of non-equivalent Lie subalgebras of given Lie algebra, ideally sorted down with respect to inner automorphism.

2 Subalgebras of A_2

The study of non-equivalent subalgebras of a given Lie algebra is not easy task. It's easy if we restrict ourselves down to a certain class of subalgebras, e.g. semisimple subalgebras of given simple algebra. Much harder is to classify nonsemisimple, i. e. solvable and Levi decomposable subalgebras.

The classification of all subalgebras, up to inner automorphism, was not known until recently but for only few cases of low dimensional Lie algebras. The first attempt of classification of subalgebras of the simple Lie algebra $A_2 = \mathrm{sl}(3,\mathbb{C})$ was done in [7]. Some inaccuracies were corrected in the important article [1], the classification which we use here.

It is stated that the Lie algebra A_2, i. e. the Lie algebra of complex matrices 3×3 having zero trace, having the basis $\{h_1,h_2,x_1,x_2,x_3,y_1,y_2,y_3\}$, where

$$h_1=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix},\ h_2=\begin{pmatrix}0&0&0\\0&1&0\\0&0&-1\end{pmatrix},\ x_1=\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix},\ x_2=\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix},$$
$$x_3=\begin{pmatrix}0&0&-1\\0&0&0\\0&0&0\end{pmatrix},\ y_1=\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix},\ y_2=\begin{pmatrix}0&0&0\\0&0&0\\0&1&0\end{pmatrix},\ y_3=\begin{pmatrix}0&0&0\\0&0&0\\-1&0&0\end{pmatrix},$$

have the non-equivalent subalgebras with the following bases (we make use of the original notation [1]):

$$(A_1 \ni L_2)^2 : \{x_3,y_3,h_1+h_2,h_1-h_2,x_2,y_1\},\qquad (A_1 \ni L_2)^1 : \{x_3,y_3,h_1+h_2,h_1-h_2,x_1,y_2\},$$
$$B : \{x_1,x_2,x_3,h_1,h_2\},\qquad (A_1 \ni K_1)^2 : \{x_3,y_3,h_1+h_2,x_2,y_1\},$$
$$(A_1 \ni K_1)^1 : \{x_3,y_3,h_1+h_2,x_1,y_2\},\qquad A_1\oplus J : \{x_3,y_3,h_1+h_2,h_1-h_2\},$$
$$M^1_{14} : \{x_1,x_2,x_3,h_1-h_2\},\qquad M^2_{13,2} : \{x_1,x_2,x_3,h_1\},$$

$$M^{2}_{13,0}:\{x_2,y_1,y_3,2h_1+h_2\}\,,\qquad M^{1,\alpha}_{13,\frac{(2\alpha-1)(\alpha-2)}{(\alpha+1)^2}}:\{x_1,x_2,x_3,\alpha h_1+h_2\}\ \ (\alpha\neq\pm1),$$
$$M^{1}_{12}:\{x_1,x_2,x_3,h_1+h_2\}\,,\qquad M^{2}_{8}:\{x_1,y_2,h_1,h_2\}\,,$$
$$M^{1}_{8}:\{x_1,x_3,h_1,h_2\}\,,\qquad L^{1}_{5}:\{x_1,x_2,x_3\}\,,$$
$$L^{2}_{4}:\{x_1,y_2,h_1+h_2\}\,,\qquad L^{1}_{4}:\{x_1,x_3,h_2\}\,,$$
$$L^{2,\alpha}_{3,-\frac{(2\alpha-1)(\alpha-2)}{9(\alpha-1)^2}}:\{x_1,y_2,\alpha h_2+h_1\}\ \ (\alpha\neq\pm1),\qquad L^{1,\alpha}_{3,-\frac{(2\alpha-1)(\alpha-2)}{9(\alpha-1)^2}}:\{x_1,x_3,(\alpha-1)h_1+\alpha h_2\}\ \ (\alpha\neq\pm1),$$
$$L^{1}_{3,0}:\{x_1,h_1,h_2\}\,,\qquad L^{1}_{3,-\frac{2}{9}}:\{x_1+x_2,x_3,h_1+h_2\}\,,$$
$$L^{2}_{3,-\frac{1}{4}}:\{y_1,y_3,2h_1+h_2+x_2\}\,,\qquad L^{1}_{3,-\frac{1}{4}}:\{x_1,x_3,2h_1+h_2+x_2\}\,,$$
$$L^{2}_{2}:\{x_1,y_2,h_1-h_2\}\,,\qquad L^{1}_{2}:\{x_1,x_3,2h_1+h_2\}\,,$$
$$A^{2}_{1}:\{x_1+x_2,2\left(y_1+y_2\right),2\left(h_1+h_2\right)\}\,,\qquad A^{1}_{1}:\{x_3,y_3,h_1+h_2\}\,,$$
$$K^{4,\alpha}_{2}:\{x_1,\alpha h_1+(2\alpha+1)h_2\}\ \ (\alpha\in\mathbb{C}),\qquad K^{3}_{2}:\left\{x_1,-\frac{2h_1}{3}-\frac{h_2}{3}+y_2\right\},$$
$$K^{2}_{2}:\left\{x_1,-\frac{h_1}{3}+\frac{h_2}{3}+x_3\right\},\qquad K^{1}_{2}:\{x_1+x_2,h_1+h_2\}\,,$$
$$K^{5}_{1}:\{h_1,h_2\}\,,\qquad K^{4}_{1}:\{x_1,y_2\}\,,$$
$$K^{3}_{1}:\{x_1,x_3\}\,,\qquad K^{2}_{1}:\{x_1,h_1+2h_2\}\,,$$
$$K^{1}_{1}:\{x_1+x_2,x_3\}\,,\qquad J^{4,\alpha}:\{\alpha h_2+h_1\}\ \ (\alpha\in\mathbb{C}),$$
$$J^{3}:\{h_1+2h_2+x_1\}\,,\qquad J^{2}:\{x_1\}\,,$$
$$J^{1}:\{x_1+x_2\}\,.$$

3 List of realizations

In this section, we list the realizations corresponding to the subalgebras above. For each subalgebra, the complement of the subalgebra and the realization obtained with the help of that complement are given.

3.1 $(A_1 \ni\!\!\!\!\!\!+ L_2)^2$

Complement: $\{x_1, y_2\}$.
Realization:

$$h_1=-2z_1\partial_1-z_2\partial_2,\qquad h_2=z_1\partial_1+2z_2\partial_2,\qquad x_1=\partial_1,$$
$$x_2=-z_1z_2\partial_1-z_2^2\partial_2,\qquad x_3=z_2\partial_1,\qquad y_1=-z_1^2\partial_1-z_1z_2\partial_2,$$
$$y_2=\partial_2,\qquad y_3=z_1\partial_2.$$

3.2 $(A_1 \ni\!\!\!\!\!\!+ L_2)^1$

Complement: $\{x_2, y_1\}$.
Realization:

$$h_1=z_1\partial_1+2z_2\partial_2,\qquad h_2=-2z_1\partial_1-z_2\partial_2,\qquad x_1=-z_1z_2\partial_1-z_2^2\partial_2,$$
$$x_2=\partial_1,\qquad x_3=-z_2\partial_1,\qquad y_1=\partial_2,$$
$$y_2=-z_1^2\partial_1-z_1z_2\partial_2,\qquad y_3=-z_1\partial_2.$$

3.3 B

Complement: $\{y_1, y_2, y_3\}$.
Realization:

$$h_1=2z_1\partial_1-z_2\partial_2+z_3\partial_3,\qquad h_2=-z_1\partial_1+2z_2\partial_2+z_3\partial_3,$$
$$x_1=-z_1^2\partial_1+(z_1z_2-z_3)\partial_2-z_1z_3\partial_3,\qquad x_2=z_3\partial_1-z_2^2\partial_2,$$
$$x_3=-z_1z_3\partial_1+z_2\left(z_1z_2-z_3\right)\partial_2-z_3^2\partial_3,\qquad y_1=\partial_1,$$
$$y_2=\partial_2+z_1\partial_3,\qquad y_3=\partial_3.$$

3.4 $(A_1 \ni K_1)^2$

Complement: $\{x_1, y_2, h_1\}$.
Realization:

$$h_1 = -2z_1\partial_1 - z_2\partial_2 + \partial_3, \quad h_2 = z_1\partial_1 + 2z_2\partial_2 - \partial_3, \quad x_1 = \partial_1,$$
$$x_2 = -z_1z_2\partial_1 - z_2^2\partial_2 + z_2\partial_3, \quad x_3 = z_2\partial_1, \quad y_1 = -z_1^2\partial_1 - z_1z_2\partial_2 + z_1\partial_3,$$
$$y_2 = \partial_2, \quad y_3 = z_1\partial_2.$$

3.5 $(A_1 \ni K_1)^1$

Complement: $\{x_2, y_1, h_2\}$.
Realization:

$$h_1 = z_1\partial_1 + 2z_2\partial_2 - \partial_3, \quad h_2 = -2z_1\partial_1 - z_2\partial_2 + \partial_3, \quad x_1 = -z_1z_2\partial_1 - z_2^2\partial_2 + z_2\partial_3,$$
$$x_2 = \partial_1, \quad x_3 = -z_2\partial_1, \quad y_1 = \partial_2,$$
$$y_2 = -z_1^2\partial_1 - z_1z_2\partial_2 + z_1\partial_3, \quad y_3 = -z_1\partial_2.$$

3.6 $A_1 \oplus J$

Complement: $\{x_1, y_2, x_2, y_1\}$
Realization:

$$h_1 = -2z_1\partial_1 - z_2\partial_2 + z_3\partial_3 + 2z_4\partial_4, \quad h_2 = z_1\partial_1 + 2z_2\partial_2 - 2z_3\partial_3 - z_4\partial_4,$$
$$x_1 = \partial_1, \quad x_2 = -z_1z_2\partial_1 - z_2^2\partial_2 + (2z_2z_3 + z_1z_4 + 1)\partial_3 + z_2z_4\partial_4,$$
$$x_3 = z_2\partial_1 - z_4\partial_3, \quad y_1 = -z_1^2\partial_1 - z_1z_2\partial_2 + z_1z_3\partial_3 + (z_2z_3 + 2z_1z_4 + 1)\partial_4,$$
$$y_2 = \partial_2, \quad y_3 = z_1\partial_2 - z_3\partial_4.$$

3.7 M_{14}^1

Complement: $\{y_1, y_2, y_3, h_1\}$
Realization:

$$h_1 = 2z_1\partial_1 - z_2\partial_2 + z_3\partial_3 + \partial_4, \quad h_2 = -z_1\partial_1 + 2z_2\partial_2 + z_3\partial_3 + \partial_4,$$
$$x_1 = -z_1^2\partial_1 + (z_1z_2 - z_3)\partial_2 - z_1z_3\partial_3 - z_1\partial_4, \quad x_2 = z_3\partial_1 - z_2^2\partial_2 - z_2\partial_4,$$
$$x_3 = -z_1z_3\partial_1 + z_2\left(z_1z_2 - z_3\right)\partial_2 - z_3^2\partial_3 + (z_1z_2 - 2z_3)\partial_4, \quad y_1 = \partial_1,$$
$$y_2 = \partial_2 + z_1\partial_3, \quad y_3 = \partial_3.$$

3.8 $M_{13,2}^2$

Complement: $\{y_2, y_1, y_3, h_2\}$
Realization:

$$h_1 = -z_1\partial_1 + 2z_2\partial_2 + z_3\partial_3, \quad h_2 = 2z_1\partial_1 - z_2\partial_2 + z_3\partial_3 + \partial_4,$$
$$x_1 = -z_3\partial_1 - z_2^2\partial_2, \quad x_2 = -z_1^2\partial_1 + (z_1z_2 + z_3)\partial_2 - z_1z_3\partial_3 - z_1\partial_4,$$
$$x_3 = -z_1z_3\partial_1 - z_2\left(z_1z_2 + z_3\right)\partial_2 - z_3^2\partial_3 - z_3\partial_4, \quad y_1 = \partial_2 - z_1\partial_3,$$
$$y_2 = \partial_1, \quad y_3 = \partial_3.$$

3.9 $M_{13,0}^2$

Complement: $\{x_1, y_2, x_3, h_1\}$
Realization:

$$h_1 = -2z_1\partial_1 - z_2\partial_2 - z_3\partial_3 + \partial_4, \quad h_2 = z_1\partial_1 + 2z_2\partial_2 - z_3\partial_3 - 2\partial_4,$$
$$x_1 = \partial_1, \quad x_2 = -z_1z_2\partial_1 - z_2^2\partial_2 + (z_2z_3 - z_1)\partial_3 + 2z_2\partial_4,$$
$$x_3 = z_2\partial_1 + \partial_3, \quad y_1 = -z_1^2\partial_1 - z_1z_2\partial_2 + z_3\left(z_2z_3 - z_1\right)\partial_3 + (z_1 + z_2z_3)\partial_4,$$
$$y_2 = \partial_2, \quad y_3 = z_1\partial_2 - z_3^2\partial_3 - z_3\partial_4.$$

3.10 $M^{1,\alpha}_{13,\frac{(2\alpha-1)(\alpha-2)}{(\alpha+1)^2}}$ $(\alpha \neq \pm 1)$

Complement: $\{y_1, y_2, y_3, h_1\}$
Realization:

$$h_1 = 2z_1\partial_1 - z_2\partial_2 + z_3\partial_3 + \partial_4, \qquad h_2 = -z_1\partial_1 + 2z_2\partial_2 + z_3\partial_3 - \alpha\partial_4,$$
$$x_1 = -z_1^2\partial_1 + (z_1z_2 - z_3)\partial_2 - z_1z_3\partial_3 - z_1\partial_4, \qquad x_2 = z_3\partial_1 - z_2^2\partial_2 + \alpha z_2\partial_4,$$
$$x_3 = -z_1z_3\partial_1 + z_2\,(z_1z_2 - z_3)\,\partial_2 - z_3^2\partial_3 + ((\alpha-1)z_3 - \alpha z_1z_2)\partial_4, \qquad y_1 = \partial_1,$$
$$y_2 = \partial_2 + z_1\partial_3, \qquad y_3 = \partial_3.$$

3.11 M^1_{12}

Complement: $\{y_1, y_3, h_1, y_2\}$
Realization:

$$h_1 = 2z_1\partial_1 + z_2\partial_2 + \partial_3, \qquad h_2 = -z_1\partial_1 + z_2\partial_2 - \partial_3 + z_4\partial_4,$$
$$x_1 = -z_1^2\partial_1 - z_1z_2\partial_2 - z_1\partial_3 - e^{z_3}z_2\partial_4, \qquad x_2 = z_2\partial_1 + e^{-z_3}z_4\partial_3,$$
$$x_3 = -z_1z_2\partial_1 - z_2^2\partial_2 - e^{-z_3}z_1z_4\partial_3 - z_2z_4\partial_4, \qquad y_1 = \partial_1,$$
$$y_2 = z_1\partial_2 + e^{z_3}\partial_4, \qquad y_3 = \partial_2.$$

3.12 M^2_8

Complement: $\{x_2, x_3, y_1, y_3\}$
Realization:

$$h_1 = z_1\partial_1 - z_2\partial_2 + 2z_3\partial_3 + z_4\partial_4, \qquad h_2 = -2z_1\partial_1 - z_2\partial_2 - z_3\partial_3 + z_4\partial_4,$$
$$x_1 = z_1\partial_2 - z_3^2\partial_3 - z_3z_4\partial_4, \qquad x_2 = \partial_1,$$
$$x_3 = \partial_2, \qquad y_1 = z_2\partial_1 + \partial_3,$$
$$y_2 = -z_1^2\partial_1 - z_1z_2\partial_2 + z_3\,(z_2z_3 - z_1)\,\partial_3 + (z_2z_4z_3 + z_3 + z_1z_4)\partial_4, \qquad y_3 = -z_1z_2\partial_1 - z_2^2\partial_2 + (z_2z_3 - z_1)\partial_3 + (2z_2z_4 + 1)\partial_4.$$

3.13 M^1_8

Complement: $\{x_2, y_1, y_2, y_3\}$
Realization:

$$h_1 = z_1\partial_1 + 2z_2\partial_2 - z_3\partial_3 + z_4\partial_4, \qquad h_2 = -2z_1\partial_1 - z_2\partial_2 + 2z_3\partial_3 + z_4\partial_4,$$
$$x_1 = -z_1z_2\partial_1 - z_2^2\partial_2 + (z_2z_3 - (z_1z_3 + 1)\,z_4)\partial_3 - z_4\,(z_2 + z_1z_4)\,\partial_4, \qquad x_2 = \partial_1,$$
$$x_3 = -z_2\partial_1 - z_3z_4\partial_3 - z_4^2\partial_4, \qquad y_1 = \partial_2,$$
$$y_2 = -z_1^2\partial_1 - z_1z_2\partial_2 + (2z_1z_3 + 1)\partial_3 + (z_2 + z_1z_4)\partial_4, \qquad y_3 = -z_1\partial_2 + \partial_4.$$

3.14 L^1_5

Complement: $\{y_1, y_2, y_3, h_1, h_2\}$
Realization:

$$h_1 = 2z_1\partial_1 - z_2\partial_2 + z_3\partial_3 + \partial_4, \qquad h_2 = -z_1\partial_1 + 2z_2\partial_2 + z_3\partial_3 + \partial_5,$$
$$x_1 = -z_1^2\partial_1 + (z_1z_2 - z_3)\partial_2 - z_1z_3\partial_3 - z_1\partial_4, \qquad x_2 = z_3\partial_1 - z_2^2\partial_2 - z_2\partial_5,$$
$$x_3 = -z_1z_3\partial_1 + z_2\,(z_1z_2 - z_3)\,\partial_2 - z_3^2\partial_3 - z_3\partial_4 + (z_1z_2 - z_3)\partial_5, \qquad y_1 = \partial_1,$$
$$y_2 = \partial_2 + z_1\partial_3, \qquad y_3 = \partial_3.$$

3.15 L^2_4

Complement: $\{x_2, y_1, x_3, y_3, h_2\}$
Realization:

$$h_1 = z_1\partial_1 + 2z_2\partial_2 - z_3\partial_3 + z_4\partial_4 - \partial_5, \qquad h_2 = -2z_1\partial_1 - z_2\partial_2 - z_3\partial_3 + z_4\partial_4 + \partial_5,$$

$$x_1 = -z_1z_2\partial_1 - z_2^2\partial_2 + (z_1 + z_2z_3)\partial_3 - z_2z_4\partial_4 + z_2\partial_5, \quad x_2 = \partial_1,$$
$$x_3 = -z_2\partial_1 + \partial_3, \quad y_1 = \partial_2,$$
$$y_2 = -z_1^2\partial_1 - z_1z_2\partial_2 - z_3(z_1 + z_2z_3)\partial_3 + (2z_3z_4z_2 + z_2 + z_1z_4)\partial_4 + z_1\partial_5, \quad y_3 = -z_1\partial_2 - z_3^2\partial_3 + (2z_3z_4 + 1)\partial_4.$$

3.16 L_4^1

Complement: $\{y_1, y_3, x_2, h_1, y_2\}$
Realization:

$$h_1 = 2z_1\partial_1 + z_2\partial_2 + z_3\partial_3 + \partial_4, \quad h_2 = -z_1\partial_1 + z_2\partial_2 - 2z_3\partial_3 + 2z_5\partial_5,$$
$$x_1 = -z_1^2\partial_1 - z_1z_2\partial_2 + z_3(z_2z_3 - z_1)\partial_3 - z_1\partial_4 - z_2(2z_3z_5 + e^{z_4})\partial_5, \quad x_2 = z_2\partial_1 + \partial_3,$$
$$x_3 = -z_1z_2\partial_1 - z_2^2\partial_2 + (z_2z_3 - z_1)\partial_3 - z_2\partial_4 - 2z_2z_5\partial_5, \quad y_1 = \partial_1,$$
$$y_2 = z_1\partial_2 - z_3^2\partial_3 + (2z_3z_5 + e^{z_4})\partial_5, \quad y_3 = \partial_2.$$

3.17 $L^{2,\alpha}_{3,-\frac{(2\alpha-1)(\alpha-2)}{9(\alpha-1)^2}}$ $(\alpha \neq \pm 1)$

Complement: $\{y_1, y_3, x_2, x_3, h_2\}$
Realization:

$$h_1 = 2z_1\partial_1 + z_2\partial_2 + z_3\partial_3 - z_4\partial_4 - \alpha\partial_5, \quad h_2 = -z_1\partial_1 + z_2\partial_2 - 2z_3\partial_3 - z_4\partial_4 + \partial_5,$$
$$x_1 = -z_1^2\partial_1 - z_1z_2\partial_2 + z_3(z_2z_3 - z_1)\partial_3 + (z_2z_4z_3 + z_3 + z_1z_4)\partial_4 + (\alpha z_1 - z_2z_3)\partial_5, \quad x_2 = z_2\partial_1 + \partial_3,$$
$$x_3 = -z_1z_2\partial_1 - z_2^2\partial_2 + (z_2z_3 - z_1)\partial_3 + (2z_2z_4 + 1)\partial_4 + (\alpha - 1)z_2\partial_5, \quad y_1 = \partial_1,$$
$$y_2 = z_1\partial_2 - z_3^2\partial_3 - z_3z_4\partial_4 + z_3\partial_5, \quad y_3 = \partial_2.$$

3.18 $L^{1,\alpha}_{3,-\frac{(2\alpha-1)(\alpha-2)}{9(\alpha-1)^2}}$ $(\alpha \neq \pm 1)$

Complement: $\{y_2, y_1, y_3, x_2, h_1\}$
Realization:

$$h_1 = -z_1\partial_1 + 2z_2\partial_2 + z_3\partial_3 + z_4\partial_4 + \partial_5,$$
$$h_2 = 2z_1\partial_1 - z_2\partial_2 + z_3\partial_3 - 2z_4\partial_4 + (\frac{1}{\alpha} - 1)\partial_5,$$
$$x_1 = -z_3\partial_1 - z_2^2\partial_2 - z_2z_4\partial_4 - z_2\partial_5,$$
$$x_2 = -z_1^2\partial_1 + (z_1z_2 + z_3)\partial_2 - z_1z_3\partial_3 + (2z_1z_4 + 1)\partial_4 + \frac{(\alpha - 1)z_1}{\alpha}\partial_5,$$
$$x_3 = -z_1z_3\partial_1 - z_2(z_1z_2 + z_3)\partial_2 - z_3^2\partial_3 + (z_3z_4 - z_2(z_1z_4 + 1))\partial_4 - \frac{\alpha z_1z_2 + z_3}{\alpha}\partial_5,$$
$$y_1 = \partial_2 - z_1\partial_3,$$
$$y_2 = \partial_1,$$
$$y_3 = \partial_3.$$

3.19 $L^1_{3,0}$

Complement: $\{y_2, y_3, x_2, x_3, y_1\}$
Realization:

$$h_1 = -z_1\partial_1 + z_2\partial_2 + z_3\partial_3 - z_4\partial_4 + 2z_5\partial_5,$$
$$h_2 = 2z_1\partial_1 + z_2\partial_2 - 2z_3\partial_3 - z_4\partial_4 - z_5\partial_5,$$
$$x_1 = -z_2\partial_1 + z_3\partial_4 - z_5^2\partial_5,$$
$$x_2 = -z_1^2\partial_1 - z_1z_2\partial_2 + (2z_1z_3 + z_2z_4 + 1)\partial_3 + z_1z_4\partial_4 + (z_2 + z_1z_5)\partial_5,$$
$$x_3 = -z_1z_2\partial_1 - z_2^2\partial_2 + z_2z_3\partial_3 + (z_1z_3 + 2z_2z_4 + 1)\partial_4 - z_5(z_2 + z_1z_5)\partial_5,$$
$$y_1 = -z_1\partial_2 + z_4\partial_3 + \partial_5,$$
$$y_2 = \partial_1,$$
$$y_3 = \partial_2.$$

3.20 $L^1_{3,-\frac{2}{9}}$

Complement: $\{y_1, y_2, y_3, x_1, h_2\}$
Realization:

$$
\begin{aligned}
h_1 &= 2z_1\partial_1 - z_2\partial_2 + z_3\partial_3 - 2z_4\partial_4 - \partial_5,\\
h_2 &= -z_1\partial_1 + 2z_2\partial_2 + z_3\partial_3 + z_4\partial_4 + \partial_5,\\
x_1 &= -z_1^2\partial_1 + (z_1z_2 - z_3)\partial_2 - z_1z_3\partial_3 + (2z_1z_4 + 1)\partial_4 + z_1\partial_5,\\
x_2 &= z_3\partial_1 - z_2^2\partial_2(-z_2z_4 - e^{3z_5})\partial_4 - z_2\partial_5,\\
x_3 &= -z_1z_3\partial_1 + z_2\left(z_1z_2 - z_3\right)\partial_2 - z_3^2\partial_3 + (e^{3z_5}z_1 + z_2z_4z_1 + z_2 + z_3z_4)\partial_4 + z_1z_2\partial_5,\\
y_1 &= \partial_1,\\
y_2 &= \partial_2 + z_1\partial_3,\\
y_3 &= \partial_3.
\end{aligned}
$$

3.21 $L^2_{3,-\frac{1}{4}}$

Complement: $\{x_1, y_2, x_3, h_1, x_2\}$
Realization:

$$
\begin{aligned}
h_1 &= -2z_1\partial_1 - z_2\partial_2 - z_3\partial_3 + \partial_4, & h_2 &= z_1\partial_1 + 2z_2\partial_2 - z_3\partial_3 - 2\partial_4 - \partial_5,\\
x_1 &= \partial_1, & x_2 &= -z_1z_2\partial_1 - z_2^2\partial_2 + (z_2z_3 - z_1)\partial_3 + 2z_2\partial_4 + (z_2 + e^{-z_4})\partial_5,\\
x_3 &= z_2\partial_1 + \partial_3, & y_1 &= -z_1^2\partial_1 - z_1z_2\partial_2 + z_3\left(z_2z_3 - z_1\right)\partial_3 + (z_1 + z_2z_3)\partial_4 + \left(z_2 + e^{-z_4}\right)z_3\partial_5,\\
y_2 &= \partial_2, & y_3 &= z_1\partial_2 - z_3^2\partial_3 - z_3\partial_4 - z_3\partial_5.
\end{aligned}
$$

3.22 $L^1_{3,-\frac{1}{4}}$

Complement: $\{y_2, y_1, y_3, h_1, x_2\}$
Realization:

$$
\begin{aligned}
h_1 &= -z_1\partial_1 + 2z_2\partial_2 + z_3\partial_3 + \partial_4,\\
h_2 &= 2z_1\partial_1 - z_2\partial_2 + z_3\partial_3 - 2\partial_4 - \partial_5,\\
x_1 &= -z_3\partial_1 - z_2^2\partial_2 - z_2\partial_4,\\
x_2 &= -z_1^2\partial_1 + (z_1z_2 + z_3)\partial_2 - z_1z_3\partial_3 + 2z_1\partial_4 + (z_1 + e^{-z_4})\partial_5,\\
x_3 &= -z_1z_3\partial_1 - z_2\left(z_1z_2 + z_3\right)\partial_2 - z_3^2\partial_3 + (z_3 - z_1z_2)\partial_4 + (z_3 - e^{-z_4}z_2)\partial_5,\\
y_1 &= \partial_2 - z_1\partial_3,\\
y_2 &= \partial_1,\\
y_3 &= \partial_3.
\end{aligned}
$$

3.23 L^2_2

Complement: $\{y_1, y_3, x_2, h_1, x_3\}$
Realization:

$$
\begin{aligned}
h_1 &= 2z_1\partial_1 + z_2\partial_2 + z_3\partial_3 + \partial_4, & h_2 &= -z_1\partial_1 + z_2\partial_2 - 2z_3\partial_3 + \partial_4,\\
x_1 &= -z_1^2\partial_1 - z_1z_2\partial_2 + z_3\left(z_2z_3 - z_1\right)\partial_3(-z_1 - z_2z_3)\partial_4 + e^{z_4}z_3\partial_5, & x_2 &= z_2\partial_1 + \partial_3,\\
x_3 &= -z_1z_2\partial_1 - z_2^2\partial_2 + (z_2z_3 - z_1)\partial_3 - 2z_2\partial_4 + e^{z_4}\partial_5, & y_1 &= \partial_1,\\
y_2 &= z_1\partial_2 - z_3^2\partial_3 + z_3\partial_4, & y_3 &= \partial_2.
\end{aligned}
$$

3.24 L^1_2

Complement: $\{x_2, y_1, y_3, h_1, y_2\}$
Realization:

$$
h_1 = z_1\partial_1 + 2z_2\partial_2 + z_3\partial_3 + \partial_4, \qquad h_2 = -2z_1\partial_1 - z_2\partial_2 + z_3\partial_3 - 2\partial_4,
$$

$$x_1 = -z_1z_2\partial_1 - z_2^2\partial_2 - z_3\left(z_2 + z_1z_3\right)\partial_3 + (z_1z_3 - z_2)\partial_4 - e^{z_4}z_3\partial_5, \qquad x_2 = \partial_1,$$
$$x_3 = -z_2\partial_1 - z_3^2\partial_3 + z_3\partial_4, \qquad y_1 = \partial_2,$$
$$y_2 = -z_1^2\partial_1 - z_1z_2\partial_2 + (z_2 + z_1z_3)\partial_3 - 2z_1\partial_4 + e^{z_4}\partial_5, \qquad y_3 = -z_1\partial_2 + \partial_3.$$

3.25 A_1^2

Complement: $\{x_2, x_3, y_3, y_1, h_2\}$
Realization:

$$h_1 = z_1\partial_1 - z_2\partial_2 + z_3\partial_3 + 2z_4\partial_4 - \partial_5,$$
$$h_2 = -2z_1\partial_1 - z_2\partial_2 + z_3\partial_3 - z_4\partial_4 + \partial_5,$$
$$x_1 = -e^{-3z_5}\partial_1 + z_1\partial_2 - z_3z_4\partial_3 + (2e^{-3z_5}z_3 - z_4^2)\partial_4 + z_4\partial_5,$$
$$x_2 = \partial_1,$$
$$x_3 = \partial_2,$$
$$y_1 = z_2\partial_1 + \partial_4,$$
$$y_2 = (e^{-3z_5}z_2 - z_1^2)\partial_1 - z_1z_2\partial_2 + (z_1z_3 + z_2z_4z_3 + z_4)\partial_3 + (z_4\left(z_2z_4 - z_1\right) - e^{-3z_5}\left(2z_2z_3 + 1\right))\partial_4 + (z_1 - z_2z_4)\partial_5,$$
$$y_3 = -z_1z_2\partial_1 - z_2^2\partial_2 + (2z_2z_3 + 1)\partial_3 + (z_2z_4 - z_1)\partial_4.$$

3.26 A_1^1

Complement: $\{x_1, y_2, x_2, y_1, h_1\}$
Realization:

$$h_1 = -2z_1\partial_1 - z_2\partial_2 + z_3\partial_3 + 2z_4\partial_4 + \partial_5, \qquad h_2 = z_1\partial_1 + 2z_2\partial_2 - 2z_3\partial_3 - z_4\partial_4 - \partial_5,$$
$$x_1 = \partial_1, \qquad x_2 = -z_1z_2\partial_1 - z_2^2\partial_2 + (2z_2z_3 + z_1z_4 + 1)\partial_3 + z_2z_4\partial_4 + z_2\partial_5,$$
$$x_3 = z_2\partial_1 - z_4\partial_3, \qquad y_1 = -z_1^2\partial_1 - z_1z_2\partial_2 + z_1z_3\partial_3 + (z_2z_3 + 2z_1z_4 + 1)\partial_4 + z_1\partial_5,$$
$$y_2 = \partial_2, \qquad y_3 = z_1\partial_2 - z_3\partial_4.$$

3.27 $K_2^{4,\alpha}$ $(\alpha \in \mathbb{C})$

Complement: $\{y_2, y_3, x_2, x_3, y_1, h_2\}$
Realization:

$$h_1 = -z_1\partial_1 + z_2\partial_2 + z_3\partial_3 - z_4\partial_4 + 2z_5\partial_5 - \left(\frac{1}{\alpha} + 2\right)\partial_6,$$
$$h_2 = 2z_1\partial_1 + z_2\partial_2 - 2z_3\partial_3 - z_4\partial_4 - z_5\partial_5 + \partial_6,$$
$$x_1 = -z_2\partial_1 + z_3\partial_4 - z_5^2\partial_5 + \left(\frac{1}{\alpha} + 2\right)z_5\partial_6,$$
$$x_2 = -z_1^2\partial_1 - z_1z_2\partial_2 + (2z_1z_3 + z_2z_4 + 1)\partial_3 + z_1z_4\partial_4 + (z_2 + z_1z_5)\partial_5 - z_1\partial_6,$$
$$x_3 = -z_1z_2\partial_1 - z_2^2\partial_2 + z_2z_3\partial_3 + (z_1z_3 + 2z_2z_4 + 1)\partial_4 - z_5\left(z_2 + z_1z_5\right)\partial_5 + \frac{1}{\alpha}(\alpha z_2 + 2\alpha z_1z_5 + z_2 + z_1z_5)\partial_6,$$
$$y_1 = -z_1\partial_2 + z_4\partial_3 + \partial_5,$$
$$y_2 = \partial_1,$$
$$y_3 = \partial_2.$$

3.28 K_2^3

Complement: $\{y_1, y_3, x_2, x_3, h_1, y_2\}$
Realization:

$$h_1 = 2z_1\partial_1 + z_2\partial_2 + z_3\partial_3 - z_4\partial_4 + \partial_5,$$
$$h_2 = -z_1\partial_1 + z_2\partial_2 - 2z_3\partial_3 - z_4\partial_4 - 2\partial_5 + 3\partial_6,$$
$$x_1 = -z_1^2\partial_1 - z_1z_2\partial_2 + z_3\left(z_2z_3 - z_1\right)\partial_3 + (z_2z_4z_3 + z_3 + z_1z_4)\partial_4 + (2z_2z_3 - z_1)\partial_5 - z_2\left(3z_3 + e^{z_5}\right)\partial_6,$$
$$x_2 = z_2\partial_1 + \partial_3,$$
$$x_3 = -z_1z_2\partial_1 - z_2^2\partial_2 + (z_2z_3 - z_1)\partial_3 + (2z_2z_4 + 1)\partial_4 + z_2\partial_5 - 3z_2\partial_6,$$

$$y_1 = \partial_1,$$
$$y_2 = z_1\partial_2 - z_3^2\partial_3 - z_3z_4\partial_4 - 2z_3\partial_5 + (3z_3 + e^{z_5})\partial_6,$$
$$y_3 = \partial_2.$$

3.29 K_2^2

Complement: $\{y_1, y_3, x_2, h_2, x_3, y_2\}$
Realization:

$$h_1 = 2z_1\partial_1 + z_2\partial_2 + z_3\partial_3 + \partial_4 + 3\partial_5 - 3z_6\partial_6,$$
$$h_2 = -z_1\partial_1 + z_2\partial_2 - 2z_3\partial_3 + \partial_4,$$
$$x_1 = -z_1^2\partial_1 - z_1z_2\partial_2 + z_3\left(z_2z_3 - z_1\right)\partial_3(-z_1 - z_2z_3)\partial_4 + (e^{z_4}z_3 - 3z_1)\partial_5 + (3z_1z_6 - e^{-2z_4}z_2)\partial_6,$$
$$x_2 = z_2\partial_1 + \partial_3,$$
$$x_3 = -z_1z_2\partial_1 - z_2^2\partial_2 + (z_2z_3 - z_1)\partial_3 - 2z_2\partial_4 + (e^{z_4} - 3z_2)\partial_5 + 3z_2z_6\partial_6,$$
$$y_1 = \partial_1,$$
$$y_2 = z_1\partial_2 - z_3^2\partial_3 + z_3\partial_4 + e^{-2z_4}\partial_6,$$
$$y_3 = \partial_2.$$

3.30 K_2^1

Complement: $\{y_2, y_3, x_2, x_3, h_2, y_1\}$
Realization:

$$h_1 = -z_1\partial_1 + z_2\partial_2 + z_3\partial_3 - z_4\partial_4 - \partial_5 + z_6\partial_6,$$
$$h_2 = 2z_1\partial_1 + z_2\partial_2 - 2z_3\partial_3 - z_4\partial_4 + \partial_5,$$
$$x_1 = -z_2\partial_1 - e^{-3z_5}\partial_3 + z_3\partial_4 + e^{-z_5}z_6\partial_5,$$
$$x_2 = -z_1^2\partial_1 - z_1z_2\partial_2 + (2z_1z_3 + z_2z_4 + 1)\partial_3 + z_1z_4\partial_4 - z_1\partial_5 + e^{z_5}z_2\partial_6,$$
$$x_3 = -z_1z_2\partial_1 - z_2^2\partial_2 + (z_2z_3 - e^{-3z_5}z_1)\partial_3 + (z_1z_3 + 2z_2z_4 + 1)\partial_4 + e^{-z_5}z_1z_6\partial_5 - z_2z_6\partial_6,$$
$$y_1 = -z_1\partial_2 + z_4\partial_3 + e^{z_5}\partial_6,$$
$$y_2 = \partial_1,$$
$$y_3 = \partial_2.$$

3.31 K_1^5

Complement: $\{x_1, x_3, y_1, y_2, y_3, x_2\}$
Realization:

$$h_1 = -2z_1\partial_1 - z_2\partial_2 + 2z_3\partial_3 - z_4\partial_4 + z_5\partial_5 + z_6\partial_6,$$
$$h_2 = z_1\partial_1 - z_2\partial_2 - z_3\partial_3 + 2z_4\partial_4 + z_5\partial_5 - 2z_6\partial_6,$$
$$x_1 = \partial_1,$$
$$x_2 = -z_1\partial_2 + z_5\partial_3 - z_4^2\partial_4 + (2z_4z_6 + 1)\partial_6,$$
$$x_3 = \partial_2,$$
$$y_1 = -z_1^2\partial_1 - z_1z_2\partial_2 + (2z_1z_3 + z_2z_5 + 1)\partial_3 - z_4\left(z_1 + z_2z_4\right)\partial_4 + z_1z_5\partial_5 + (2z_4z_6z_2 + z_2 + z_1z_6)\partial_6,$$
$$y_2 = -z_2\partial_1 + \partial_4 + z_3\partial_5,$$
$$y_3 = -z_1z_2\partial_1 - z_2^2\partial_2 + z_2z_3\partial_3 + (z_1 + z_2z_4)\partial_4 + (z_1z_3 + 2z_2z_5 + 1)\partial_5 - z_2z_6\partial_6.$$

3.32 K_1^4

Complement: $\{y_1, y_3, x_2, h_1, h_2, x_3\}$
Realization:

$$h_1 = 2z_1\partial_1 + z_2\partial_2 + z_3\partial_3 + \partial_4,$$
$$h_2 = -z_1\partial_1 + z_2\partial_2 - 2z_3\partial_3 + \partial_5,$$
$$x_1 = -z_1^2\partial_1 - z_1z_2\partial_2 + z_3\left(z_2z_3 - z_1\right)\partial_3 - z_1\partial_4 - z_2z_3\partial_5 + e^{z_4+z_5}z_3\partial_6,$$
$$x_2 = z_2\partial_1 + \partial_3,$$

$$x_3 = -z_1z_2\partial_1 - z_2^2\partial_2 + (z_2z_3 - z_1)\partial_3 - z_2\partial_4 - z_2\partial_5 + e^{z_4+z_5}\partial_6, \qquad y_1 = \partial_1,$$
$$y_2 = z_1\partial_2 - z_3^2\partial_3 + z_3\partial_5, \qquad y_3 = \partial_2.$$

3.33 K_1^3

Complement: $\{y_1, y_3, x_2, h_1, h_2, y_2\}$
Realization:

$$h_1 = 2z_1\partial_1 + z_2\partial_2 + z_3\partial_3 + \partial_4, \qquad h_2 = -z_1\partial_1 + z_2\partial_2 - 2z_3\partial_3 + \partial_5,$$
$$x_1 = -z_1^2\partial_1 - z_1z_2\partial_2 + z_3(z_2z_3 - z_1)\partial_3 - z_1\partial_4 - z_2z_3\partial_5 - e^{z_4-2z_5}z_2\partial_6, \qquad x_2 = z_2\partial_1 + \partial_3,$$
$$x_3 = -z_1z_2\partial_1 - z_2^2\partial_2 + (z_2z_3 - z_1)\partial_3 - z_2\partial_4 - z_2\partial_5, \qquad y_1 = \partial_1,$$
$$y_2 = z_1\partial_2 - z_3^2\partial_3 + z_3\partial_5 + e^{z_4-2z_5}\partial_6, \qquad y_3 = \partial_2.$$

3.34 K_1^2

Complement: $\{y_2, y_3, x_2, x_3, y_1, h_2\}$
Realization:

$$h_1 = -z_1\partial_1 + z_2\partial_2 + z_3\partial_3 - z_4\partial_4 + 2z_5\partial_5 - 2\partial_6,$$
$$h_2 = 2z_1\partial_1 + z_2\partial_2 - 2z_3\partial_3 - z_4\partial_4 - z_5\partial_5 + \partial_6,$$
$$x_1 = -z_2\partial_1 + z_3\partial_4 - z_5^2\partial_5 + 2z_5\partial_6,$$
$$x_2 = -z_1^2\partial_1 - z_1z_2\partial_2 + (2z_1z_3 + z_2z_4 + 1)\partial_3 + z_1z_4\partial_4 + (z_2 + z_1z_5)\partial_5 - z_1\partial_6,$$
$$x_3 = -z_1z_2\partial_1 - z_2^2\partial_2 + z_2z_3\partial_3 + (z_1z_3 + 2z_2z_4 + 1)\partial_4 - z_5(z_2 + z_1z_5)\partial_5 + (z_2 + 2z_1z_5)\partial_6,$$
$$y_1 = -z_1\partial_2 + z_4\partial_3 + \partial_5,$$
$$y_2 = \partial_1,$$
$$y_3 = \partial_2.$$

3.35 K_1^1

Complement: $\{y_1, y_2, y_3, x_1, h_1, h_2\}$
Realization:

$$h_1 = 2z_1\partial_1 - z_2\partial_2 + z_3\partial_3 - 2z_4\partial_4 + \partial_5,$$
$$h_2 = -z_1\partial_1 + 2z_2\partial_2 + z_3\partial_3 + z_4\partial_4 + \partial_6,$$
$$x_1 = -z_1^2\partial_1 + (z_1z_2 - z_3)\partial_2 - z_1z_3\partial_3 + (2z_1z_4 + 1)\partial_4 - z_1\partial_5,$$
$$x_2 = z_3\partial_1 - z_2^2\partial_2 + (-z_2z_4 - e^{3z_6-3z_5})\partial_4 - z_2\partial_6,$$
$$x_3 = -z_1z_3\partial_1 + z_2(z_1z_2 - z_3)\partial_2 - z_3^2\partial_3 + (e^{3z_6-3z_5}z_1 + z_2z_4z_1 + z_2 + z_3z_4)\partial_4 - z_3\partial_5 + (z_1z_2 - z_3)\partial_6,$$
$$y_1 = \partial_1,$$
$$y_2 = \partial_2 + z_1\partial_3,$$
$$y_3 = \partial_3.$$

3.36 $J^{4,\alpha}$ $(\alpha \in \mathbb{C})$

Complement: $\{x_1, x_3, y_1, y_3, x_2, y_2, h_2\}$
Realization:

$$h_1 = -2z_1\partial_1 - z_2\partial_2 + 2z_3\partial_3 + z_4\partial_4 + z_5\partial_5 - z_6\partial_6 - \alpha\partial_7,$$
$$h_2 = z_1\partial_1 - z_2\partial_2 - z_3\partial_3 + z_4\partial_4 - 2z_5\partial_5 + 2z_6\partial_6 + \partial_7,$$
$$x_1 = \partial_1,$$
$$x_2 = -z_1\partial_2 + z_4\partial_3 + \partial_5,$$
$$x_3 = \partial_2,$$
$$y_1 = -z_1^2\partial_1 - z_1z_2\partial_2 + (2z_1z_3 + z_2z_4 + 1)\partial_3 + z_1z_4\partial_4 + (z_2 + z_1z_5)\partial_5 - z_1z_6\partial_6 - \alpha z_1\partial_7,$$
$$y_2 = -z_2\partial_1 + z_3\partial_4 - z_5^2\partial_5 + (2z_5z_6 + 1)\partial_6 + z_5\partial_7,$$
$$y_3 = -z_1z_2\partial_1 - z_2^2\partial_2 + z_2z_3\partial_3 + (z_1z_3 + 2z_2z_4 + 1)\partial_4 - z_5(z_2 + z_1z_5)\partial_5 + (2z_5z_6z_1 + z_1 + z_2z_6)\partial_6 + (-\alpha z_2 + z_2 + z_1z_5)\partial_7.$$

3.37 J^3

Complement: $\{y_3, y_2, x_2, x_3, y_1, h_2, x_1\}$
Realization:

$$
\begin{aligned}
h_1 &= z_1\partial_1 - z_2\partial_2 + z_3\partial_3 - z_4\partial_4 + 2z_5\partial_5 - 2\partial_6 - \partial_7,\\
h_2 &= z_1\partial_1 + 2z_2\partial_2 - 2z_3\partial_3 - z_4\partial_4 - z_5\partial_5 + \partial_6,\\
x_1 &= -z_1\partial_2 + z_3\partial_4 - z_5^2\partial_5 + 2z_5\partial_6 + (z_5 + e^{-z_6})\partial_7,\\
x_2 &= -z_1z_2\partial_1 - z_2^2\partial_2 + (2z_2z_3 + z_1z_4 + 1)\partial_3 + z_2z_4\partial_4 + (z_1 + z_2z_5)\partial_5 - z_2\partial_6,\\
x_3 &= -z_1^2\partial_1 - z_1z_2\partial_2 + z_1z_3\partial_3 + (z_2z_3 + 2z_1z_4 + 1)\partial_4 - z_5\left(z_1 + z_2z_5\right)\partial_5 +\\
&\quad (z_1 + 2z_2z_5)\partial_6 + (z_1 + z_2(z_5 + e^{-z_6}))\partial_7,\\
y_1 &= -z_2\partial_1 + z_4\partial_3 + \partial_5,\\
y_2 &= \partial_2,\\
y_3 &= \partial_1.
\end{aligned}
$$

3.38 J^2

Complement: $\{x_2, x_3, y_2, y_1, y_3, h_1, h_2\}$
Realization:

$$
\begin{aligned}
h_1 &= z_1\partial_1 - z_2\partial_2 - z_3\partial_3 + 2z_4\partial_4 + z_5\partial_5 + \partial_6,\\
h_2 &= -2z_1\partial_1 - z_2\partial_2 + 2z_3\partial_3 - z_4\partial_4 + z_5\partial_5 + \partial_7,\\
x_1 &= z_1\partial_2 - z_5\partial_3 - z_4^2\partial_4 - z_4\partial_6,\\
x_2 &= \partial_1,\\
x_3 &= \partial_2,\\
y_1 &= z_2\partial_1 + \partial_4 - z_3\partial_5,\\
y_2 &= -z_1^2\partial_1 - z_1z_2\partial_2 + (2z_1z_3 + z_2z_5 + 1)\partial_3 + z_4\left(z_2z_4 - z_1\right)\partial_4 + z_1z_5\partial_5 + z_2z_4\partial_6 + z_1\partial_7,\\
y_3 &= -z_1z_2\partial_1 - z_2^2\partial_2 + z_2z_3\partial_3 + (z_2z_4 - z_1)\partial_4 + (z_1z_3 + 2z_2z_5 + 1)\partial_5 + z_2\partial_6 + z_2\partial_7.
\end{aligned}
$$

3.39 J^1

Complement: $\{y_2, y_3, x_2, x_3, y_1, h_1, h_2\}$
Realization:

$$
\begin{aligned}
h_1 &= -z_1\partial_1 + z_2\partial_2 + z_3\partial_3 - z_4\partial_4 + 2z_5\partial_5 + \partial_6,\\
h_2 &= 2z_1\partial_1 + z_2\partial_2 - 2z_3\partial_3 - z_4\partial_4 - z_5\partial_5 + \partial_7,\\
x_1 &= -z_2\partial_1 - e^{3z_6 - 3z7}\partial_3 + z_3\partial_4 - z_5^2\partial_5 - z_5\partial_6,\\
x_2 &= -z_1^2\partial_1 - z_1z_2\partial_2 + (2z_1z_3 + z_2z_4 + 1)\partial_3 + z_1z_4\partial_4 + (z_2 + z_1z_5)\partial_5 - z_1\partial_7,\\
x_3 &= -z_1z_2\partial_1 - z_2^2\partial_2 + (z_2z_3 - z_1e^{3z_6 - 3z_7})\partial_3 + (z_1z_3 + 2z_2z_4 + 1)\partial_4 -\\
&\quad z_5\left(z_2 + z_1z_5\right)\partial_5 + (-z_2 - z_1z_5)\partial_6 - z_2\partial_7,\\
y_1 &= -z_1\partial_2 + z_4\partial_3 + \partial_5,\\
y_2 &= \partial_1,\\
y_3 &= \partial_2.
\end{aligned}
$$

4 Adjoint realization of A_2

It is interesting to compare the above realization generated by Shirokov's method with adjoint realization of A_2. The latter reads

$$
\begin{aligned}
h_1 &= 2z_3\partial_3 - z_4\partial_4 + z_5\partial_5 - 2z_6\partial_6 + z_7\partial_7 - z_8\partial_8,\\
h_2 &= -z_3\partial_3 + 2z_4\partial_4 + z_5\partial_5 + z_6\partial_6 - 2z_7\partial_7 - z_8\partial_8,\\
x_1 &= -2z_3\partial_1 + z_3\partial_2 - z_5\partial_4 + z_1\partial_6 + z_7\partial_8,\\
x_2 &= z_4\partial_1 - 2z_4\partial_2 + z_5\partial_3 + z_2\partial_7 - z_6\partial_8,\\
x_3 &= -z_5\partial_1 - z_5\partial_2 + z_4\partial_6 - z_3\partial_7 + (z_1 + z_2)\partial_8,\\
y_1 &= 2z_6\partial_1 - z_6\partial_2 - z_1\partial_3 - z_4\partial_5 + z_8\partial_7,\\
y_2 &= -z_7\partial_1 + 2z_7\partial_2 - z_2\partial_4 + z_3\partial_5 - z_8\partial_6,\\
y_3 &= z_8\partial_1 + z_8\partial_2 - z_7\partial_3 + z_6\partial_4 - (z_1 + z_2)\partial_5.
\end{aligned}
$$

5 Havlíček's realization of A_2

In [2], the following realization of A_2 was given:

$$h_1 = z_1\partial_1 - z_2\partial_2 + 2z_3\partial_3, \qquad h_2 = z_1\partial_1 + 2z_2\partial_2 - z_3\partial_3,$$
$$x_1 = z_1\partial_2 + z_3^2\partial_3, \qquad x_2 = z_1z_2\partial_1 + z_2^2\partial_2 - (z_1 + z_2z_3)\partial_3,$$
$$x_3 = -z_1^2\partial_1 - z_1z_2\partial_2 - z_3\left(z_1 + z_2z_3\right)\partial_3, \qquad y_1 = z_2\partial_1 - \partial_3,$$
$$y_2 = -\partial_2, \qquad y_3 = \partial_1.$$

6 Final remarks

- It is not clear wheter adjoint or Havlíček's realizations of A_2 are equivalent to some of Shirokov's realization given in the list. See [4] for the comments to this regarding the algebra A_1.
- Some of realizations act on spaces of smooth functions, which can be narrowed down to a suitable subpace, i. e. subspace of polynomials or even of finite-dimension.

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