Metriplectic Geometries: Definitions and Applications

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In Memory of Arno Bohm

Metriplectic 4-Bracket: undergrad, **Michael Updike** (geometry), now Princeton grad student, grad student intern, **Azeddine Zaidni** (UT Algorithm), UM6, Marrakesh-Safi, Morocco; **Naoki Sato** (collision operators), NIFS, Nagoya, Japan, **William Barham** (numerics), UT Austin.

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Theory of thermodynamically consistent theories. (Theory = dynamical system).

Overview

I. Motivation

II. Metriplectic 4-Bracket

III. Unified Thermodynamic (UT) Algorithm

IV. Examples

V. Final Comments

I. Motivation

Thermodynamic Consistency (TC)– Examples

Navier-Stokes is **inconsistent:**

$$\partial_t \boldsymbol{v} = -\boldsymbol{v}\cdot\nabla\boldsymbol{v} - \frac{1}{\rho_0}\nabla p + \nu\nabla^2\boldsymbol{v}\,, \qquad \nabla\cdot\boldsymbol{v} = 0 \quad \Rightarrow \quad p[\boldsymbol{v}]$$

$$\boxed{H = \int_\Omega \rho_0\,|\boldsymbol{v}|^2/2 \quad \text{and} \quad \dot{H} \le 0\,, \quad \nexists \text{ any thermodynamics!}}$$

Navier-Stokes-Fourier (NSF) is **consistent:** (Eckart 1940):

$$\begin{aligned}
\partial_t \boldsymbol{v} &= -\boldsymbol{v}\cdot\nabla\boldsymbol{v} - \frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla\cdot\mathcal{T} \\
\partial_t \rho &= -\nabla\cdot(\rho\boldsymbol{v}) \\
\partial_t s &= -\boldsymbol{v}\cdot\nabla s - \frac{1}{\rho T}\nabla\cdot\boldsymbol{q} + \frac{1}{\rho T}\mathcal{T}:\nabla\boldsymbol{v} \qquad \text{heat flux \& viscous heating}
\end{aligned}$$

$$\boxed{H = \int_\Omega \rho\,|\boldsymbol{v}|^2/2 + \rho u(\rho, s)\,, \quad \dot{H} = 0 \quad \text{and} \quad S = \int_\Omega \rho s \;\rightarrow\; \dot{S} \ge 0}$$

Example of **Thermodynamic Completion**, i.e. NS $\rightarrow$ NSF.

Thermodynamic Consistency

The realization in a **dynamical system** of the first and second laws of thermodynamics:

First Law is energy conservation:

$$\dot{H} = 0$$

Second Law is entropy production:

$$\dot{S} \geq 0$$

Good models lift thermodynamics to **dynamical systems.** They have two functions H, S.

Theories & Models as Dynamical Systems

Main Scientific Goal:

Predict the future or explain the past $\Rightarrow$

$$\dot{z} = V(z)\,, \qquad \text{dynamical variable } z \in \mathcal{Z} \text{ the Phase Space}$$

Ultimately a dynamical system. Vector fields on manifolds and Cauchy problem (IVP).

Examples: Maps, ODEs, PDEs, etc. finite-dimensional, infinite-dimensional (field theories)

Whence vector field V?

• Fundamental parent theory (microscopic, N interacting gravitating or charged particles, BBGKY hierarchy, Vlasov-Maxwell system, ...). Identify small parameters, limits, rigorous asymptotics, Hilbert's 6th $\rightarrow$ Reduced Computable Model for V.

• Phenomena based modeling using known properties, constraints, symmetries, etc. used to intuit $\rightarrow$ Reduced Computable Model V. $\leftarrow$ Here metriplectic structure can be useful.

Vector Field Splitting

$$V(z) = V_{nondissipative} + V_{dissipative}$$

How?

Vector Field Splitting

$$V(z) = V_{nondissipative} + V_{dissipative}$$

How?

$V_{nondissipative} \equiv$ Hamiltonian and $V_{dissipative} \equiv$?

What is Dissipation?

- Not all conservative systems are Hamiltonian
- Not all Hamiltonian systems are conservative
- Not all reversible systems are Hamiltonian
- All finite dynamical systems (autonomous ODEs) are reversible (1 parameter Lie group)
- Some infinite systems (PDEs) are reversible and some irreversible (group vs. semigroup)
- Not all Hamiltonian systems have time reversal symmetry
- Not all systems with time reversal symmetry are Hamiltonian
- $\exists$ systems with time reversible symmetry and global asymptotic stability

Thermodynamically Consistent Dissipation:
<u>Energy conserving</u> systems with an <u>increasing entropy</u> that implies <u>global asymptotic stability</u>.

Such systems have a 'vector field' that naturally splits in Hamiltonian and dissipative parts. Hamiltonian is an unambiguous way to define nondissipative. The Metriplectic 4-bracket is an unambiguous way to define dissipative. Together they $\Rightarrow$ thermodynamic consistency.

Toward a Thermodynamically Consistent Split

$$V(z) = V_H + V_D$$

Hamiltonian Form:

$$V_H = J\,\frac{\partial H}{\partial z}$$

where $J(z)$ is Poisson tensor/operator and H is the Hamiltonian. Basic product decomposition.

Dissipative Form:

$$V_D = \ldots\ ? \qquad \rightarrow \qquad V_D = G\,\frac{\partial F}{\partial z}$$

General degenerate 'metric tensor' of some kind for gradient flow?

Frameworks for Dissipation – Some History

Lagrangian/Action Based: Rayleigh (1873),: $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_\nu}\right) - \left(\frac{\partial \mathcal{L}}{\partial q_\nu}\right) + \left(\frac{\partial \mathcal{F}}{\partial \dot{q}_\nu}\right) = 0$
Linear dissipation e.g. of sound waves. *Theory of Sound*.
Gay-Balmaz & Yoshimura (2017) & Eldred (2020).

Gradient Based: Cahn-Hilliard (1958): $\frac{\partial n}{\partial t} = \nabla^2 \frac{\delta \mathcal{F}}{\delta n} = \nabla^2 \left(n^3 - n - \nabla^2 n\right)$
Phase separation, nonlinear diffusive dissipation, binary fluid ..
Otto, Ricci Flows, Poincarè conjecture on S^3, Hamilton, Perelman (2002)... : $\frac{\partial \psi}{\partial t} = \mathcal{G}\,\frac{\delta \mathcal{F}}{\delta \psi}$

Bracket Based: Kaufman & pjm (1982, 1984), Grmela (1984), pjm (1986), Öttinger & Grmela (1997), ... : $\frac{\partial \psi}{\partial t} = (\psi, \mathcal{F})$ $\leftarrow$ emerges here
Plasma models, kinetic theory, fluids.

New Metriplectic 4-Bracket Based: pjm, Updike, Zaidni (2024,2025):
An encompassing theory.

Metriplectic Dynamics

(Metric $\cup$ Symplectic Flows (pjm 1986) $\leftrightarrow V_D + V_H$)

• Formalism for natural split of vector fields

• Enforces thermodynamic consistency: $\dot{H} = 0$ the 1st Law and $\dot{S} \geq 0$ the 2nd Law.

• Other invariants? E.g., collision operators preserve, mass, momentum, ... There exists some theory for building in, but won't discuss today.

• **Encompassing 4-bracket**: Entropy is a Casimir is & "curvature" is dissipation rate

Ideas of Casimirs are candidates for entropy, multibracket, curvature, etc. in pjm (1984). Metriplectic in pjm (1986).

Metriplectic 4-Bracket Dynamics

Dynamical System (finite or infinite):

$$\dot{z} = \{z, H\} + (z, H; S, H)$$

Dynamics for any observable (functional of dynamical variables), z, is generated by multi-linear brackets, Poisson bracket + 4-bracket (2024), with Hamiltonian H and entropy = Casimir S.

Hamiltonian Review

Poisson Bracket: $\{f, g\}$

Noncanonical Poisson Brackets – Flows on Poisson Manifolds

Definition. A Poisson manifold $\mathcal{Z}$ has bracket

$$\{\,,\}\colon C^\infty(\mathcal{Z}) \times C^\infty(\mathcal{Z}) \to C^\infty(\mathcal{Z})$$

st $C^\infty(\mathcal{Z})$ with $\{\,,\}$ is a Lie algebra realization, i.e., is

- $\mathbb{R}$-bilinear,
- antisymmetric,
- Jacobi identity
- Leibniz, i.e., acts as a derivation $\Rightarrow$ vector field.

Geometrically $C^\infty(\mathcal{Z}) \equiv \Lambda^0(\mathcal{Z})$ and $\boldsymbol{d}$ exterior derivative.

$$\{f,g\} = \langle \boldsymbol{d}f, J\boldsymbol{d}g\rangle = J(\boldsymbol{d}f,\boldsymbol{d}g) = \frac{\partial f}{\partial z^i} J^{ij} \frac{\partial g}{\partial z^j}\,.$$

J the Poisson tensor/operator. Flows are integral curves of noncanonical Hamiltonian vector fields, $J\boldsymbol{d}H$, i.e.,

$$\dot z^i = J^{ij}(z)\frac{\partial H(z)}{\partial z^j}\,, \qquad \mathcal{Z}'s \text{ coordinate patch } z = (z^1,\ldots,z^N)$$

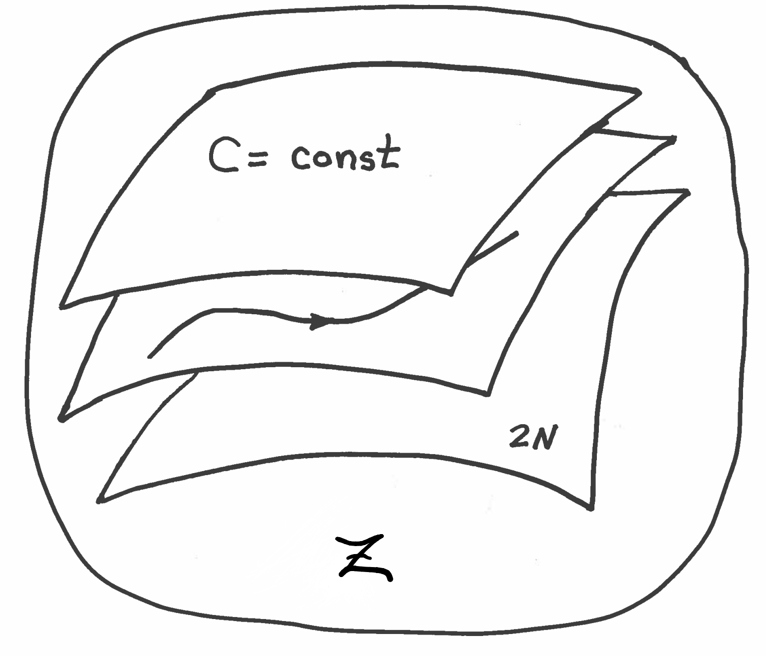
Because of degeneracy, $\exists$ functions C st $\{f,C\} = 0$ for all $f \in C^\infty(\mathcal{Z})$, called Casimir invariants. (Lie's distinguished functions!) **Casimir are candidate entropies!**

Poisson Manifold (phase space) $\mathcal{Z}$ Cartoon

Degeneracy in J $\Rightarrow$ Casimirs:

$$\{f, C\} = 0 \quad \forall \ f : \mathcal{Z} \to \mathbb{R}$$

Lie-Darboux Foliation by Casimir (symplectic) leaves:



Noncanonical Hamiltonian Structure

Sophus Lie (1890) $\longrightarrow$ PJM (noncanonical 1980) $\longrightarrow$ Poisson Manifolds etc.

Noncanonical Coordinates:

$$\dot{z}^i = \{z^i, H\} = J^{ij}(z)\frac{\partial H}{\partial z^j}$$

Noncanonical Poisson Bracket:

$$\{f,g\} = \frac{\partial f}{\partial z^i}J^{ij}(z)\frac{\partial g}{\partial z^j}$$

Bilinear Poisson Bracket Properties:

antisymmetry $\rightarrow$ $\{f,g\} = -\{g,f\}$ $\rightarrow$ $J^{ij} = -J^{ji}$

Jacobi identity $\rightarrow$ $\{f,\{g,h\}\} + \{b,\{h,f\}\} + \{h,\{f,g\}\} = 0$ $\rightarrow$ Jacobiator $S^{ijk} = J^{i\ell}\partial_\ell J^{jk} + cyc \equiv 0$

Leibniz $\rightarrow$ $\{fh,g\} = f\{h,g\} + \{h,g\}f\,,$ fg pointwise

G. Darboux: $detJ \neq 0 \Longrightarrow J \rightarrow J_c$ Canonical Coordinates

Sophus Lie: $detJ = 0 \Longrightarrow$ Canonical Coordinates plus Casimirs $\leftarrow$ G. Sudarshan
(Lie's distinguished functions!)

Hamilton's Canonical Equations

Phase Space with Canonical Coordinates: (q, p)

Hamiltonian function: $H(q, p)$ $\leftarrow$ the energy

Equations of Motion:

$$\boxed{\dot{p}_\alpha = -\frac{\partial H}{\partial q^\alpha}\,, \qquad \dot{q}^\alpha = \frac{\partial H}{\partial p_i}\,,} \qquad \alpha = 1, 2, \ldots N$$

Phase Space Coordinate Rewrite: $z = (q, p)\,, \quad i, j = 1, 2, \ldots 2N$

$$\dot{z}^i = J_c^{ij}\frac{\partial H}{\partial z^j} = \{z^i, H\}_c\,, \qquad (J_c^{ij}) = \begin{pmatrix} 0_N & I_N \\ -I_N & 0_N \end{pmatrix},$$

$J_c :=$ Poisson tensor, Hamiltonian bi-vector, cosymplectic form

II. Metriplectic 4-Bracket: $(f, k; g, n)$

Why a 4-Bracket?

- One slot for dynamical variables (observables), z.

- Two slots for two fundamental functions: Hamiltonian, H, and Entropy (Casimir), S.

- There remains one slot for $\mathcal{F}$, free energy like generator $\mathcal{F} = H - TS$. Better argument: Needed to have multilinearity.

Comments:

- Provides natural reductions to other bilinear & binary brackets.

- The three slot brackets of pjm 1984 were not trilinear. Four needed to be multilinear.

The Metriplectic 4-Bracket

4-bracket on 0-forms (functions):

$$(\,\cdot\,,\,\cdot\,;\,\cdot\,,\,\cdot\,)\colon C^\infty(\mathcal{Z})\times \mathbb{C}^\infty(\mathcal{Z})\times C^\infty(\mathcal{Z})\times C^\infty(\mathcal{Z})\rightarrow C^\infty(\mathcal{Z})$$

For functions $f,k,g,n\in C^\infty(\mathcal{Z})$ in a coordinate patch the 4-bracket has the form:

$$(f,k;g,n) = R^{ijkl}(z)\frac{\partial f}{\partial z^i}\frac{\partial k}{\partial z^j}\frac{\partial g}{\partial z^k}\frac{\partial n}{\partial z^l}\,. \qquad \leftarrow \text{quadravector?}$$

• Metriplectic manifolds have both Poisson tensor, J^{ij}, and compatible quadravector R^{ijkl}, where S (selected from set of Casimirs) and H come from Hamiltonian part.

A blend of my previous early ideas 1980s: Two important functions H and S, symmetries, curvature idea, multi-brackets.

Metriplectic 4-Bracket Properties

(i) $\mathbb{R}$-linearity in all arguments, e.g, for $\lambda \in \mathbb{R}$

$$(f + \lambda h, k; g, n) = (f, k; g, n) + \lambda(h, k; g, n)$$

(ii) algebraic identities/symmetries

$$(f, k; g, n) = -(k, f; g, n)\,, \qquad (f, k; g, n) = -(f, k; n, g)\,, \qquad (f, k; g, n) = (g, n; f, k)$$

(iii) derivation in all arguments, e.g.,

$$(fh, k; g, n) = f(h, k; g, n) + (f, k; g, n)h$$

where as usual, fh denotes pointwise multiplication.

Symmetries of algebraic curvature without torsion identity. **Minimal Metriplectic.**

Observation: Often see $R^{l}{}_{ijk}$ or R_{lijk} but not R^{lijk}! Never 4-bracket, i.e. action on 1-forms?

Properties – Existence – General Construction Methods

• Thermodynamic Consistency Built-in:

$$\dot{H} = \{H, H\} + (H, H; S, H) = 0 \qquad \text{and} \qquad \dot{S} = (S, H; S, H) \geq 0$$

Reduces to metriplectic 2-bracket (1984): $(F, G)_H = (F, H; G, H)$.

• For any Riemannian manifold $\exists$ metriplectic 4-bracket. This means there is a wide class of them, but the bracket tensor does not need to come from Riemann tensor only needs to satisfy the bracket properties.

• If Riemannian, entropy production rate is positive contravariant sectional curvature. For closed $\sigma, \eta \in \Lambda^1(\mathcal{Z})$, entropy production by

$$\dot{S} = K(\sigma, \eta) := (S, H; S, H) \geq 0\,,$$

where the second equality follows from $\sigma = \boldsymbol{d}S$ and $\eta = \boldsymbol{d}H$.

• Two methods of construction? **Kulkarni-Nomizu** (K-N) product and **Lie algebra** based. $K(\sigma, \eta) \geq 0$ automatic for K-N and easily made minimally degenerate!

Methods of Construction

Construction via Kulkarni-Nomizu Product

Given σ and μ, two symmetric rank-2 tensor fields operating on 1-forms (assumed exact) $\boldsymbol{d}f, \boldsymbol{d}k$ and $\boldsymbol{d}g, \boldsymbol{d}n$, the K-N product is

$$\begin{aligned}\sigma \owedge \mu\,(\boldsymbol{d}f, \boldsymbol{d}k, \boldsymbol{d}g, \boldsymbol{d}n) &= \sigma(\boldsymbol{d}f, \boldsymbol{d}g)\,\mu(\boldsymbol{d}k, \boldsymbol{d}n) - \sigma(\boldsymbol{d}f, \boldsymbol{d}n)\,\mu(\boldsymbol{d}k, \boldsymbol{d}g)\\ &+ \mu(\boldsymbol{d}f, \boldsymbol{d}g)\,\sigma(\boldsymbol{d}k, \boldsymbol{d}n) - \mu(\boldsymbol{d}f, \boldsymbol{d}n)\,\sigma(\boldsymbol{d}k, \boldsymbol{d}g)\,.\end{aligned}$$

Metriplectic 4-bracket:

$$(f, k; g, n) = \sigma \owedge \mu(\boldsymbol{d}f, \boldsymbol{d}k, \boldsymbol{d}g, \boldsymbol{d}n)\,.$$

In coordinates:

$$R^{ijkl} = \sigma^{ik}\mu^{jl} - \sigma^{il}\mu^{jk} + \mu^{ik}\sigma^{jl} - \mu^{il}\sigma^{jk}\,.$$

If σ or μ defines inner product, then minimally degenerate, one fixed point on $H=$ constant.

Infinite dimensions: $\mu \to M\,, \quad \sigma \to \Sigma$.

Lie Algebra Based Metriplectic 4-Brackets

- For structure constants $c^{kl}{}_{s}$:

$$(f, k; g, n) = c^{ij}{}_{r} c^{kl}{}_{s}\, g^{rs} \frac{\partial f}{\partial z^i} \frac{\partial k}{\partial z^j} \frac{\partial g}{\partial z^k} \frac{\partial n}{\partial z^l}\,.$$

Lacks cyclic symmetry, but $\exists$ procedure to remove torsion (Bianchi identity) for any symmetric 'metric' g^{rs}. Dynamics does not see torsion, but manifold does.

- For $g^{rs}_{CK} = c^{rl}{}_{k}\, c^{sk}{}_{l}$ the Cartan-Killing metric, torsion vanishes automatically. Completely determined by Lie algebra. For $\mathfrak{so}(3)$ reproduces relaxing free rigid body (pjm 1986).

- Covariant connection $\nabla \colon \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$. A contravariant connection $D \colon \Lambda^1(\mathcal{Z}) \times \Lambda^1(\mathcal{Z}) \to \Lambda^1(\mathcal{Z})$ satisfying Koszul identities, but Leibniz becomes $D_\alpha(f\gamma) = fD_\alpha\gamma + J(\alpha)[f]\gamma$ where $J(\alpha)[f] = \alpha_i J^{ij} \partial f / \partial z^j$ is a 0-form that replaces the term $\mathbf{X}(f)$ (Fernandes, 2000). Here $\alpha, \beta, \gamma \in \Lambda^1(\mathcal{Z})$, $f \in \Lambda^0(\mathcal{Z})$. Build 4-bracket like curvature from connection.

III. Unified Thermodynamic (UT) Algorithm

UT Algorithm is an algorithm for constructing metriplectic systems! Applied to many systems. **So far UT Algorithm either reproduces, corrects, or extends for every case considered!**

Examples: Cahn-Hilliard-Navier-Stokes, Brenner-Navier-Stokes, Generalized Brenner-Navier-Stokes, generalization of Landau collision operator ... .

Four Steps of the UT Algorithm

1. Identify dynamical variables defined on $\Omega \subset \mathbb{R}^3$; e.g. for NSF

$$\boldsymbol{\xi} = (\boldsymbol{m} = \rho \boldsymbol{v}, \rho, \sigma = \rho s)$$

2. Propose energy and entropy functionals, $H[\boldsymbol{\xi}]$ and $S[\boldsymbol{\xi}]$; for NSF

$$H = \int_\Omega \frac{|\mathbf{m}|^2}{2\rho} + \rho u(\rho, \sigma/\rho) \quad \text{and} \quad S = \int_\Omega \sigma$$

3. Find Poisson bracket $\{F, G\}$ for which entropy S is a Casimir invariant, $\{F, S\} = 0 \,\forall\, F$

4. Construct metriplectic 4-bracket $(F, K; G, N)$ via Kulkarni-Nomizu product by a **new method that separates local thermodynamics from phenomenological quantities**, giving the EoMs as Poisson bracket + 4-bracket:

$$\partial_t \boldsymbol{\xi} = \{\boldsymbol{\xi}, H\} + (\boldsymbol{\xi}, H; S, H)$$

Result automatically Thermodynamically consistent!

3. For NSF Ideal Fluid Poisson Bracket Dynamics

Hamiltonian:

$$H = \int_\Omega \frac{\rho|\mathbf{v}|^2}{2} + \rho u\,(\rho, s)\ , \qquad T = \frac{\partial u}{\partial s}\,, \qquad p = \rho^2\frac{\partial u}{\partial \rho}\,.$$

Lie-Poisson Bracket (pjm-Greene, 1980):

$$\{F,G\} = -\int_\Omega \mathbf{m}\cdot[F_{\boldsymbol{m}}\cdot\nabla G_{\boldsymbol{m}} - G_{\boldsymbol{m}}\cdot\nabla F_{\boldsymbol{m}}] + \rho\,[F_{\boldsymbol{m}}\cdot\nabla G_\rho - G_{\boldsymbol{m}}\cdot\nabla F_\rho]$$
$$+\,\sigma\,[F_{\boldsymbol{m}}\cdot\nabla G_\sigma - G_{\boldsymbol{m}}\cdot\nabla F_\sigma]\,.$$

Equations of Motion:

$$\partial_t\mathbf{v} = \{\boldsymbol{v}, H\} = -\boldsymbol{v}\cdot\nabla\boldsymbol{v} - \nabla p/\rho\,, \quad \partial_t\rho = \{\rho, H\} = -\nabla\cdot(\rho\,\boldsymbol{v})\,, \quad \partial_t\sigma = \{\sigma, H\} = -\nabla\cdot(\sigma\,\boldsymbol{v})\,.$$

Casimir:

$$S = \int_\Omega \rho s = \int_\Omega \sigma\,.$$

Note: $F_{\boldsymbol{m}} = \delta F/\delta \boldsymbol{m}$, etc., functional derivatives.

4. Metriplectic 4-Bracket

Old method (early 2024): guess M and Σ.

New Method

Theorem: Order dynamical variables st

$$\partial_t \xi^\alpha = \{\xi^\alpha, H\} + \nabla \cdot \boldsymbol{J}^\alpha\,, \qquad \alpha = 1, \ldots, N-1\,,$$
$$\partial_t \xi^N = \{\xi^N, H\} + \nabla \cdot \boldsymbol{J}^N + \boldsymbol{Z}_\alpha \cdot \tilde{L}^{\alpha\beta} \cdot \boldsymbol{Z}_\beta\,.$$

where $\xi^N = \sigma$, the entropy density. Above splits Hamiltonian and conservative.

Then

$$\dot{S} = \int_\Omega \boldsymbol{Z}_\alpha \cdot \tilde{L}^{\alpha\beta} \cdot \boldsymbol{Z}_\beta =: \int_\Omega \dot{\sigma}^{prod} \geq 0\,.$$

and $\dot{H} \Rightarrow$

$$\boldsymbol{Z}_\alpha = \nabla H_{\xi^\alpha}\,, \qquad \boldsymbol{J}^\alpha = -H_{\xi^N} \tilde{L}^{\alpha\beta} \nabla H_{\xi^\beta} = -L^{\alpha\beta} \nabla H_{\xi^\beta}\,.$$

which leads naturally to

$$M(dF, dG) = F_{\xi^N}\, G_{\xi^N}\,, \qquad \Sigma(dF, dG) = \nabla(F_{\xi^\alpha}) \frac{L^{\alpha\beta}}{H_{\xi^N}} \nabla(G_{\xi^\beta})\,.$$

Important Byproduct of UT Algorithm

- Special **ordering of dynamical variables** and concomitant 'Force-Flux' relations of nonequilibrium thermodynamics:

$$\boldsymbol{J}^\alpha = L^{\alpha\beta} X_\beta \quad \to \quad \boldsymbol{J}^\alpha = -L^{\alpha\beta}\,\nabla(\delta H/\delta\xi^\beta)$$

'Forces': $\boldsymbol{X} \sim \nabla T, \nabla p, \nabla \boldsymbol{v}$ etc., UT Algorithm removes ambiguous selection of forces and provides definition of phenomenological coefficients, $L^{\alpha\beta}$, for dynamical variables ξ^β.

- Separates dependences on thermodynamical variables that come from internal energy U (local thermodynamic equilibrium) from those that appear in the phenomenological coefficients $L^{\alpha\beta}$. For example in in the Fourier heat law entropy production expression

$$\dot{\sigma}^{prod} = \nabla T \cdot \frac{\bar{\kappa}}{T^2} \cdot \nabla T$$

one T comes from Fourier's law $q = -\bar{\kappa}\nabla T/T$ while the other comes from the phenomenological coefficient.

- Physically identify the sectional curvature

$$\dot{S} = (S, H; S, H) = K(H, S) = \int_\Omega \Sigma(dH, dH) = \int_\Omega \nabla H_{\xi^\alpha} \cdot \tilde{L}^{\alpha\beta} \cdot \nabla H_{\xi^\beta} \geq 0\,.$$

4. Metriplectic 4-Bracket: General and NSF

General flux expressions:

$$
\begin{aligned}
\boldsymbol{J}_\rho &= -L^{\rho\rho}\cdot\nabla H_\rho - L^{\rho \boldsymbol{m}}:\nabla H_{\boldsymbol{m}} - L^{\rho\sigma}\cdot\nabla H_\sigma\,,\\
\bar{J}_{\mathbf{m}} &= -L^{\boldsymbol{m}\rho}\otimes\nabla H_\rho - L^{\boldsymbol{mm}}:\nabla H_{\boldsymbol{m}} - L^{\boldsymbol{m}\sigma}\otimes\nabla H_\sigma\,,\\
\boldsymbol{J}_\sigma &= -L^{\sigma\rho}\cdot\nabla H_\rho - L^{\sigma \boldsymbol{m}}:\nabla H_{\boldsymbol{m}} - L^{\sigma\sigma}\cdot\nabla H_\sigma\,,
\end{aligned}
$$

where $\boldsymbol{J}_\rho$ is mass flux, $\bar{J}_{\mathbf{m}}$ is momentum flux 2-tensor, and $\boldsymbol{J}_\sigma$ is entropy flux.

For **NSF** all zero except:

$$
L^{\boldsymbol{mm}} = \bar{\bar{\Lambda}} \quad \text{and} \quad L^{\sigma\sigma} = \frac{\bar{\kappa}}{T}\,.
$$

$\bar{\bar{\Lambda}}$ isotropic 4-tensor, $\bar{\kappa}$ conduction 2-tensor

$$
\dot{S} = (S,H;S,H) = \int_\Omega \Sigma(dH,dH) = \int_\Omega \nabla\mathbf{v} : \frac{\bar{\bar{\Lambda}}}{T} : \nabla\mathbf{v} + \nabla T\cdot\frac{\bar{\kappa}}{T^2}\cdot\nabla T \geq 0\,.
$$

Note in $\bar{\kappa}/T^2$ one T from H one from $L^{\alpha\beta}$. Σ sectional curvature density?

4. Metriplectic 4-Bracket for NSF Generalizations

For **Brenner NSF** all zero except:

$$L^{m\rho} = \tilde{D}\rho\, \boldsymbol{m}\,, \quad L^{m\sigma} = \tilde{D}\hat{\sigma}\, \boldsymbol{m}\,, \quad L^{mm} = \bar{\bar{\Lambda}} + \tilde{D}\, \boldsymbol{m} \otimes \bar{I} \otimes \boldsymbol{m}\,.$$

$$L^{\sigma\rho} = \tilde{D}\rho\hat{\sigma}\, \bar{I}\,, \quad L^{\sigma\sigma} = \frac{\bar{\kappa}}{T} + \tilde{D}\hat{\sigma}^2\, \bar{I} \quad L^{\sigma m} = \tilde{D}\hat{\sigma}\, \bar{I} \otimes \boldsymbol{m}$$

$$\dot{S} = \int_\Omega \frac{1}{T}\left[\frac{\tilde{D}}{\kappa_T^2\rho^2}|\nabla\rho|^2 + \nabla T \cdot \frac{\bar{\kappa}}{T} \cdot \nabla T \; + \nabla \boldsymbol{v} : \bar{\bar{\Lambda}} : \nabla \boldsymbol{v}\right] \geq 0\,.$$

Generalization of Brenner by **Reddy et al.** (2019) falls out. We further generalized.

V. Final Comments

- The UT Algorithm based on the metriplectic 4-bracket, is a proven framework, provides a direct method for constructing thermodynamically consistent systems.

- Tons of interesting geometry already ... more to explore.

- Metriplectic 4-brackets are easy to discretize while maintaining symmetries. First numerical implementation via 4-bracket discretization (Barham et al. 2025) for 1-D Navier-Stokes-Fourier. Finite element projection of PDE to thermodynamically consistent finite-dimensional 4-bracket, i.e., ODEs. For example, for the density $\rho(x,t)$

$$\rho_h(x,t) = \sum_{i=1}^{N} \rho_i(t)\phi_i(x) \quad \rightarrow \quad \dot{\rho}_i(t) = \{\rho_i, H\} + (\rho_i, H; S, H)\ldots$$

Results use Firedrake library, implicit midpoint, Irksome module ...