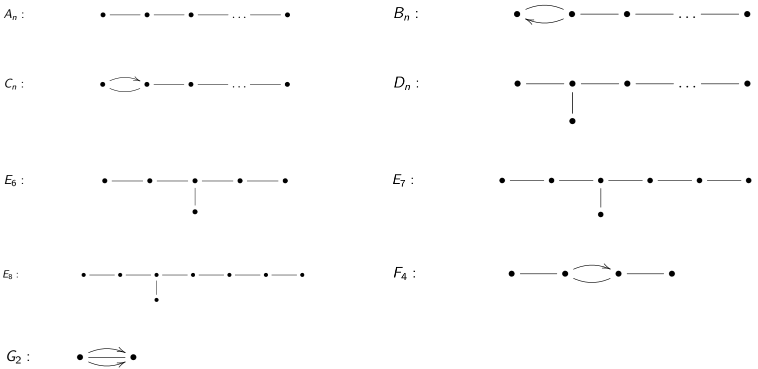
Infinite Dynkin diagrams and monoidal actions in Lie-theoretic context

Volodymyr Mazorchuk

(Uppsala University)

XLII Workshop on Geometric Methods in Physics, Bialystok

Dynkin diagrams



Some classifications using Dynkin diagrams

Classifications on the nose:

- irreducible finite root systems,
- simple complex finite dimensional Lie algebras,
- simple algebraic groups,
- simply connected complex Lie groups which are simple modulo centers,
- simply connected compact Lie groups which are simple modulo centers.

Closely related classification:

- finite irreducible Weyl groups (e.g. types B and C give the same group).

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Simply laced Dynkin diagrams

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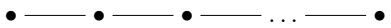
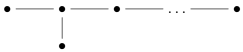
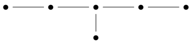


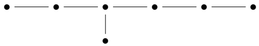
D_n :



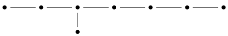
E_6 :



E_7 :



E_8 :



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Classifications on the nose:

- simply laced root systems,
- finite subgroups of $SL_2(\mathbb{C})$ (McKay),
- discrete subgroups of $SU(2)$ (McKay),
- underlying graphs for quivers of finite representation type (Gabriel),
- simple surface singularities (Arnol'd),
- simple finite graphs for which the spectral radius of the adjacency matrix is < 2 (Godsil, McKay),
- minimal conformal invariant theories (Cappelli, Itzykson, Zuber),
- simple transitive 2-representations of Soergel bimodules with middle apex in finite dihedral types (Mackaay, Tubbenhauer).

Closely related classification:

- "Finite subgroups" of $U_q(\mathfrak{sl}(2))$ (Kirillov-Ostrik) (the types D_{2n+1} and E_7 are missing).

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McKay correspondence

Let $\mathscr{C}$ be the monoidal category of finite dimensional $SL(2, \mathbb{C})$-modules.

Note: it is generated (in some weak sense) by the natural module $V := \mathbb{C}^2$, under the monoidal and additive structures.

Let G be a finite subgroup of $SL(2, \mathbb{C})$.

Then $\mathscr{C}$ acts on G-mod naturally (using restriction from $SL(2, \mathbb{C})$ to G).

Also, G-mod has finitely many indecomposables (=simples), say $X_1, X_2, \dots, X_k$.

Define the action matrix: $([V \otimes_{\mathbb{C}} X_i : X_j])_{i,j=1}^{k}$.

This is just a bookkeeping tool that records the multiplicities.

As it turns out, these matrices are exactly the adjacency matrices for extended Dynkin diagrams of type ADE.

Why? Essentially because $\dim V = 2$ is the Perron-Frobenius eigenvalue and hence the spectral radius of such a matrix.

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Extended simply laced Dynkin diagrams

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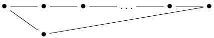
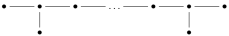
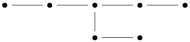


$\widetilde{D}_n$:



$\widetilde{E}_6$:



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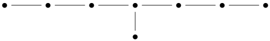

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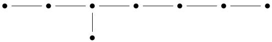

Note: These are in bijection with the usual simply laced Dynkin diagrams.

Finite rank $\mathscr{C}$-module categories

Recall: $\mathscr{C}$ be the monoidal category of finite dimensional $SL(2,\mathbb{C})$-modules.

If G is a finite subgroup of $SL(2,\mathbb{C})$, then G-mod has the following properties:

- it is an $\mathscr{C}$-module category;
- it is semi-simple (with finite dimensional morphism spaces);
- it has finitely many simples (up to isomorphism).

All $\mathscr{C}$-module categories with the above properties can be classified:

Theorem. (Etingof-Ostrik)

All $\mathscr{C}$-module categories with the above properties are classified by the following data (up to isomorphism):

- a finite set I;
- a collection of finite dimensional vector spaces V_{ij}, for $i, j \in I$;
- a collection of non-degenerate bilinear forms $B_{ij} : V_{ij} \otimes V_{ji} \to \mathbb{C}$ such that, for each $i \in I$, we have $\sum_j \mathrm{trace}(E_{ij}(E_{ji}^t)^{-1}) = -2$.

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If G is a finite subgroup of $SL(2,\mathbb{C})$, then G-mod has the following properties:

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Remark and Example

Remark. This theorem implies that there are a lot of such $\mathscr{C}$-module categories.

Example. Take $I = \{i\}$ to be a singleton and V_{ii} of dimension two.

Let our form have the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Non-degeneracy condition: $ad - bc \neq 0$.

Trace equation: $\frac{1}{ad-bc}(2ad - b^2 - c^2) = -2$.

Equivalently: $4ad = (b+c)^2$.

Put together: $4(ad - bc) = (b-c)^2 \neq 0$.

This has many non-equivalent solutions.

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Lie algebra setup

Let $\mathscr{D}$ be the monoidal category of finite dimensional $\mathfrak{sl}_2(\mathbb{C})$-modules.

Note: $\mathscr{C}$ and $\mathscr{D}$ are monoidally equivalent.

Question: What kind of $\mathscr{D}$-module categories appear naturally in (=can be constructed intrinsically using) $\mathfrak{sl}_2(\mathbb{C})$-Mod?

For example: start with an $\mathfrak{sl}_2(\mathbb{C})$-module N.

Consider $\mathscr{D}N := \{V \otimes_{\mathbb{C}} N \,:\, V \in \mathscr{D}\}$.

Take its additive closure $\mathrm{add}(\mathscr{D}N)$, i.e. add summands.

Then $\mathscr{D}$ acts on $\mathrm{add}(\mathscr{D}N)$ in the obvious way (by acting on the "V" part of the expression and using additivity).

So $\mathrm{add}(\mathscr{D}N)$ is a $\mathscr{D}$-module category.

If N is simple (not necessarily fin. dim.), then $\mathrm{add}(\mathscr{D}N)$ is "nice", for example, it has fin. dim. hom-spaces (but infinitely many indecomposables, in general).

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Combinatorics

Let $\mathscr{M}$ be a "nice" $\mathscr{D}$-module category with indecomposables $\{M_i \,:\, i \in I\}$.

Here "nice" means additive, idempotent split with finite dimensional morphism spaces.

Recall that $\mathscr{D}$ is generated, as a monoidal category, by the 2-dimensional natural $\mathfrak{sl}_2$-module $\mathbb{C}^2$.

Define the action matrix M as $([\mathbb{C}^2(M_j) : M_i])_{i,j\in I}$.

Note: this is an infinite matrix, in general.

This matrix captures the combinatorial shadow of the action.

As the entries are non-negative integers, we an visualize the matrix as an oriented graph, call it the action graph:

Vertices: elements of I.

We have $[\mathbb{C}^2(M_j) : M_i]$ oriented arrows from j to i, for $i, j \in I$.

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Example

Example. Take $N = \mathbb{C}$ to be the trivial $\mathfrak{sl}_2$-module.

Then $\mathscr{M} = \mathrm{add}(\mathscr{D}\, N) = \mathscr{D}$.

Indecomposables: $L_0, L_1, L_2, \ldots$ (simple fin.dim $\mathfrak{sl}_2$-modules), where L_i has dimension $i+1$.

Action: $\mathbb{C}^2 \otimes_{\mathbb{C}} L_0 = L_1$ and $\mathbb{C}^2 \otimes_{\mathbb{C}} L_i = L_{i+1} \oplus L_{i-1}$, for $i > 0$.

Action matrix: $\mathrm{M} = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots \\ 1 & 0 & 1 & 0 & \ldots \\ 0 & 1 & 0 & 1 & \ldots \\ 0 & 0 & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$.

Action graph: . . . .

Simplify to

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Let N be (any) simple $\mathfrak{sl}_2$-module.

Then combinatorics of a simple $\mathscr{D}$-invariant subquotient of $\mathrm{add}(\mathscr{D} \cdot N)$ is given by one of the following action graphs:

Note: These are four out of six infinite Dynkin diagrams.

The missing ones:

Remark. The D_∞ diagram is realizable in a similar way using $\mathfrak{sl}_2 \ltimes L_4$-modules.

Remark. The B_∞ diagram can only be realized via a "dual point of view" at the C_∞-realization (whose underlying category is not semi-simple).

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Recall: $\mathscr{D}$ is a semi-simple monoidal category (but it has infinitely many indecomposables=simples).

Observation 1. Simple infinite rank $\mathscr{D}$-module categories might have non semi-simple underlying categories (happens, e.g. sometimes in type A_∞^∞ and always in type C_∞, but never in types A_∞ and D_∞).

Observation 2. In type A_∞, the simple $\mathscr{D}$-module category is unique, up to equivalence.

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Observation 4. In type A_∞^∞, we construct a simple $\mathscr{D}$-module category for which the action of $\mathscr{D}$ never maps simples to projectives (in sharp contrast to the finite rank case).

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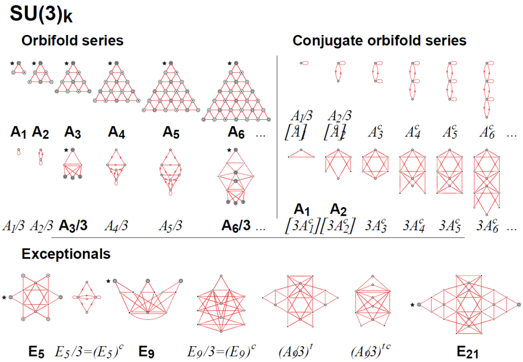
3D-generalizations?

A. Ocneanu, 2000: Classification of subgroups of quantum $SU(N)$. Contemp. Math., 294 American Mathematical Society, Providence, RI, 2002, 133–159.

The $SU(3)$-example from this paper:



Another example

J.-B. Zuber, 1998: Generalized Dynkin diagrams and root systems and their folding. Progr. Math., 160 Birkhäuser Boston, Inc., Boston, MA, 1998, 453–493.

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30 *J.-B. Zuber*

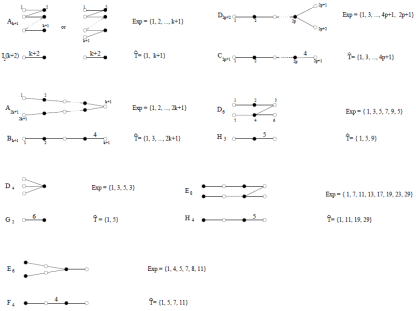

Fig. 6: The folding of *ADE* Dynkin diagrams. Classes T_i encompass vertices on the same vertical.

Let $\mathscr{B}$ be the monoidal category of finite dimensional $\mathfrak{sl}_3(\mathbb{C})$-modules.

Let N be a simple $\mathfrak{sl}_3(\mathbb{C})$-module.

Consider $\mathscr{B}\,N := \{V \otimes_{\mathbb{C}} N \,:\, V \in \mathscr{B}\}$.

Take its additive closure $\mathrm{add}(\mathscr{B}\,N)$.

Then $\mathrm{add}(\mathscr{B}\,N)$ is a $\mathscr{B}$-module category.

The category $\mathrm{add}(\mathscr{B}\,N)$ is additive, idempotent split, with finite dimensional morphism and countably many indecomposable objects.

Example. If N is finite dimensional, then $\mathrm{add}(\mathscr{B}\,N) = \mathscr{B}$.

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Very weak sense: every object of $\mathscr{B}$ is isomorphic to a summand of $(\mathbb{C}^3)^{\otimes k}$, for some k.

However, the class of $\mathbb{C}^3$ does not generate the split Grothendieck ring of $\mathscr{B}$.

In fact, the latter is generated by the classes of $\mathbb{C}^3$ and its dual $(\mathbb{C}^3)^*$.

So, combinatorics is completely determined by the combinatorics of the action of these two objects.

Just like for $\mathfrak{sl}_2$, we cen define the action matrices for $\mathbb{C}^3$ and $(\mathbb{C}^3)^*$ which bookkeep the multiplicities of the action on indecomposable objects.

And we can visualize those matrices as graphs.

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Just like for $\mathfrak{sl}_2$, we cen define the action matrices for $\mathbb{C}^3$ and $(\mathbb{C}^3)^*$ which bookkeep the multiplicities of the action on indecomposable objects.

And we can visualize those matrices as graphs.

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The combinatorics of the action of $\mathscr{B}$ on any simple subquotient of $\mathrm{add}(\mathscr{B}\,N)$ is given by one of the following eight graphs:

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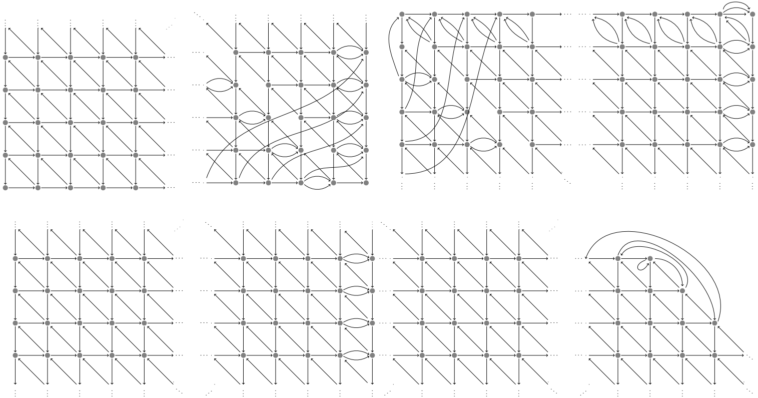

Remarks

Remark 1. It is natural to view these graphs as $3D$-analogues of infinite Dynkin diagrams.

Remark 2. The combinatorics of the action of $(\mathbb{C}^3)^*$ is described by similar eight graphs.

Remark 3. The regular $\mathscr{B}$-module category ${}_{\mathscr{B}}\mathscr{B}$ is unique, up to isomorphism, for its combinatorics.

Remark 4. If the graph does not have double oriented arrows (this holds for four out of eight graphs), then the underlying category of the corresponding $\mathscr{B}$-module category is semi-simple.

Remark 5. We expect that infinitely many pairwise non-equivalent simples $\mathscr{B}$-module categories appear, but we do not know how to prove that. In the case of $\mathfrak{sl}_2$ the corresponding result was established using very particular theorem of Dixmier about pairwise non-isomorphism of certain primitive quotients of $U(\mathfrak{sl}_2)$.

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Speculations on further research

Speculation 1. For any simple Lie algebra $\mathfrak{g}$ one can ask a similar question for the action of $\mathfrak{g}$-fdmod on $\mathrm{add}(\mathfrak{g}\text{-fdmod}\, N)$, where N is a simple $\mathfrak{g}$-module.

It is natural to expect that, for a fixed $\mathfrak{g}$, there are only finitely many corresponding combinatorial pictures. Not very clear how to prove this and, in particular, which discrete invariant(s) index(es) the answer.

Speculation 2. Similarly to the idea of the classical McKay correspondence, it would be interesting to understand the combinatorics of the action of $\mathfrak{g}$-fdmod on module categories arising from Lie subalgebras of $\mathfrak{g}$.

Speculation 3. It would be also interesting to understand the combinatorics of the action of $\mathfrak{g}$-fdmod on module categories arising from non semi-simple Lie algebras $\mathfrak{a}$ for which $\mathfrak{g}$ is the simple quotient.

Note that this idea was used to realize the type D_∞ Dynkin diagram for $\mathfrak{sl}_2$.

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References

Paper 1. V. M., Xiaoyu Zhu. Infinite rank module categories over finite dimensional $\mathfrak{sl}_2$-modules in Lie-algebraic context. Preprint arXiv:2405.19894.

Paper 2. V. M., Xiaoyu Zhu. Combinatorics of infinite rank module categories over finite dimensional $\mathfrak{sl}_3$-modules in Lie-algebraic context. Preprint arXiv:2501.00291.

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THANK YOU!!!

Check out: Uppsala Algebra on YouTube:
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