Stäckel systems, soliton hierarchies and Painlevé-type systems

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Lecture 1

Stäckel systems

Paul Stäckel

- Paul Gustav Samuel Stäckel (20 August 1862, Berlin – 12 December 1919, Heidelberg) was a German mathematician, active in the areas of differential geometry, number theory, and non-Euclidean geometry.
- Works:
  1. Über die Bewegung eines Punktes auf einer Fläche, 1885, Dissertation
  2. Die Integration der Hamilton-Jacobischen Differentialgleichung mittelst Separation der Variablen, 1891, Habilitation

The main message

The goal of lectures

The goal of these lectures is to present an overall (and recent) picture of relations between three major classes of integrable differential equations.

The following relations can be demonstrated

soliton hierarchies

autonomous constraints $\swarrow$ $\searrow$ non-autonomous constraints

Stäckel systems $\xrightarrow{\text{deformation}}$ *Painlevé-type systems*

Moreover, the diagram above is commutative (in a certain sense, explainded at the end of last lecture)

Poisson manifolds

- Consider a smooth manifold M. $C^\infty(M)$ denotes the set of all smooth real valued function on M.
- M is Poisson if it is equipped with a Poisson tensor π i.e. a (2,0)-tensor such that
  i) π antisymmetric $\pi^T = -\pi$
  ii) the bilinear mapping $\{\cdot,\cdot\}_\pi : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ given by

  $$\{f,g\}_\pi = (df, \pi dg)$$

  satisfies the Jacobi identity, i.e. for all $f, g, h \in C^\infty(M)$

  $$\{\{f,g\}_\pi, h\}_\pi + \{\{g,h\}_\pi, f\}_\pi + \{\{h,f\}_\pi, g\}_\pi = 0$$

  $(\cdot,\cdot)$ is the dual map between cotangent and tangent spaces
- Thus, $\{\cdot,\cdot\}_\pi$ turns $C^\infty(M)$ into a Lie algebra
- Note that the Leibniz rule $\{fg,h\}_\pi = f\{g,h\}_\pi + g\{f,h\}_\pi$ follows from $\{f,g\}_\pi = (df,\pi dg)$ as $d(fg) = fdg + gdf$.

Poisson manifolds

Theorem

(Darboux) For any Poisson operator π there exists (locally, i.e. in a neighbourhood of ever point) a coordinate system $(x_1, \dots, x_n, y_1, \dots, y_n, c_1, \dots, c_k)$ (Darboux coordinates, canonical coordinates) such that in this system π attains the form

$$\pi = \begin{pmatrix} 0 & I_n & 0 \\ -I_n & 0 & \vdots \\ 0 \dots & & 0_k \end{pmatrix}$$

In operator form

$$\pi = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}$$

The variables $(c_1, \dots, c_k)$ are called Casimir coordinates. For any $f \in C^\infty(M)$

$$\{f, g(c_1, \dots, c_k)\} = 0$$

Note: $\mathrm{corank}(\pi) = \mathrm{k}$.

Hamiltonian systems

- Vector fields of the form
$$X = \pi dh$$
where $h \in C^\infty(M)$, are called Hamiltonian vector fields on M.
- At each $x \in M$, π maps T_xM to T_x^*M (not necessarily bijectively)
- The system of ODE's of the form
$$\frac{dx}{dt} = \pi dh$$
(where $x = (x_1, \ldots, x_{\dim M})^T$ is a point on M) is called a Hamiltonian system on M generated by the Hamiltonian h.

Completely integrable systems

- Assume $\dim M = 2n$. The Hamiltonian system

$$\frac{dx}{dt} = \pi dh$$

is completely integrable (Liouville integrable, Arnold-Liouville integrable) if there exist n functionally independent Poisson-commuting functions h_i (i.e. such that $\{h_i, h_j\}_\pi = 0$ for all $i, j = 1, \ldots, n$) such that h is one of them, say $h = h_1$

- It is more proper to call $\{h_1, \ldots, h_n\}$ a completely integrable system (no need to distinguish h_1)
- The systems

$$\frac{dx}{dt_i} = \pi dh_i \equiv X_i, \; i = 1, \ldots, n$$

commute: $[X_i, X_j] = 0$ for $i, j = 1, \ldots, n$ (i.e. are Frobenius integrable)

Completely integrable systems

Theorem

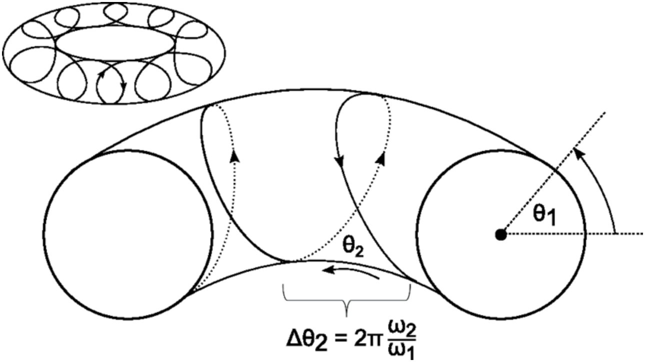
(Arnold-Liouville) if the Hamiltonian system $\frac{dx}{dt} = \pi dh$ *has n functionally idependent integrals of motion* h_i, $i = 1, \ldots, n$ $(h = h_1)$ *in involution (i.e.* $\{h_i, h_j\}_\pi = 0$*) then it is integrable in quadratures (Liouville) in the following sense (Arnold): if the invariant submanifold*

$$M_f = \{x \in M : h_i(x) = f_i, \ i = 1, \ldots n\}$$

is compact and connected then it is diffeomorphic to n-dimensional torus $T^n = \{(\varphi_1, \ldots, \varphi_n) \ mod\, 2\pi\}$ *and the phase flow (integral curves) of h is almost periodic i.e. in a suitably chosen coordinate system* $(I, \varphi) = (I_1, \ldots, I_n, \varphi_1, \ldots, \varphi_n)$ *(action-angle variables,* $I = I(h)$*) it attains the form*

$$\frac{dI_i}{dt} = 0, \ \frac{d\varphi_i}{dt} = \omega_i(I),$$

A trajectory of a completely integrable system



A non-complete list of completely integrable systems

- The 2 body problem (Kepler problem, Coulomb problem)
- The simple pendulum,
- The double pendulum
- Free rigid body
- Rigid body with a fixed point (= top - Euler top, Lagrange top, Kovalevskaya top),
- The harmonic oscillator
- The an-harmonic oscillator in 2 dim
- The motion of a particle in a central potential
- The motion on a sphere with a harmonic potential
- The geodesic motion on an ellipsoid (requires Jacobi elliptic coordinates)
- The geodesic motion on a surface of revolution
- The geodesic motion on a torus
- The geodesic motion on a quartic
- The geodesic motion on SO(3)

A non-complete list of completely integrable systems

- The Moser system,
- The Calogero systems
- The Calogero-Moser systems
- The Toda lattices (periodic, non-periodic, non-abelian),
- The Clebsh rigid body in an ideal fluid,
- The Garnier system,
- The Gaudin systems

Remark

It is not easy to determine whether a given Hamiltonian system is integrable. Some partial results exist (such us Stäckel theorem or complete criterium of separability for natural Hamiltonian systems (i.e. with Euclidean metric)) but there is no general criterium. It is therefore worth to start from the end of the journey and travel it backwards.

Separation relations on Poisson manifolds

(Sklyanin approach)

Separation relations

A system of n relations of the form

$$\varphi_i(x, a_1, \ldots, a_n) = 0, \quad i = 1, \ldots, n, \quad n \text{ arbitrary} \tag{1}$$

on a Poisson manifold (M, π), with $x \in M$ and a_i being real parameters, is called separations relations on M.

Solving (1) w.r.t. a_i yields $a_i = h_i(x)$ and thus n functions h_i on M.

Theorem

Suppose that $\{\varphi_i, \varphi_j\} \overset{\text{def}}{=} \pi(d\varphi_i, d\varphi_j) = 0$ for $i, j = 1, \ldots, n$ and for all values of a_r. Suppose also $\det\left(\frac{\partial \varphi_i}{\partial a_j}\right) \neq 0$. Then the functions h_i also Poisson commute: $\{h_i, h_j\} = 0$ for $i, j = 1, \ldots, n$.

Separation relations on Poisson manifolds

Proof

Differentiating separation relations we get

$$\frac{\partial \varphi_i}{\partial x_j} + \sum_{k=1}^{n} \frac{\partial \varphi_i}{\partial a_k} \frac{\partial h_k}{\partial x_j} = 0 \Rightarrow \frac{\partial h_k}{\partial x_j} = -\sum_{s=1}^{n} A_{ks} \frac{\partial \varphi_s}{\partial x_j}$$

Thus

$$\{h_i, h_j\} = \pi\,(dh_i, dh_j) = \sum_{s,r=1}^{n} A_{is} A_{jr}\,\{\varphi_s, \varphi_r\} = 0$$

So, we obtain a system of commuting Hamiltonian flows

$$\frac{dx}{dt_i} = \pi dh_i, \quad i = 1 \ldots . n$$

Smooth curves - source of separation relations

Take any smooth (preferrably algebraic) n-parameter curve in the plane

$$\varphi(\lambda, \mu, a_1, \ldots, a_n) = 0$$

Consider: M manifold, $\dim M = 2n$, $(\mu_1, \ldots, \mu_n, \lambda_1, \ldots, \lambda_n)$ - coordinates on M and the Poisson operator $\pi = \sum_{i=1}^{n} \frac{\partial}{\partial \lambda_i} \wedge \frac{\partial}{\partial \mu_i}$.
Take as separation relations n copies of the above curve with (λ_i, μ_i) in each copy:

$$\varphi(\lambda_i, \mu_i, a_1, \ldots, a_n) = 0, \qquad i = 1, \ldots, n \tag{2}$$

Then $\varphi_i \equiv \varphi(\lambda_i, \mu_i, a_1, \ldots, a_n)$ mutually Poisson-commute.

Smooth curves - a source of separation relations

Solving (2) w.r.t. a_i yields n functions h_i that mutually commute: $\{h_i, h_j\}_\pi = 0$

We obtain n commuting Hamiltonian systems

$$\frac{d}{dt_i}\begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} \frac{\partial h_i}{\partial \lambda} \\ \frac{\partial h_i}{\partial \mu} \end{pmatrix}, \quad i = 1, \ldots, n \tag{3}$$

They are all separable in the variables $(\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n)$.

Separability of the obtained systems

Substitute $\mu_i = \frac{\partial W_i(\lambda_i, a)}{\partial \lambda_i}$ into separation relations (2)

We obtain

$$\varphi\left(\lambda_i, \frac{\partial W_i(\lambda_i,a)}{\partial \lambda_i}, a_1, \ldots, a_n\right) = 0 \quad , i = 1, \ldots, n \quad (n \text{ decoupled ODE's})$$

If $W_i(\lambda_i, a)$ solves eq. i then $W(\lambda, a) = \sum_{i=1}^{n} W_i(\lambda_i, a)$ is a complete integral of

$$a_i = h_i\left(\lambda, \mu = \partial W(\lambda, a)/\partial \lambda\right) \qquad (\text{H-J for } h_i)$$

The canonical transformation $(\lambda, \mu) \overset{W(\lambda,a)}{\rightarrow} (b, a)$ linearizes all (3):

$$\frac{db_i}{dt_j} = \delta_{ij}, \quad \frac{da_i}{dt_j} = 0 \Rightarrow b_i = t_i + c_i, \quad a_i = const_i$$

(= Jacobi theorem)

Jacobi theorem

Theorem

Assume 1) h_i constitute integrable system 2) $W(\lambda_1, \ldots, \lambda_n, a_1, \ldots, a_n)$ simultaneously solves all the H-J equations

$$h_i\left(\lambda_1, \ldots, \lambda_n, \frac{\partial W(\lambda, a)}{\partial \lambda_1}, \ldots, \frac{\partial W(\lambda, a)}{\partial \lambda_n}\right) = a_i \tag{4}$$

Then the functions

$$b_i = \frac{\partial W(\lambda, a)}{\partial a_i} \tag{5}$$

satisfy $b_i = t_i + const_i$.

Note: (5) can be used to obtain the functions $\lambda_i = \lambda_i(a, b)$ (so called inverse Jacobi problem) i.e. half of the map $(\lambda, \mu) \to (b, a)$. Then

$$\mu_i(a, b) = \left.\frac{\partial W(\lambda, a)}{\partial \lambda_i}\right|_{\lambda=\lambda(a,b)}$$

Jacobi theorem

Proof

Differentiation of (4) w.r.t. a_k yields

$$\sum_{s=1}^{n} \frac{\partial h_i}{\partial \mu_s} \frac{\partial^2 W(\lambda, a)}{\partial a_k \partial \lambda_s} = \delta_{ik}$$

while differentiating b_k in (5) w.r. t. t_i we get

$$\frac{\partial b_k}{\partial t_i} = \sum_{s=1}^{n} \frac{\partial^2 W(\lambda, a)}{\partial a_k \partial \lambda_s} \frac{\partial \lambda_s}{\partial t_i}$$

But

$$\frac{\partial \lambda_s}{\partial t_i} = \frac{\partial h_i}{\partial \mu_s}$$

so that $\frac{\partial b_k}{\partial t_i} = \delta_{ik}$

Stäckel systems

Consider the following algebraic curve in the (λ, μ)-plane

$$\sigma(\lambda) + \sum_{r=1}^{n} h_r \lambda^{n-r} = f(\lambda)\mu^2 \qquad (6)$$

where f and σ are arbitrary Laurent polynomials in λ.

Taking n copies of (6) at points $(\lambda, \mu) = (\lambda_i, \mu_i)$, $i = 1, \ldots, n$, we obtain a system of n linear equations (separation relations) for h_r :

$$\sigma(\lambda_i) + \sum_{r=1}^{n} h_r \lambda_i^{n-r} = f(\lambda_i)\mu_i^2, \qquad i = 1, \ldots, n,$$

or, in matrix form,

$$\begin{pmatrix} \sigma(\lambda_1) \\ \vdots \\ \sigma(\lambda_n) \end{pmatrix} + \begin{pmatrix} \lambda_1^{n-1} & \lambda_1^{n-2} & \ldots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_n^{n-1} & \lambda_n^{n-2} & \ldots & 1 \end{pmatrix} \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} = \begin{pmatrix} f(\lambda_1)\mu_1^2 \\ \vdots \\ f(\lambda_n)\mu_n^2 \end{pmatrix}$$

The matrix above is a Stäckel matrix (of Vandermonde type).

Stäckel systems

Solving this system yields n functions (Hamiltonians)

$$h_r = E_r + V_r^{(\sigma)}, \quad r = 1, \ldots, n \tag{7}$$

on a $2n$-dimensional manifold (phase space) $M = \mathbb{R}^{2n}$ parametrized by the coordinates $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ with $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)^T$ and $\boldsymbol{\mu} = (\mu_1, \ldots \mu_n)^T$.

In geodesic part

$$E_r = \boldsymbol{\mu}^T A_r G \boldsymbol{\mu}, \quad r = 1, \ldots, n,$$

G is a contravariant metric tensor on the configurational space Q (such that $M = T^*Q$), K_r are $(1,1)$-Killing tensors of G while $V_r^{(\sigma)}$ are separable potentials on Q.

In the separation coordinates λ_i on Q, the geometric objects G, K and $V_r^{(\sigma)}$ are explicitly given by

$$G^{ij} = \frac{f(\lambda_i)}{\Delta_i}\delta^{ij}, \ (A_r)^i_j = -\frac{\partial \rho_r}{\partial \lambda_i}\delta^i_j, \quad V_r^{(\sigma)} = \sum_{j=1}^n \frac{\partial \rho_r}{\partial \lambda_j}\frac{\sigma(\lambda_j)}{\Delta_j}$$

Stäckel systems

where $\Delta_j = \prod_{k\neq j}(\lambda_j - \lambda_k)$ and $\rho_r = (-1)^r s_r$ where s_r are elementary symmetric polynomials in n variables λ_i.

The curvature of G

- Thus:

$$G = \operatorname{diag}\left(\frac{f(\lambda_1)}{\Delta_1}, \ldots, \frac{f(\lambda_n)}{\Delta_n}\right), \quad \Delta_i = \prod_{j\neq i}(\lambda_i - \lambda_j)$$

- If f is a polynomial of degree $\leq n$ then G is flat
- if f is a polynomial of degree $n+1$ then G has constant but non-zero curvature

Stäckel systems in Viète coordinates

Consider the (point) transformation to the so called Viète (canonical) coordinates:

$$q_i = (-1)^i s_i, \qquad p_i = -\sum_{k=1}^{n} \frac{\lambda_k^{n-i}\mu_k}{\Delta_k}, \qquad i = 1, \ldots, n.$$

where s_k are the elementary symmetric polynomials in λ_i and $\Delta_i = \prod_{j\neq i}(\lambda_i - \lambda_j)$.

Let $\boldsymbol{p} = (p_1, \ldots, p_n)^T$ and $\boldsymbol{q} = (q_1, \ldots, q_n)^T$.

The Stäckel Hamiltonians take in Viète coordinates the following form:

$$h_k = \frac{1}{2}\boldsymbol{p}^T A_k G \boldsymbol{p} + V_k, \qquad k = 1, \ldots, n,$$

and the respective Hamiltonian evolution equations are

$$\boldsymbol{q}_{t_k} = \frac{\partial h_k}{\partial \boldsymbol{p}}, \qquad \boldsymbol{p}_{t_k} = -\frac{\partial h_k}{\partial \boldsymbol{q}}, \qquad k = 1, \ldots, n$$

Stäckel systems

Explicitly:

$$G_0^{ij} = q_{i+j-n-1}, \qquad (K_r)^i_j = \begin{cases} q_{i-j+r-1}, & i \leq j \text{ and } r \leq j \\ -q_{i-j+r-1}, & i > j \text{ and } r > j \\ 0, & \text{otherwise} \end{cases}$$

where G_0 corresponds to $f(\lambda) \equiv 1$; we set $q_0 = 1$, $q_k = 0$ for $k < 0$ or $k > n$.

Moreover,

$$G_f = f(L)G_0, \qquad L = \begin{pmatrix} -q_1 & 1 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ \vdots & 0 & 0 & 1 \\ -q_n & 0 & 0 & 0 \end{pmatrix},$$

where L i so called *special conformal killing tensor*.

Stäckel systems

Elementary separable potentials $V_r^{(\alpha)}$ for $\sigma(\lambda)=\lambda^\alpha$ can be explicitly constructed by the recursion formula

$$V^{(\alpha)}=R^\alpha V^{(0)},\qquad V^{(\alpha)}=(V_1^{(\alpha)},\dots,V_n^{(\alpha)})^T,\qquad R=\begin{pmatrix}-q_1 & 1 & 0 & 0\\ \vdots & 0 & \ddots & 0\\ \vdots & 0 & 0 & 1\\ -q_n & 0 & 0 & 0\end{pmatrix},$$

with $V^{(0)}=(0,\dots,0,-1)^T$.

The first n basic separable potentials are trivial

$$V_k^{(\alpha)}=-\delta_{k,n-\alpha},\quad \alpha=0,\dots,n-1.$$

The first nontrivial positive and negative potentials are

$$V^{(n)}=(q_1,\dots,q_n)^T,\qquad V^{(-1)}=\left(\frac{1}{q_n},\dots,\frac{q_{n-1}}{q_n}\right)^T$$

and higher positive and negative potentials are more complicated polynomials and rational functions in all q_i.

Stäckel systems

By construction, all the Hamiltonian functions h_r are in involution

$$\{h_r, h_s\} \equiv \pi(dh_r, dh_s) = 0, \qquad r, s = 1, ..., n$$

with respect to the Poisson bracket

$$\pi = \sum_{i=1}^{n} \frac{\partial}{\partial \lambda_i} \wedge \frac{\partial}{\partial \mu_i}$$

on M. They also separate in coordinates $(\lambda_i, \mu_i)_{i=1,\ldots,n}$.

The Hamiltonians (7) constitute a Liouville integrable system on M:

$$\frac{d\xi}{dt_r} = \pi dh_r, \qquad r = 1, \ldots, n.$$

that is called a Stäckel system.

Stäckel systems

Thus, Stäckel system is a set of n *autonomous* Hamiltonian evolution equations on $2n$-dimensional phase M

$$\frac{d\xi}{dt_r} = X_r(\xi) = \pi dh_r(\xi), \quad r = 1, \ldots, n \tag{8}$$

where $h_r(\xi)$ are Hamiltonian functions, $\xi = (q_1, ..., q_n, p_1, ..., p_n)^T$ denote points on M and π is a canonical Poisson tensor.

The system (8) is Frobenius integrable, i.e. there exist a common, unique (local) solution $\xi(t_1, \ldots, t_n, \xi_0)$ through each point $\xi_0 \in M$, depending in general on all the evolution parameters t_k.

The necessary and sufficient condition for Frobenius integrability of autonomous system reads

$$[X_s, X_r] = 0 \quad \text{or} \quad \{\mathrm{h_r}, \mathrm{h_s}\}_\pi = 0, \quad \mathrm{r}, \mathrm{s} = 1, \ldots, \mathrm{n}$$

Isospectral Lax representation for Stäckel systems

Assume now $f(\lambda) = \lambda^m$, $m \in \mathbb{Z}$. Consider the system of linear equations

$$L(\lambda;\xi)\Psi = \lambda^m \mu \Psi, \quad \frac{d\Psi}{dt_k} = U_k(\lambda;\xi)\Psi, \quad \frac{d\lambda}{dt_k} = \frac{d\mu}{dt_k} = 0, \qquad k = 1, \dots, n,$$

where $L(\lambda;\xi)$ and $U_k(\lambda;\xi)$ are 2×2 matrices which depend rationally on the spectral parameter λ. As usual, $\xi \in M$

Specifically

$$L(\lambda) = \begin{pmatrix} \mathfrak{v} & \mathfrak{u} \\ \mathfrak{w} & -\mathfrak{v} \end{pmatrix}$$

with the elements defined by

$$\mathfrak{u} = \lambda^n + \sum_{k=1}^{n} q_k \lambda^{n-k}, \qquad \mathfrak{v} = -\frac{1}{2} \sum_{k=1}^{n} \Big[\sum_{i=1}^{n} (G_m)^{ki} p_i \Big] \lambda^{n-k}$$

and

$$\mathfrak{w} := \frac{1}{\mathfrak{u}} \Big[\lambda^m \Big(\sigma(\lambda) + \sum_{k=1}^{n} H_k \lambda^{n-k} \Big) - \mathfrak{v}^2 \Big].$$

Isospectral Lax representation for Stäckel systems

while the auxiliary matrices $U_k(\lambda)$ are

$$U_k(\lambda) := \left[\frac{\mathfrak{u}_k L}{\mathfrak{u}}\right]_+ = \begin{pmatrix} \mathfrak{v}_k & \mathfrak{u}_k \\ \mathfrak{w}_k & -\mathfrak{v}_k \end{pmatrix}, \qquad k = 1, \ldots, n,$$

where

$$\mathfrak{u}_k := \left[\frac{\mathfrak{u}}{\lambda^{n-k+1}}\right]_+ \equiv \lambda^{k-1} + \sum_{i=1}^{k-1} q_k \lambda^{k-i-1}, \quad \mathfrak{v}_k := \ldots, \quad \mathfrak{w}_k := \ldots$$

Equivalent form for $U_k(\lambda)$:

$$U_k(\lambda) = \left[\frac{B_k(\lambda)}{u(\lambda)}\right]_+, \quad B_k(\lambda) = \frac{1}{2}\left[\frac{u(\lambda)}{\lambda^{n-k+1}}\right]_+ L(\lambda).$$

Here $[\cdot]_+$ is the projection on the part consisting of non-negative degree terms in the expansion into its Laurent series at ∞.

Isospectral Lax representation for Stäckel systems

The compatibility conditions take the form of so called *isospectral Lax representations*

$$\frac{dL(\lambda;\xi)}{dt_k} \equiv L'[X_k] = [U_k(\lambda;\xi), L(\lambda;\xi)], \qquad k = 1, \ldots, n,$$

which are differential consequences of the Stäckel system (8).

The characteristic equation of L is equivalent with the spectral curve (6):

$$0 = \det[L - f(\lambda)\mu I] = -f(\lambda)\left[\sigma(\lambda) + \sum_{r=1}^{n} h_r \lambda^{n-r} - f(\lambda)\mu^2\right].$$

(remind: $f(\lambda) = \lambda^m$, $m \in \mathbb{Z}$)

Example: Hénon-Heiles system is a Stäckel system

Consider Liouville integrable Hénon-Heiles system on $M = \mathbb{R}^4$, generated by two Hamiltonian functions in involution

$$h_1 = E_1 + V_1(x) = \tfrac{1}{2}p_1^2 + \tfrac{1}{2}p_2^2 + x_1^3 + \tfrac{1}{2}x_1x_2^2,$$
$$h_2 = E_2 + V_2(x) = \tfrac{1}{2}x_2p_1p_2 - \tfrac{1}{2}x_1p_2^2 + \tfrac{1}{16}x_2^4 + \tfrac{1}{4}x_1^2x_2^2$$

written in Cartesian coordinates (x_1, x_2) and conjugate momenta (p_1, p_2).

Its separation curve has the form

$$h_1\lambda + h_2 = \tfrac{1}{2}\lambda\mu^2 + \lambda^4,$$

i.e. it is the case of $n = 2$, $m = 1$ and $V(x) = -V^{(4)}(x)$.

The point transformation between Cartesian, Viète and separation coordinates is as follows

$$\lambda_1 + \lambda_2 = -q_1 = x_1, \quad \lambda_1\lambda_2 = q_2 = -\frac{1}{4}x_2^2.$$

Example: Hénon-Heiles system

The Hénon-Heiles Hamiltonian is h_1, so for the canonical form of the Poisson tensor $\{x_i, p_j\}_\pi = \delta_{ij}$, the related autonomous evolution equations are

$$(x_1)_{t_1} = p_1, \qquad (x_2)_{t_1} = p_2,$$

$$(p_1)_{t_1} = -3x_1^2 - \tfrac{1}{2}x_2^2, \qquad (p_1)_{t_1} = -x_1x_2.$$

Here h_2 is the first integral so its Hamiltonian flow equations

$$(x_1)_{t_2} = \frac{1}{2}x_2p_2, \qquad (x_2)_{t_2} = \frac{1}{2}x_2p_1 - x_1p_2$$

$$(p_1)_{t_2} = \frac{1}{2}p_2^2 - \frac{1}{2}x_1x_2^2, \qquad (p_2)_{t_2} = -\frac{1}{2}p_1p_2 - \frac{1}{4}x_2^3 - \frac{1}{2}x_1^2x_2$$

represent the symmetry of the Hénon-Heiles.

Example: Hénon-Heiles system

This Stäckel system has the Lax representation:

$$L(\lambda)=\begin{pmatrix} p_1\lambda+\frac{1}{2}x_2p_2 & \lambda^2-x_1\lambda-\frac{1}{4}x_2^2 \\ -2\lambda^3-2x_1\lambda^2-\left(2x_1^2+\frac{1}{2}x_2^2\right)\lambda+p_2^2 & -p_1\lambda-\frac{1}{2}x_2p_2 \end{pmatrix},$$

$$U_1(\lambda)=\begin{pmatrix} 0 & \frac{1}{2} \\ -\lambda-2x_1 & 0 \end{pmatrix},\quad U_2(\lambda)=\begin{pmatrix} \frac{1}{2}p_1 & \frac{1}{2}\lambda-\frac{1}{2}x_1 \\ -\lambda^2-x_1\lambda-x_1^2-\frac{1}{2}x_2^2 & -\frac{1}{2}p_1 \end{pmatrix}.$$

Lecture 2

Deforming Stäckel systems to Painlevé-type systems

Frobenius integrable systems

Definition

A set of n non-autonomous dynamical systems on a smooth manifold $\mathcal{M}$

$$\frac{d\xi}{dt_i} = Y_i(\xi, t_1, \ldots, t_n), \qquad i = 1, \ldots, n, \quad \xi \in \mathcal{M} \tag{9}$$

is Frobenius integrable (compatible, satisfies zero-curvature condition) if

$$\frac{\partial Y_i}{\partial t_j} - \frac{\partial Y_j}{\partial t_i} - [Y_i, Y_j] = 0, \text{ for all } i, j = 1, \ldots, n$$

- The Frobenius condition means simply that $\frac{d}{dt_j} Y_i = \frac{d}{dt_i} Y_j$ so it is a necessary and sufficient condition for (9) to possess a common, multi-time solution $\xi = \xi(t_1, \ldots, t_n, \xi_0)$ through each point $\xi_0 \in \mathcal{M}$.
- For time-independent fields $Y_i(\xi)$ this simplifies to $[Y_i, Y_j] = 0$.
- There is no assumed relation between n and $\dim \mathcal{M}$

Frobenius integrable Hamiltonian systems

- Suppose π is a Poisson tensor on $\mathcal{M}$. If $Y_i = \pi dH_i$ for some Hamiltonian functions H_i depending explicitly, in general, on all times t_k: $H_i = H_i(\xi, t_1, \ldots, t_n)$, then the Frobenius condition is equivalent to

$$\frac{\partial H_i}{\partial t_j} - \frac{\partial H_j}{\partial t_i} + \{H_i, H_j\} = f_{ij}(t_1, \ldots, t_n), \quad i, j = 1, \ldots, n$$

  where $f_{ij}(t_1, \ldots, t_n)$ are some functions of parameters t_i only.
- Notation: $\{H_i, H_j\} = \langle dH_i, \pi dH_j \rangle$ so that

$$[Y_i, Y_j] = [\pi dH_i, \pi dH_j] = -\pi d \{H_i, H_j\}$$

Painlevé equations

Linear differential equations

What can be said about solutions of linear ODEs?
1. They are single-valued.
2. All singular points are fixed, i.e. do not depend on initial condition (constants of integration).
As a result **solutions of linear ODE's can be a source of new functions**.

Consider the second order linear equation

$$\frac{d^2w}{dz^2} + p(z)\frac{dw}{dz} + q(z)w = 0.$$

Fuchs' theorem: w can only be singular (non-analytic) at points where $p(z)$ and $q(z)$ are singular. Such singularities are called **fixed** because their positions are determined a priori, before solving the equations, and their location is common throughout the space of all possible solutions.

Painlevé equations

Example: Bessel's equation

Bessel's equation

$$\frac{d^2 w}{dz^2} + \frac{1}{z}\frac{dw}{dz} + \left(1 - \frac{\nu^2}{z^2}\right) w = 0.$$

Solutions (Bessel functions) only have singularities at 0 and ∞.

Again: such singularities are said to be **fixed** since they depend upon the equation and not the particular solution. Hence, solutions of particular linear ODE's can define special functions that play an important role in mathematical physics.

For example: Airy functions, Bessel functions, parabolic cylinder functions, Whittaker functions, hypergeometric functions.

Painlevé equations

For nonlinear equations the situation is drastically different.

Example: Riccati equation

Riccati equation $w'(z) + w^2(z) = 0$ has the general solution

$$u(z) = \frac{1}{z - z_0}, \quad z_0 = -\frac{1}{u(0)}$$

Thus, the location of singularity **moves** with initial conditions. Such singularities are called **movable**.

Painlevé equations

Consider second order nonlinear equation

$$\frac{d^2w}{dz^2} = F\left(\frac{dw}{dz}, w, z\right) \tag{10}$$

where F is rational in $w,\ w'$ and locally analytic in z.

Critical point is a point where multi-valuedness can occur. For linear equations (10) only fixed singularities can occur, independent of initial condition. For nonlinear equations (10) there appear movable singularities, i.e. depending on initial condition.

Examples of solutions with movable critical point

$$\begin{aligned} w(z) &= \sqrt{z - z_0} && \text{algebraic branch point,} \\ w(z) &= \ln(z - z_0) && \text{logarithmic branch point,} \\ w(z) &= \tan[\ln(z - z_0)] && \text{multi-valued essential singularity.} \end{aligned}$$

Painlevé equations

Picard (1887) posed the problem of determining which ODE's (10) have only single-valuedness solutions with no movable critical points, thus allowing movable poles, movable single-valued essential singularities and all fixed singularities. This is now known as the *Painlevé property*.

Painlevé property

ODE satisfies Painlevé property if all solutions are single-valued about all movable singularities, which means that the only movable singularities are poles.

Thus, the solutions of the Painlevé ODE's are 'regular' single-valued functions around movable poles, and as such are good candidates that define new special (*transcendental*) functions, like it is in the case of linear equations.

Painlevé and Gambier (1893-1906) have found 50 canonical types of second order equations (10), whose solutions have no movable critical points. 44 of these are integrable in terms of previously known classical special functions, such as elliptic functions, trigonometric functions etc. or can be linearized.

Painlevé transcendents

Painlevé, Gambier, Fuchs (1900-1910): (up to equivalence classes under Möbius transformations), there are only six new ODE's with Painlevé property. Their general solutions are higher transcendental functions. Obs: $w = y$, $z = t$ below.

$$P_I : \quad \frac{d^2y}{dt^2} = 6y^2 + t$$

$$P_{II} : \quad \frac{d^2y}{dt^2} = 2y^3 + ty + \alpha$$

$$P_{III} : \quad \frac{d^2y}{dt^2} = \frac{1}{y}\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} + \frac{1}{t}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y}$$

Painlevé transcendents

$$P_{IV}: \quad \frac{d^2y}{dt^2} = \frac{1}{2y}\left(\frac{dy}{dt}\right)^2 + \tfrac{3}{2}y^3 + 4ty^2 + 2(t^2-\alpha)y + \frac{\beta}{y}$$

$$P_{V}: \quad \frac{d^2y}{dt^2} = \left(\frac{1}{2y} + \frac{1}{y-1}\right)\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} + \frac{(y-1)^2}{t^2}\left(\alpha y + \frac{\beta}{y}\right) + \gamma\frac{y}{t} + \delta\frac{y(y+1)}{y-1}$$

$$P_{VI}: \quad \frac{d^2y}{dt^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt} + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left\{\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right\}$$

α, β, γ and δ are arbitrary parameters.

Observation

All (completely) integrable ODE's appear to have the Painlevé property.

Painlevé transcendents

- All above Painlevé equations are Hamiltonian and have the so-called isomonodromy representation.

Isomonodromic (Lax) representation

$$\partial_\lambda \psi = L(t,\lambda)\psi, \quad \psi_t = V(t,\lambda)\psi \qquad \Longleftrightarrow \qquad L_t = [V, L] + V_\lambda$$

For instance, if $p = y',\ q = y$, then

$$\dot{q} = \frac{\partial H}{\partial p},\ \dot{p} = -\frac{\partial H}{\partial q} \qquad \Longleftrightarrow \qquad P_I :\ y'' = 6y^2 + t\,,$$

where $H = \frac{1}{2}p^2 - 2q^3 - tq$ and

$$L = \begin{pmatrix} -p & \lambda^2 + q\lambda + \frac{1}{2}t \\ 4\lambda - 4q & p \end{pmatrix}, \quad V = \begin{pmatrix} 0 & \frac{1}{2}\lambda + q \\ 2 & 0 \end{pmatrix}.$$

Painlevé-type systems

Definition (Painlevé-type systems)

These are non-autonomous, finite dimensional Hamiltonian systems that are i) Frobenius integrable and ii) have isomonodromy Lax representations.

Thus, a Painlevé-type system is a set of n non-autonomous Hamiltonian evolution equations on a $2n$-dimensional phase M

$$\frac{d\xi}{dt_r} = Y_{h_r}(\xi, t) = \pi dh_r(\xi, t), \quad r = 1, \ldots, n \tag{11}$$

where $t = (t_1, ..., t_n)$ and $\xi = (q_1, ..., q_n, p_1, ..., p_n)^T$ that are Frobenius integrable:

$$[Y_s, Y_r] + \frac{\partial Y_r}{\partial t_s} - \frac{\partial Y_s}{\partial t_r} = 0, \quad r, s = 1, \ldots, n,$$

or, on the Hamiltonian level,

$$\{h_r, h_s\} + \frac{\partial h_s}{\partial t_r} - \frac{\partial h_r}{\partial t_s} = 0, \quad r, s = 1, \ldots, n$$

Painlevé-type systems

AND, moreover, such that there exists a related a system of linear equations

$$\lambda^s \frac{\partial \Psi}{\partial \lambda} = L(\lambda; \xi, t)\Psi, \quad \frac{d\Psi}{dt_r} = V_k(\lambda; \xi, t)\Psi, \quad \frac{d\lambda}{dt_k} = 0, \quad r = 1, \ldots, n,$$

such that its compatibility conditions

$$\frac{dL(\lambda; \xi, t)}{dt_r} = [V(\lambda; \xi, t), L(\lambda; \xi, t)] + \lambda^s \frac{\partial V_r(\lambda; \xi, t)}{\partial \lambda}, \qquad r = 1, \ldots, n,$$

are differential consequences of (11).

Isomonodromic Lax representation

The equations above are called the isomonodromic Lax representation of (11).

Thus, Painlevé-type systems are natural generalizations of Liouville integrable systems to the non-autonomous case.

Painlevé-type systems are deformations of Stäckel systems

Nowadays we have full control over Stäckel systems. We know how to construct them from separation relations together with their Lax representations for arbitrary n. On the contrary, very little is known about the Painlevé type systems for $n > 1$.

A significant progress in construction of new multi-component Painlevé equations took place since the modern theory of nonlinear integrable PDE's has been born (the so-called *soliton theory*). The Painlevé equations are constructed under particular similarity reductions of soliton PDE's hierarchies.

In this lecture I will present an alternative way of construction of already known and new Painlevé-type ODE's for $n > 1$ by an appropriate deformations of Stäckel systems:

$$\text{Stäckel system} \overset{\text{deformation}}{\longrightarrow} \text{Painlevé-type system}$$

The method consists of few steps which we present in some detail below.

Deformation of Stäckel systems to Painlevé-type systems

The construction is presented in our articles:

1. Błaszak, K. Marciniak, A. Sergyeyev, *Deforming Lie algebras to Frobenius integrable non-autonomous Hamiltonian systems*, Rep. Math, Phys. **87** (2021) 249-263
2. M. Błaszak, K. Marciniak, Z. Domański, *Systematic construction of non-autonomous Hamiltonian equations of Painlevé-type. I. Frobenius integrability,* Stud. Appl. Math. **148** (2022) 1208
3. M. Błaszak, Z. Domański, K. Marciniak, *Systematic construction of non-autonomous Hamiltonian equations of Painlevé-type. II. Isomonodromic Lax representation*, Stud. Appl. Math. **149** (2022) 364
4. M. Błaszak, Z. Domański, K. Marciniak, *Systematic construction of non-autonomous Hamiltonian equations of Painlevé-type. III. Quantization* , Stud. Appl. Math. **149** (2022) 416

Deformation of geodesic flows

Consider the following separation curve

$$\sum_{r=1}^{n} \lambda^{n-r} E_{m,r} = \tfrac{1}{2}\lambda^m \mu^2$$

where $f(\lambda) = \lambda^m$ and $m \in \{0, \dots, n+1\}$. It yields a geodesic Stäckel system (i.e. without potentials).

Now let us deform geodesic Hamiltonians $E_{r,m} = \frac{1}{2}\boldsymbol{p}^T K_r G_m \boldsymbol{p}$

$$\begin{aligned}\mathcal{E}_{1,m} &= E_{1,m} = \tfrac{1}{2}\boldsymbol{p}^T G_m \boldsymbol{p}, \\ \mathcal{E}_{r,m} &= E_{r,m} + W_{r.m} = \tfrac{1}{2}\boldsymbol{p}^T G_m K_r \boldsymbol{p} + \boldsymbol{p}^T Z_{r,m}(\boldsymbol{q}) \qquad r = 2, ..., n,\end{aligned}$$

where the additional terms $W_{1,m} = 0$, $W_{r,m} = \boldsymbol{p}^T Z_{r,mr} = \sum_{i=1}^{n} Z^i_{r,m} p_i$ contain components of vector fields $Z_{r,m} = Z^i_{r,m}\frac{\partial}{\partial q_i}$ on Q which are particular Killing vectors of metric G_m. Thus

$$\{E_{1,m}, W_{r,m}\} = 0, \qquad r = 2, \dots, n.$$

Deformation of geodesic flows

Our demand concerning the choice of Killing vectors is that pseudo-geodesic Hamiltonians $\mathcal{E}_{r,m}$ constitute a Lie algebra.

Theorem

For the metric G_m the following set of functions $W_{r,m}$ turns $\mathcal{E}_{r,m}$ into a (nilpotetnt) Lie algebra: (in Viéte coordinates)

$$W_{r,m} = \sum_{k=n-m-r+2}^{n-m} (n+1-m-k) q_{m+r-n-2+k}\, p_k, \quad r \in \{2, \ldots, n-m+1\} \equiv I_1^m$$

and

$$W_{r,m} = -\sum_{k=n-m+2}^{2n-m+2-r} (n+1-m-k) q_{m+r-n-2+k}\, p_k, \quad r \in \{n-m+2, \ldots, n\} \equiv I_2^m.$$

Deformation of geodesic flows

Example

For $n = 3$ linear in momenta terms $W_{r,m}$ are as follows

$$\begin{aligned} m = 0: &\quad W_{2,0} = p_3, \quad W_{3,0} = q_1 p_3 + 2p_2 \\ m = 1: &\quad W_{2,1} = p_2, \quad W_{3,1} = q_1 p_2 + 2p_1 \\ m = 2: &\quad W_{2,2} = p_1, \quad W_{3,2} = q_3 p_3 \\ m = 3: &\quad W_{2,3} = q_2 p_2 + 2q_3 p_3, \quad W_{3,3} = q_3 p_2 \\ m = 4: &\quad W_{2,4} = q_2 p_1 + 2q_3 p_2, \quad W_{3,4} = q_3 p_1 \end{aligned}$$

The quasi-geodesic Hamiltonians $\mathcal{E}_{r,m}$ span a Lie algebra $\mathfrak{g}_m$ with the following commutation relations:

$$\{\mathcal{E}_{m,1}, \mathcal{E}_{m,r}\} = 0, \quad r = 2, \ldots, n,$$

and

$$\{\mathcal{E}_{m,r}, \mathcal{E}_{m,s}\} = \begin{cases} 0, & \text{for } r \in I_1^m \text{ and } s \in I_2^m, \\ (s-r)\mathcal{E}_{m,r+s-(n-m+2)}, & \text{for } r, s \in I_1^m, \\ -(s-r)\mathcal{E}_{m,r+s-(n-m+2)}, & \text{for } r, s \in I_2^m, \end{cases} \tag{12}$$

We use the convention that $\mathcal{E}_{m,r} = 0$ for $r \leq 0$ or $r > n$.

Deformation of geodesic flows

As the Hamiltonians $\mathcal{E}_{m,r}$ in (12) do not commute, they do not constitute a Liouville integrable system. In particular, there is no reason to expect that they will possess common, multi-time solutions for any initial data ξ_0. In (Błaszak, Marciniak, Sergyeyev 2017) we found polynomial-in-times deformations $H_{r,m}(t_1, \ldots, t_{r-1})$ of the pseudo-geodesic Hamiltonians $\mathcal{E}_{r,m}$ such that the Hamiltonians $H_{r,m}$ satisfy the Frobenius integrability condition.

$$
\begin{aligned}
H_i \quad = \quad & \mathcal{E}_i - \sum_{r_1=1}^{i-1} \left(\mathrm{ad}_{\mathcal{E}_{r_1}} \mathcal{E}_i\right) t_{r_1} + \sum_{r_1=1}^{i-1} \sum_{r_2=r_1}^{i-1} \alpha_{ir_1 r_2} \left(\mathrm{ad}_{\mathcal{E}_{r_2}} \mathrm{ad}_{\mathcal{E}_{r_1}} \mathcal{E}_i\right) t_{r_1} t_{r_2} \\
& - \sum_{r_1=1}^{i-1} \sum_{r_2=r_1}^{i-1} \sum_{r_3=r_2}^{i-1} \alpha_{ir_1 r_2 r_3} \left(\mathrm{ad}_{\mathcal{E}_{r_3}} \mathrm{ad}_{\mathcal{E}_{r_2}} \mathrm{ad}_{\mathcal{E}_{r_1}} \mathcal{E}_i\right) t_{r_1} t_{r_2} t_{r_3} + \cdots ,
\end{aligned}
\tag{13}
$$

and the real constants $\alpha_{ir_1 \ldots r_k}$ can be uniquely determined from the Frobenius integrability condition and where $\mathrm{ad}_{\mathcal{E}_i} \mathcal{E}_j = \{\mathcal{E}_i, \mathcal{E}_j\}$. Due to the nilpotency of the algebra, the expressions on the right hand side of (13) terminate.

Deformation of geodesic flows

The algebra $\mathfrak{g}$ has an Abelian subalgebra

$$\mathfrak{a} = \operatorname{span}\{\mathcal{E}_1, \ldots, \mathcal{E}_{\kappa_1}, \mathcal{E}_{n-\kappa_2+1}, \ldots, \mathcal{E}_n\}$$

where

$$\kappa_1 = \left[\frac{n+3-m}{2}\right], \qquad \kappa_2 = \left[\frac{m}{2}\right].$$

(so that $\mathfrak{a}$ depends on m as well). Note that $\mathfrak{g} = \mathfrak{a}$ precisely when $\kappa_1 + \kappa_2 = n$ as $\dim \mathfrak{a} = \kappa_1 + \kappa_2$.

for $r \in I_1^m$

$$H_r = \mathcal{E}_r, \quad \text{for } r = 1, \ldots, \kappa_1,$$

$$H_r = \sum_{j=1}^{r} \zeta_{r,j}(t_1, \ldots, t_{r-1})\mathcal{E}_j, \quad \zeta_{r,r} = 1, \quad \text{for } r = \kappa_1 + 1, \ldots, n-m+1$$

and for $r \in I_2^m$

$$H_r = \sum_{j=0}^{n-r} \zeta_{r,r+j}(t_{r+1}, \ldots, t_n)\mathcal{E}_{r+j}, \quad \zeta_{r,r} = 1, \quad \text{for } r = n-m+2, \ldots, n-\kappa_2,$$

$$H_r = \mathcal{E}_r, \quad \text{for } r = n-\kappa_2+1, \ldots, n.$$

where ζ_j are polynomials determined from the Frobenius conditions.

Deformation of geodesic flows

Example

Let us take $n = 11$, $m = 6$. Then

$$\kappa_1 = \left[\frac{n+3-m}{2}\right] = 4, \qquad \kappa_2 = \left[\frac{m}{2}\right] = 3$$

and (the index m is below omitted)

$$\begin{aligned}
H_r &= \mathcal{E}_r, \quad r = 1, \ldots, 4, 9, \ldots, 11, \\
H_5 &= \mathcal{E}_5 + t_4\mathcal{E}_2 + 2t_3\mathcal{E}_1, \\
H_6 &= \mathcal{E}_6 + 4t_2\mathcal{E}_1 + (3t_3 - \tfrac{1}{2}t_5^2)\mathcal{E}_2 + 2t_4\mathcal{E}_3 + t_5\mathcal{E}_4, \\
H_7 &= \mathcal{E}_7 + t_8\mathcal{E}_8 + 2t_9\mathcal{E}_9 + (3t_{10} + t_8 t_9)\mathcal{E}_{10} + (4t_{11} + 2t_8 t_{10})\mathcal{E}_{11}, \\
H_8 &= \mathcal{E}_8 + t_9\mathcal{E}_{10} + 2t_{10}\mathcal{E}_{11}.
\end{aligned}$$

$H_1, \ldots, H_{11}$ satisfy the Frobenius condition (45) with all $f_{rs} = 0$.

Deformations with non-zero potentials

Let us now use the separable potentials.

Consider the following set of Hamiltonians

$$h_{m,r} = \mathcal{E}_{m,r} + V_r = E_{m,r} + W_{m,r} + \sum_{\alpha=-m}^{2n-m+2} c_\alpha(t_1, ..., t_n) V_r^{(\alpha)}, \quad r = 1, \ldots, n.$$

which naturally generalizes the geodesic case.

The question is what is the explicit form of functions $c_\alpha(t_1, ..., t_n)$ such that the appropriate deformations $H_{m,r}$ of $h_{m,r}$ fulfill Frobenius conditions.

As we have proved, the functions $c_\alpha(t_1, ..., t_n)$ are determined by a system of first order PDE's that can be recursively solved.

Deformations with non-zero potentials

In general our procedure leads to a $(n+3)$-parameter family of Frobenius integrable non-autonomous systems with potentials. Although the obtained systems are parametrized by $2n+3$ integration constants $a_{-m}, \ldots, a_{2n-m+2}$, the n constants $a_0, \ldots, a_{n-1}$ are integration constants that originate in the trivial potentials $V_r^{(0)}, \ldots, V_r^{(n-1)}$ and as such enter the Hamiltonians only in a trivial way, through some undetermined functions of times only, not affecting the dynamics of the systems. We can thus say that our systems are parametrized by $n+3$ dynamical parameters $(a_{-m}, \ldots, a_{-1}, a_n, \ldots, a_{2n-m+2})$ and by n non-dynamical parameters $(a_0, \ldots, a_{n-1})$.

Deformations with non-zero potentials

For $n = 2$ and $m = 1$ we find

$$h_{1,r} = E_{1,r} + W_{1,r} + a_5 V_r^{(5)} + (a_4 + 4a_5 t_2) V_r^{(4)} + [a_3 + 3a_4 t_2 + 2a_5(t_1 + 3t_2^2)] V_r^{(3)} + [a_2 + 2a_3 t_2 + a_4(t_1 + 3t_2^2) + 4a_5(t_1 t_2 + t_2^3)] V_r^{(2)} + a_{-1} V_r^{(-1)}, \qquad r =$$

where

$$E_{1,1} = \tfrac{1}{2} p_1^2 - \tfrac{1}{2} q_2 p_2^2, \qquad E_{1,2} + W_{1,2} = -q_2 p_2 p_1 - \tfrac{1}{2} q_1 q_2 p_2^2 + p_1.$$

Here $H_{1,1} = h_{1,1}$, $H_{1,2} = h_{1,2}$ and

$$\{H_1, H_2\} + \frac{\partial H_1}{\partial t_2} - \frac{\partial H_2}{\partial t_1} = a_5(4t_2^3 + 4t_1 t_2) + a_4(3t_2^2 + t_1) + 2a_3 t_2 + a_2.$$

Separable potentials are generated recursively by

$$V^{(k)} = \begin{pmatrix} V_1^{(k)} \\ V_2^{(k)} \end{pmatrix} = R^k \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad R = \begin{pmatrix} -q_1 & 1 \\ -q_2 & 0 \end{pmatrix}, \qquad k \in \mathbb{Z}.$$

Deformations with non-zero potentials

For $n = 2$ and $m = 3$ we find

$$h_{3,r} = E_{3,r} + W_{3,r} + a_3 \mathrm{e}^{2t_1} V_r^{(3)} + a_2 \mathrm{e}^{t_1} V_r^{(2)} + (a_{-1} + a_{-2} t_2 + a_{-3} t_2^2) V_r^{(-1)}$$
$$+ (a_{-2} + 2a_{-3} t_2) V_r^{(-2)} + a_{-3} V_r^{(-3)}, \qquad r = 1, 2,$$

where

$$E_{3,1} = \tfrac{1}{2}\left(q_1^2 - q_2\right) p_1^2 + \tfrac{1}{2} q_2^2\, p_2^2 + q_1 q_2 p_1 p_2, \quad E_{3,2} + W_{3,2} = q_2^2 p_1 p_2 + \tfrac{1}{2}\, q_1 q_2 p_1^2 + q_2 p_1.$$

Here again $H_{3,1} = h_{3,1}$, $H_{3,2} = h_{3,2}$. They satisfy the Frobenius condition:

$$\{H_1, H_2\} + \frac{\partial H_1}{\partial t_2} - \frac{\partial H_2}{\partial t_1} = 0.$$

For $n = 3$ and $m = 1$ we get

$$h_{1,r} = E_{1,r} + W_{1,r} + a_7 V_r^{(7)} + (a_6 + 6a_7 t_3) V_r^{(6)} + [a_5 + 5a_6 t_3 + a_7(4t_2 + 15t_3^2)] V_r^{(5)}$$
$$+ [a_4 + 4\, a_5 t_3 + a_6(3t_2 + 10t_3^2) + 2a_7(t_1 + 9t_2 t_3 + 10t_3^3)] V_r^{(4)}$$
$$+ [a_3 + 3\, a_4 t_3 + 2\, a_5(t_2 + 3t_3^2) + a_6\left(t_1 + 10t_2 t_3 + 10t_3^3\right)$$
$$+ a_7(4t_2^2 + 6t_1 t_3 + 30t_2 t_3^2 + 15t_3^4)] V_r^{(3)} + a_{-1} V_r^{(-1)},$$

Deformations with non-zero potentials

where

$$\begin{aligned} E_{1,1} &= p_1 p_2 + \tfrac{1}{2}\, q_1 p_2^2 - \tfrac{1}{2}\, {p_3}^2 q_3, \\ E_{1,2} + W_{1,2} &= p_2 q_1 p_1 + \tfrac{1}{2} p_1^2 - q_3 p_2 p_3 + \tfrac{1}{2}\left(q_1^2 - q_2\right) p_2^2 - \tfrac{1}{2}\, q_1 q_3 p_3^2 + p_2, \\ E_{1,3} + W_{1,3} &= -q_3 p_1 p_3 - \tfrac{1}{2}\, q_3 p_2^2 - q_1 q_3 p_2 p_3 - \tfrac{1}{2}\, q_2 q_3 p_3^2 + q_1 p_2 + 2 p_1 \end{aligned}$$

The separable potentials are

$$V^{(k)} = \begin{pmatrix} V_1^{(k)} \\ V_2^{(k)} \\ V_3^{(k)} \end{pmatrix} = R^k \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \qquad R = \begin{pmatrix} -q_1 & 1 & 0 \\ -q_2 & 0 & 1 \\ -q_3 & 0 & 0 \end{pmatrix}, \qquad k \in \mathbb{Z}.$$

Now, $H_{1,1} = h_{1,1}$, $H_{1,2} = h_{1,2}$ and $H_{1,3} = h_{1,3} + t_2 h_{1,1}$. They satisfy the Frobenius condition:

$$\{H_i, H_j\} + \frac{\partial H_i}{\partial t_j} - \frac{\partial H_j}{\partial t_i} = 0, \quad i, j = 1, \ldots, 3.$$

Isomonodromic Lax representation

Theorem

Each non-autonomous Hamiltonian flow $Y_r(\xi,t) = \pi dH_r$ has the isomonodromic Lax representation

$$\frac{d}{dt_r}L(\lambda,\xi,t) = [\overline{U}_r(\lambda,\xi,t), L(\lambda,\xi,t)] + 2\lambda^m \frac{\partial}{\partial\lambda}\overline{U}_r(\lambda,\xi,t) \qquad (14)$$

where now

$$\frac{d}{dt_r} = \frac{\partial}{\partial t_r} + \{\cdot, H_r\}$$

is the evolutionary derivative along the Hamiltonian vector field. The Lax matrix $L(\lambda,\xi,t)$ is given by

$$L(\lambda,\xi,t) = \begin{pmatrix} v(\lambda) & u(\lambda) \\ w(\lambda,t) & -v(\lambda) \end{pmatrix}$$

with $u(\lambda)$ and $v(\lambda)$ are the same as in the Stäckel case, while w is given by

$$w(x,t) = -\lambda^m \left[\frac{v^2(\lambda)\lambda^{-m}}{u(\lambda)}\right]_+ + 2\lambda^m \sum_{\alpha=-m}^{2n-m+2} c_\alpha(t_1,\ldots,t_n)\left[\frac{\lambda^{\alpha-m}}{u(\lambda)}\right]_+,$$

Isomonodromic Lax representation

Further, for $r \in \{1\} \cup I_1^m$

$$\overline{U}_r(\lambda,\xi,t) = U_r(\lambda,\xi,t) \quad \text{for } r = 1,\ldots,\kappa_1,$$

$$\overline{U}_r(\lambda,\xi,t) = \sum_{j=1}^{r} \zeta_{r,j}(t_1,\ldots,t_{r-1})U_j(\lambda,\xi,t) \quad \text{for } r = \kappa_1+1,\ldots,n-m+1$$

and for $r \in I_2^m$

$$\overline{U}_r(\lambda,\xi,t) = \sum_{j=0}^{n-r} \zeta_{r,r+j}(t_{r+1},\ldots,t_n)U_{r+j}(\lambda,\xi,t) \quad \text{for } r = n-m+2,\ldots,n-\kappa_2,$$

$$\overline{U}_r(\lambda,\xi,t) = U_r(\lambda,\xi,t) \quad \text{for } r = n-\kappa_2+1,\ldots,n, \text{ where}$$

$$U_r(\lambda,\xi,t) = \left[\frac{B_r(\lambda)}{u(\lambda)}\right]_+, \qquad B_r(\lambda) = \frac{1}{2}\left[\frac{u(\lambda)}{\lambda^{n+1-r}}\right]_+ L(\lambda,\xi,t), \qquad r \in \{1\} \cup I_1^m,$$

$$U_r(\lambda,\xi,t) = \left[\frac{B_r(\lambda)}{u(\lambda)}\right]_+, \quad B_r(\lambda) = -\frac{1}{2}\left(\frac{u(\lambda)}{x^{n+1-r}} - \left[\frac{u(\lambda)}{\lambda^{n+1-r}}\right]_+\right) L(\lambda,\xi,t), \quad r \in I_2^m.$$

Example: non-autonomous Hénon-Heiles system

Consider Liouville integrable Hénon-Heiles system on $M = \mathbb{R}^4$, generated by two Hamiltonian functions in involution

$$h_1 = E_1 + V_1(x) = \tfrac{1}{2}p_1^2 + \tfrac{1}{2}p_2^2 + x_1^3 + \tfrac{1}{2}x_1x_2^2,$$
$$h_2 = E_2 + V_2(x) = \tfrac{1}{2}x_2p_1p_2 - \tfrac{1}{2}x_1p_2^2 + \tfrac{1}{16}x_2^4 + \tfrac{1}{4}x_1^2x_2^2$$

written in Cartesian coordinates (x_1, x_2) and conjugate momenta (p_1, p_2).

Its separation curve has the form

$$h_1\lambda + h_2 = \tfrac{1}{2}\lambda\mu^2 + \lambda^4,$$

i.e. it is the case of $n = 2$, $m = 1$ and $V(x) = -V^{(4)}(x)$.

The point transformation between Cartesian, Viète and separation coordinates is as follows

$$\lambda_1 + \lambda_2 = -q_1 = x_1, \quad \lambda_1\lambda_2 = q_2 = -\frac{1}{4}x_2^2.$$

Example: non-autonomous Hénon-Heiles system

The Hénon-Heiles Hamiltonian is h_1, so for the canonical form of the Poisson tensor $\{x_i, p_j\}_\pi = \delta_{ij}$, the related autonomous evolution equations are

$$(x_1)_{t_1} = p_1, \qquad (x_2)_{t_1} = p_2,$$

$$(p_1)_{t_1} = -3x_1^2 - \tfrac{1}{2}x_2^2, \qquad (p_1)_{t_1} = -x_1 x_2.$$

Here h_2 is the first integral so its Hamiltonian flow equations

$$(x_1)_{t_2} = \frac{1}{2}x_2 p_2, \qquad (x_2)_{t_2} = \frac{1}{2}x_2 p_1 - x_1 p_2$$

$$(p_1)_{t_2} = \frac{1}{2}p_2^2 - \frac{1}{2}x_1 x_2^2, \qquad (p_2)_{t_2} = -\frac{1}{2}p_1 p_2 - \frac{1}{4}x_2^3 - \frac{1}{2}x_1^2 x_2$$

represent the symmetry of the Hénon-Heiles.

Example: non-autonomous Hénon-Heiles system

Considered evolution equations have Lax representations

$$L(\lambda) = \begin{pmatrix} p_1\lambda + \frac{1}{2}x_2p_2 & \lambda^2 - x_1\lambda - \frac{1}{4}x_2^2 \\ -2\lambda^3 - 2x_1\lambda^2 - \left(2x_1^2 + \frac{1}{2}x_2^2\right)\lambda + p_2^2 & -p_1\lambda - \frac{1}{2}x_2p_2 \end{pmatrix},$$

$$U_1(\lambda) = \begin{pmatrix} 0 & \frac{1}{2} \\ -\lambda - 2x_1 & 0 \end{pmatrix}, \quad U_2(\lambda) = \begin{pmatrix} \frac{1}{2}p_1 & \frac{1}{2}\lambda - \frac{1}{2}x_1 \\ -\lambda^2 - x_1\lambda - x_1^2 - \frac{1}{2}x_2^2 & -\frac{1}{2}p_1 \end{pmatrix}.$$

Example: non-autonomous Hénon-Heiles system

Now, let us deform the original Hamiltonians in the following way. First, notice that $\{E_1, p_1\} = 0$, i.e. take $W_2 = -p_1$ generated by the Killing vector $Z = (-1, 0)^T$ of the Euclidean metric in $\mathbb{R}^2$. Second, add to both Hamiltonians the lower nontrivial positive separable potentials with coefficients depending on evolution parameters, i.e. $c_3(t_1, t_2)V^{(3)} + c_2(t_1, t_2)V^{(2)}$.

Thus, consider the following deformed Hamiltonians

$$\begin{aligned} H_1(t) &= h_1 + c_3(t_1, t_2)V_1^{(3)} + c_2(t_1, t_2)V_1^{(2)} \\ &= \tfrac{1}{2}p_1^2 + \tfrac{1}{2}p_2^2 + x_1^3 + \tfrac{1}{2}x_1x_2^2 + c_3(t_1, t_2)(x_1^2 + \tfrac{1}{4}x_2^2) + c_2(t_1, t_2)x_1, \\ H_2(t) &= h_1 - p_1 + c_3(t_1, t_2)V_2^{(3)} + c_2(t_1, t_2)V_2^{(2)} \\ &= \tfrac{1}{2}x_2p_1p_2 - \tfrac{1}{2}x_1p_2^2 - p_1 + \tfrac{1}{16}x_2^4 + \tfrac{1}{4}x_1^2x_2^2 + \tfrac{1}{4}c_3(t_1, t_2)x_1x_2^2 + \tfrac{1}{4}c_2(t_1, t_2)x_2^2. \end{aligned}$$

From the demand of the Frobenius condition (45) we immediately find that

$$c_3(t_1, t_2) = 3t_2, \qquad c_2(t_1, t_2) = t_1 + 3t_2^2, \qquad f_{12}(t_1, t_2) = -c_2(t_1, t_2).$$

Example: non-autonomous Hénon-Heiles system

Hence, the related non-autonomous evolution equations are

$$\begin{aligned}
&(x_1)_{t_1} = p_1, \qquad (x_2)_{t_1} = p_2,\\
&(p_1)_{t_1} = -3x_1^2 - \tfrac{1}{2}x_2^2 - 6t_2x_1 + t_1 + 3t_2^2,\\
&(p_2)_{t_1} = -x_1x_2 - \tfrac{3}{2}t_2x_2.
\end{aligned}$$

and

$$\begin{aligned}
&(x_1)_{t_2} = \tfrac{1}{2}x_2p_2 - 1, \qquad (x_2)_{t_2} = \tfrac{1}{2}x_2p_1 - x_1p_2,\\
&(p_1)_{t_2} = \tfrac{1}{2}p_2^2 - \tfrac{1}{2}x_1x_2^2 - \tfrac{3}{4}t_2x_2^2,\\
&(p_2)_{t_2} = -\tfrac{1}{2}p_1p_2 - \tfrac{1}{4}x_2^3 - \tfrac{1}{2}x_1^2x_2 - \tfrac{3}{2}t_2x_1x_2 - \tfrac{1}{2}\left(t_1 + 3t_2^2\right)x_2.
\end{aligned}$$

Example: non-autonomous Hénon-Heiles system

The matrices $L(\lambda, t)$, $U_1(\lambda, t)$ and $U_2(\lambda, t)$ with extra potential $3t_2 V^{(3)} + (t_1 + 3t_2^2)V^{(2)}$ are as follows

$$L(\lambda; t) = \begin{pmatrix} p_1\lambda + \frac{1}{2}x_2 p_2 & \lambda^2 - x_1\lambda - \frac{1}{4}x_2^2 \\ \begin{array}{c} -2\lambda^3 - 2(x_1 + 3t_2)\lambda^2 \\ -\left(2x_1^2 + \frac{1}{2}x_2^2 + 6x_1 t_2 + 6t_2^2 + 2t_1\right)\lambda + p_2^2 \end{array} & -p_1\lambda - \frac{1}{2}x_2 p_2 \end{pmatrix},$$

$$U_1(\lambda; t) = \begin{pmatrix} 0 & \frac{1}{2} \\ -\lambda - 2x_1 - 3t_2 & 0 \end{pmatrix},$$

$$U_2(\lambda; t) = \begin{pmatrix} \frac{1}{2}p_1 & \frac{1}{2}\lambda - \frac{1}{2}x_1 \\ -\lambda^2 - (x_1 + 3t_2)\lambda - x_1^2 - \frac{1}{2}x_2^2 - 3x_1 t_2 - 3t_2^2 - t_1 & -\frac{1}{2}p_1 \end{pmatrix}.$$

Example: non-autonomous Hénon-Heiles system

Now, because of explicit time dependence and the deformation of geodesic Hamiltonian E_2 by $W_2 = -p_1$ term, we get

$$\frac{dL(\lambda;t)}{dt_1} - [U_1(\lambda;t), L(\lambda;t)] = \begin{pmatrix} 0 & 0 \\ -2\lambda & 0 \end{pmatrix} = 2\lambda \frac{\partial U_1(\lambda;t)}{\partial \lambda},$$

$$\frac{dL(\lambda;t)}{dt_2} - [U_2(\lambda;t), L(\lambda;t)] = \begin{pmatrix} 0 & \lambda \\ -4\lambda^2 - 2(x_1 + 3t_2)\lambda & 0 \end{pmatrix} = 2\lambda \frac{\partial U_2(\lambda;t)}{\partial \lambda}$$

So, the non-autonomous evolution equations have the following isomonodromic Lax representation

$$\frac{dL(\lambda;t)}{dt_k} = [U(\lambda;t), L(\lambda;t)] + 2\lambda \frac{\partial U_k(\lambda;t)}{\partial \lambda}, \qquad k = 1, 2,$$

or the form (14) with $m = 1$.

This way, a huge family of multi-parameter Painlevé-type systems have been constructed, including $P_{34}, P_I - P_{IV}$ hierarchies.

Lecture 3

Autonomous restrictions of KdV hierarchy

Main message for lectures 3 and 4, more precise form

Autonomous restrictions of soliton hierarchies

The n-th *autonomous* restriction

$$u_{t_1} = K_1[u], \quad u_{t_2} = K_2[u], \quad \ldots, \quad u_{t_n} = K_n[u], \quad K_{n+1}[u] = 0$$

of the soliton hierarchy $u_{t_i} = K_i[u] \equiv K_i(u, u_x, u_{xx}, \ldots)$ can be parametrized as an appropriate classical Stäckel separable system (investigated so far and published: KdV, cKdV, AKNS)

Non-autonomous restrictions of soliton hierarchies

The n-th *non-autonomous* restriction

$$u_{t_1} = K_1[u], \quad u_{t_2} = K_2[u], \quad \ldots, \quad u_{t_n} = K_n[u],$$

$$\mathcal{K} \equiv \sigma + a_1(t_1, \ldots, t_n)K_1 + \ldots + a_N(t_1, \ldots, t_n)K_N = 0,$$

$(N \geq n-1)$ of the soliton hierarchy $u_{t_i} = K_i[u]$ (where σ is a suitably chosen master symmetry of the hierarchy and a_i suitably chosen functions of times t_j) can be parametrized as an appropriate Painlevé-type system (ongoing research, not published anywhere)

Bibliography

Recent results, concerning stationary systems of soliton hierarchies:

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- M. Błaszak, K. Marciniak, B.M. Szablikowski, "Stationary systems of the AKNS hierarchy",J. Nonlin. Math. Phys. **32** (2025) article 89

and construction of non-autonomous soliton hierarchies:

- M. Błaszak, K. Marciniak and B.M. Szablikowski, "Non-autonomous soliton hierarchies", Symmetry 17(7):1103, July 2025, DOI: 10.3390/sym17071103

Soliton hierarchy

By a soliton hierarchy we mean an infinite hierarchy of evolutionary PDE's on $\mathcal{M}$ with vector fields K_n that are bi-Hamiltonian with respect to two compatible Poisson structures π_0 and π_1

$$u_{t_n} = K_n[u] = \pi_0 \delta H_n = \pi_1 \delta H_{n-1}, \qquad n = 1, 2, \ldots,$$

$u = u(x, t_1, t_2, \ldots)$, $K_n[u] = K_n(u, u_x, u_{xx}, \ldots)$, with π_0 being invertible.

Then

$$K_n \equiv N^{n-1} K_1, \qquad n = 2, 3, \ldots,$$

where $N = \pi_1 \pi_0^{-1}$ has the zero Nijenhuis torsion (it is hereditary then) so that

$$[K_n, K_m] = 0, \qquad n, m = 1, 2, \ldots .$$

(K_i are mutual symmetries) and

$$\mathcal{L}_{K_n} N = 0, \qquad n = 1, 2, \ldots,$$

Cosymmetries

Corresponding sequence of 1-forms (cosymmetries):

$$\gamma_n \equiv \delta H_n = \left(N^\dagger\right)^i \gamma_0, \qquad n = 0, 1, \ldots,$$

where $\gamma_0 = \delta H_0$ and $N^\dagger = \pi_0^{-1}\pi_1$. Then $\gamma_n = \delta H_n$ with $H_n = \int h_n dx$. Define now the infinite sequence of Poisson operators

$$\pi_k = N^k \pi_0, \qquad k = 2, 3, \ldots .$$

π_k are pairwise compatible and usually non-local. It follows that K_n is $(n+1)$-Hamiltonian

$$K_n = \pi_0 \delta H_n = \pi_1 \delta H_{n-1} = \ldots = \pi_n \delta H_0, \qquad n = 1, 2, \ldots .$$

Hereditary algebra

If a scaling vector field σ_0 allowed ($\mathcal{L}_{\sigma_0} K_1 = \rho K_1$, $\rho \in \mathbf{R}$, $\mathcal{L}_{\sigma_0} N = N$) then one can define master symmetries σ_n on $\mathcal{M}$

$$\sigma_n = N^n \sigma_0, \qquad n = -1, 0, 1, \dots .$$

K_n and σ_m constitute a hereditary algerba

$$\begin{aligned} [K_n, K_m] &= 0, \qquad m, n = 1, 2, \dots, \\ [\sigma_n, K_m] &= (\rho + m - 1) K_{n+m}, \qquad n = -1, 0, 1, \dots, \quad m = 1, 2, \dots, \\ [\sigma_n, \sigma_m] &= (m - n) \sigma_{n+m}, \qquad m, n = -1, 0, 1, \dots . \end{aligned}$$

where σ_{-1} is defined by $\sigma_0 = N \sigma_{-1}$ and it is assumed to be Hamiltonian with respect to π_0

$$\sigma_{-1} = \pi_0 \delta F$$

Then, all σ_n are Hamiltonian: $\sigma_n = \pi_{n+1} \delta F$, $n = -1, 0, \dots$

Lax (isospectral) (zero-curvature) representation

Usually the hierarchy $u_{t_n} = K_n[u]$ can be obtained from the isospectral linear problem

$$\begin{cases} \Psi_x = L\Psi, \\ \Psi_{t_i} = U_i\Psi, \quad i = 1, 2, \dots, \end{cases} \tag{15}$$

where $L = L(\lambda, u)$, $U_i = U_i(\lambda, [u])$ are 2×2 matrices depending on $[u]$ and the auxiliary variable λ, s.t. $\lambda_{t_i} = 0$ for all i.

The compatibility condition, that is the condition for existence of a common multi-time solution $\Psi(x, t_1, t_2, \dots)$, for the problem (15) is

$$(\Psi_x)_{t_i} = (\Psi_{t_i})_x, \qquad i = 1, 2, \dots, \tag{16a}$$

$$(\Psi_{t_i})_{t_j} = (\Psi_{t_j})_{t_i}, \qquad i, j = 1, 2 \dots . \tag{16b}$$

What gives what

$$\text{(16a)} \iff L_{t_i} = [U_i, L] + (U_i)_x \iff u_{t_i} = K_i[u] \tag{17}$$

$$\text{(16b)} \iff (U_i)_{t_j} - (U_j)_{t_i} + [U_i, U_j] = 0 \iff [K_i, K_j] = 0$$

Lax representation for master symmetries σ_i

The (non-commuting) hierarchy of master symmetries can be obtained from the following *deformed* linear isospectral problem

$$\begin{cases} \Psi_x = L\Psi, \\ \Psi_{\tau_i} = V_i\Psi - \lambda^{i+1}\Psi_\lambda, \quad i = -1, 0, 1 \ldots, \end{cases} \tag{18}$$

where $L = L(\lambda, u)$ is the same L as in (15) while $V_i = V_i(\lambda, [u])$ are some matrices depending on $[u]$ and λ such that $\lambda_{\tau_i} = 0$.

The system (18) has, for each i, a solution $\Psi(x, \tau_i)$ so that

$$(\Psi_x)_{\tau_i} = (\Psi_{\tau_i})_x \iff L_{\tau_i} = [V_i, L] + (V_i)_x - \lambda^{i+1}L_\lambda \iff u_{\tau_i} = \sigma_i[u]$$

For a curious person: since the fields σ_i do not commute we clearly cannot expect that $(\Psi_{\tau_i})_{\tau_j} = (\Psi_{\tau_j})_{\tau_i}$. Instead we have

$$(V_i)_{\tau_j} - (V_j)_{\tau_i} + [V_i, V_j] + \lambda^{j+1}(V_i)_\lambda - \lambda^{i+1}(V_j)_\lambda = (i-j)V_{i+j}, \qquad i, j = -1, 0, \ldots.$$

KdV hierarchy

The KdV equation

$$u_t = \frac{1}{4} u_{xxx} + \frac{3}{2} u u_x$$

is a member of the bi-Hamiltonian chain of nonlinear PDE's

$$u_{t_n} = K_n =\equiv \pi_0 dH_n = \pi_1 dH_{n-1}, \qquad n = 1, 2, ... \tag{19}$$

where the two local Poisson operators are

$$\pi_0 = \partial_x, \qquad \pi_1 = \frac{1}{4}\partial_x^3 + \frac{1}{2} u \partial_x + \frac{1}{2}\partial_x u.$$

The hierarchy (19) can be generated by the recursion operator and its adjoint

$$N \equiv \pi_1 \pi_0^{-1} = \frac{1}{4}\partial_x^2 + u + \frac{1}{2} u_x \partial_x^{-1}, \qquad N^\dagger = \frac{1}{4}\partial_x^2 + u - \frac{1}{2}\partial_x^{-1} u_x,$$

in the sense that

$$K_{n+1} = N^n K_1, \qquad \gamma_n = dH_n = \left(N^\dagger\right)^n \gamma_0, \qquad n = 1, 2, \ldots .$$

KdV hierarchy

In particular, we find that that the first vector fields (symmetries) K_n are:

$$K_1 = u_x,$$

$$K_2 = \frac{1}{4}u_{xxx} + \frac{3}{2}uu_x,$$

$$K_3 = \frac{1}{16}u_{5x} + \frac{5}{8}uu_{3x} + \frac{5}{4}u_x u_{xx} + \frac{15}{8}u^2 u_x,$$

$$K_4 = \frac{1}{64}u_{7x} + \frac{7}{32}uu_{5x} + \frac{21}{32}u_x u_{4x} + \frac{35}{32}u_{xx}u_{3x} + \frac{35}{32}u_x^3 + \frac{35}{8}uu_x u_{xx} + \frac{35}{32}u^2 u_{3x} + \frac{35}{16}$$

the first conserved one-forms (co-symmetries) γ_n are

$$\gamma_0 = 2,$$

$$\gamma_1 = u,$$

$$\gamma_2 = \frac{1}{4}u_{xx} + \frac{3}{4}u^2,$$

$$\gamma_3 = \frac{1}{16}u_{4x} + \frac{5}{8}uu_{xx} + \frac{5}{16}u_x^2 + \frac{5}{8}u^3,$$

$$\gamma_4 = \frac{1}{64}u_{6x} + \frac{7}{32}uu_{4x} + \frac{7}{16}u_x u_{3x} + \frac{21}{64}u_{xx}^2 + \frac{35}{32}u^2 u_{xx} + \frac{35}{32}uu_x^2 + \frac{35}{64}u^4,$$

$$\vdots$$

KdV hierarchy

The first Hamiltonian densities h_n of $H_n = \int h_n\, dx$ are

$$h_0 = 2u, \qquad h_1 = \frac{1}{2}u^2, \qquad h_2 = -\frac{1}{8}u_x^2 + \frac{1}{4}u^3,$$
$$h_3 = \frac{1}{32}u_{xx}^2 + \frac{5}{32}u^2u_{xx} + \frac{5}{32}u^4,$$
$$h_4 = -\frac{1}{128}u_{3x}^2 + \frac{7}{64}uu_{xx}^2 - \frac{35}{64}u^2u_x^2 + \frac{7}{64}u^5.$$

First few master symmetries σ_n:

$$\sigma_{-1} = 1$$
$$\sigma_0 = u + \frac{1}{2}xu_x,$$
$$\sigma_1 = \frac{1}{2}u_{xx} + \frac{1}{8}xu_{3x} + u^2 + \frac{1}{2}xuu_x + \frac{1}{4}u_x\partial_x^{-1}u.$$

The symmetries K_i and master symmetries σ_j of the KdV equation generate the hereditary algebra (75) with

$$\rho = \frac{1}{2} \qquad \text{so that} \qquad \kappa_m = m - \frac{1}{2}.$$

Isospectral Lax representation for KdV hierarchy

It has the form (17):

$$L_{t_i} = [U_i, L] + (U_i)_x \quad i = 1, 2, \ldots$$

with

$$L = \begin{pmatrix} 0 & 1 \\ \lambda - u & 0 \end{pmatrix},$$

and with the auxiliary matrices

$$U_n = \frac{1}{2} \sum_{i=0}^{n-1} \begin{pmatrix} -\frac{1}{2}(\gamma_i)_x & \gamma_i \\ (\lambda - u)\gamma_i - \frac{1}{2}(\gamma_i)_{xx} & \frac{1}{2}(\gamma_i)_x \end{pmatrix} \lambda^{n-i-1}, \qquad n = 1, 2, \ldots.$$

In particular, $U_1 = L$,

$$U_2 = \begin{pmatrix} -\frac{1}{4}u_x & \lambda + \frac{1}{2}u \\ \lambda^2 - \frac{1}{2}u\lambda - \frac{1}{2}u^2 - \frac{1}{4}u_{xx} & \frac{1}{4}u_x \end{pmatrix}$$

and

$$U_3 = \begin{pmatrix} -\frac{1}{4}u_x\lambda - \frac{1}{16}(u_{3x} + 6uu_x) & \lambda^2 + \frac{1}{2}u\lambda + \frac{1}{8}(u_{xx} + 3u^2) \\ (U_3)_{12} & \frac{1}{4}u_x\lambda + \frac{1}{16}(u_{3x} + 6uu_x) \end{pmatrix},$$

with $(U_3)_{12} = \lambda^3 - \frac{1}{2}u\lambda^2 - \frac{1}{8}(u_{xx} + u^2)\lambda - \frac{1}{16}u_{4x} + \frac{1}{2}uu_{xx} + \frac{3}{8}u_x^2 + \frac{3}{8}u^3$.

The Lax formulation for master symmetries of KdV hierarchy

$$V_n = \frac{1}{2}\sum_{i=-1}^{n-1}\begin{pmatrix} -\frac{1}{2}\sigma_i & \partial_x^{-1}\sigma_i \\ (\lambda-u)\partial_x^{-1}\sigma_i - \frac{1}{2}\left(\sigma_i\right)_x & \frac{1}{2}\sigma_i \end{pmatrix}\lambda^{n-i-1}, \qquad n=-1,0,1\ldots,$$

so that

$$V_{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad V_0 = \begin{pmatrix} -\frac{1}{4} & \frac{1}{2}x \\ \frac{1}{2}(\lambda-u)x & \frac{1}{4} \end{pmatrix}.$$

Stationary autonomous systems of KdV hierarchy

Definition

The n-th stationary KdV system consists of the first n evolution equations from the KdV hierarchy together with its $(n+1)$-th stationary flow:

$$u_{t_1} = K_1, \qquad u_{t_2} = K_2, \qquad \dots, \qquad u_{t_n} = K_n, \qquad K_{n+1} = 0 \tag{20}$$

- The stationary condition $\mathcal{K}_{n+1} = 0$ provides a constraints/restriction on the infinite-dimensional (functional) manifold on which the KdV hierarchy is defined, reducing it to the finite-dimensional submanifold, n-th stationary manifold:

$$\mathcal{M}_n = \{[u] \in \mathcal{M} \mid K_{n+1} = 0\}\,, \qquad \dim \mathcal{M}_n = 2n+1,$$

parametrized by the $2n+1$ jest variables $u, u_x, \dots u_{(2n)x}$

The autonomous invariant manifold $\mathcal{S}$

By the Frobenius condition, the finite dimensional autonomous submanifold $\mathcal{S} \subset \mathcal{M}$ defined by our autonomous constraint

$$K_m[u] = 0$$

is invariant with respect to all the vector fields K_i, $i = 1, \ldots, n$ in the sense that the trajectories of all K_i that start in $\mathcal{S}$ remain in $\mathcal{S}$. Moreover, K_i are tangent to $\mathcal{S}$.

Important remark

Using the constraint $K_m = 0$ and its differential consequences we can eliminate higher-order jet coordinates from the fields K_i, which turns the first n equations $u_{t_i} = K_i[u]$ of the KdV hierarchy into the *autonomous* dynamical system

$$\xi_{t_1} = \mathcal{X}_1(\xi), \quad \ldots, \quad \xi_{t_n} = \mathcal{X}_n(\xi) \tag{21}$$

defined on $\mathcal{S}$. Moreover, (21) is Frobenius integrable

$$[\mathcal{X}_i, \mathcal{X}_j] = 0, \qquad i, j = 1, \ldots, n.$$

Lax equations for the n-th stationary KdV system

- The constraint $K_{n+1} = 0$ can be obtained by imposing - on the linear problems (15) - the constraint

$$\Psi_{t_{n+1}} = \lambda^m \mu \Psi$$

or equivalently

$$U_{n+1}\Psi = \lambda^m \mu \Psi. \qquad (22)$$

It quickly yields $L_{t_{n+1}} = 0$ and thus $K_{n+1} = 0$.

- The factor λ^m in (22) is a matter of later convenience.
- This yields the finite sequence of Lax equations of the stationary cKdV system

Lax equations for stationary cKdV system

$$\frac{d}{dt_k} U_{n+1} = [U_k, U_{n+1}],, \qquad k = 1, 2, \ldots, n,$$

with the constraint $\mathcal{K}_{n+1} = 0$ applied.

- Note that U_{n+1} plays now the role of the Lax matrix.

Characteristic equation of U_{n+1} - a Stäckel system is unfolding

Non-trivial solutions to (22) exist if and only if the characteristic equation

$$\det\left(U_{n+1} - \lambda^m \mu I\right) = 0,$$

is satisfied (remind: here U_{n+1} is restricted to $K_{n+1} = 0$). It has the form of the spectral curve

$$\lambda^{2n+1} + \sum_{k=0}^{n} h_k \lambda^{n-k} = \lambda^{2m}\mu^2, \tag{23}$$

where

$$h_k = -\frac{1}{16}\sum_{i=0}^{n-k}\left[2\gamma_{n-i}(\gamma_{i+k})_{xx} - (\gamma_{n-i})_x(\gamma_{i+k})_x + 4u\gamma_{n-i}\gamma_{i+k}\right] + \frac{1}{4}\sum_{i=1}^{n-k}\gamma_{n-i+1}\gamma_{i+k}.$$

Constants of motion

$h_0, \ldots, h_n$ are constants of motion of the stationary system (20)

First form of stationary KdV system

$$0 = \mathcal{K}_{n+1} = \pi_0 \gamma_{n+1} = 0 \Longrightarrow \gamma_{n+1} + c = 0 \tag{24}$$

The above constraint defines a foliation of $\mathcal{M}_n$, of codimension 1:

$$\mathcal{M}_n = \bigcup_{c \in \mathbb{R}} \mathcal{M}_{n,c}.$$

First integrated form of the n-th stationary KdV system (20)

$$u_{t_1} = K_1, \qquad u_{t_2} = K_2, \qquad \ldots, \qquad u_{t_n} = K_n, \qquad \gamma_{n+1} + c = 0, \tag{25}$$

It is a system of n ODE's on the $2n$-dimensional leaf $\mathcal{M}_{n,c}$ endowed with the jet coordinates $u, u_x, \ldots, u_{(2n-1)x}$.

Miracle I

In the "Beautiful theorem" below we will parametrize this system as the Stäckel system given by

$$\lambda^{2n+1} + c\lambda^n + \sum_{k=1}^{n} H_k \lambda^{n-k} = \mu^2, \tag{26}$$

This curve is identical with (23), with $m = 0$, and with $H_k := h_k, k = 1, \ldots, n$.

Second form of stationary KdV system

$$0 = K_{n+1} = \pi_1 \gamma_n \Rightarrow \frac{1}{2}\gamma_n(\gamma_n)_{xx} - \frac{1}{4}(\gamma_n)_x^2 + u\gamma_n^2 + 4\bar{c} = 0 \qquad (27)$$

Yields the alternative foliation

$$\mathcal{M}_n = \bigcup_{\bar{c}\in\mathbb{R}} \mathcal{M}_{n,\bar{c}}.$$

Second integrated form of the n-th stationary KdV system (20)

$$u_{t_1} = K_1, \quad u_{t_2} = K_2, \quad \ldots, \quad u_{t_n} = K_n, \qquad \frac{1}{2}\gamma_n(\gamma_n)_{xx} - \frac{1}{4}(\gamma_n)_x^2 + u\gamma_n^2 + 4\bar{c} = 0 \qquad (28)$$

on the $2n$-dimensional leaf $\bar{\mathcal{M}}_{n,\bar{c}}$ endowed with the same set of jet coordinates $u, u_x, \ldots, u_{(2n-1)x}$.

Miracle II

In theorem below we will parametrize this system as the Stäckel system

$$\lambda^{2n} + \bar{c}\lambda^{-1} + \sum_{k=1}^{n} \bar{H}_k \lambda^{n-k} = \lambda\mu^2, \qquad (29)$$

This curve is identical with (23), with $m = 1$, and with $\bar{H}_k := h_{k-1}, k = 1, \ldots, n$.

Stationary autonomous KdV systems are Stäckel systems

In (Błaszak, Szablikowski, Marciniak 2023) we proved

Beautiful theorem

The stationary systems (25) and (28) are equivalent to Stäckel systems generated by separation curves (26) and (29), respectively. The invertible transformation between Viète coordinates $(q_i, p_i)_{i=1,\dots,n}$ of Stäckel systems and jet coordinates $(u, u_x, \dots, u_{(2n-1)x})$ takes the form

$$q_i = \frac{1}{2}\gamma_i, \qquad p_i = \frac{1}{2}\sum_{j=1}^{n}\left(G_m^{-1}\right)_{ij}(\gamma_j)_x, \qquad i = 1,\dots,n, \qquad (m = 0, 1), \quad (30)$$

where G_m are contravariant metrics defined by H_1 and $\bar{H}_1$, respectively.

The map (30) has been derived from the comparison of Lax matrices of both representations, as we know U_{n+1} in jet coordinates, and in Viète coordinates

$$U_{n+1}[1,2] = \sum_{i=0}^{n} q_{n-i}\lambda^i, \qquad U_{n+1}[1,1] = -\frac{1}{2}\sum_{r=1}^{n}\left[\sum_{i=1}^{n}\left(G_m\right)^{ri}\, p_i\right]\lambda^{n-r}$$

Example 1

Autonomous constraint of KdV with π_0 and $n = 2$.

The system consists of two first KdV flows

$$u_{t_1} = K_1 \equiv u_x, \qquad u_{t_2} = K_2 \equiv \frac{1}{4} u_{xxx} + \frac{3}{2} u u_x,$$

and the autonomous constraint $\gamma_3 + c = 0$:

$$\frac{1}{16} u_{4x} + \frac{5}{8} u u_{xx} + \frac{5}{16} u_x^2 + \frac{5}{8} u^3 + c = 0.$$

It is Stäckel system generated by the separation curve

$$\lambda^5 + c\lambda^2 + H_1 \lambda + H_2 = \mu^2,$$

where in Viète coordinates (q, p)

$$H_1 = 2p_1 p_2 + q_1 p_2^2 - q_1^4 - q_2^2 + 3q_1^2 q_2 + c q_1,$$
$$H_2 = p_1^2 + (q_1^2 - q_2) p_2^2 + 2q_1 p_1 p_2 - q_1^3 q_2 + 2q_1 q_2^2 + c q_2.$$

Viète coordinates and separation coordinates λ_i are related by the following point transformation: $q_1 = -\lambda_1 - \lambda_2$, $q_2 = \lambda_1 \lambda_2$.

Example 1

The invertible transformation between jet coordinates and Viète's coordinates are

$$q_1 = \frac{1}{2}u, \quad q_2 = \frac{1}{8}u_{xx} + \frac{3}{8}u^2, \quad p_1 = \frac{1}{16}u_{xxx} + \frac{1}{4}uu_x, \quad p_2 = \frac{1}{4}u_x.$$

Lax representation

$$\frac{dL(\lambda;\xi)}{dt_k} = [U_k(\lambda;\xi), L(\lambda;\xi)], \qquad k = 1, 2,$$

where

$$L = \begin{pmatrix} -p_2\lambda - p_1 - q_1 p_2 & \lambda^2 + q_1\lambda + q_2 \\ \lambda^3 - q_1\lambda^2 + (q_1^2 - q_2)\lambda - p_2^2 - q_1^3 + 2q_1 q_2 + c & p_2\lambda + p_1 + q_1 p_2 \end{pmatrix},$$

$$U_1 = \begin{pmatrix} 0 & 1 \\ \lambda - 2q_1 & 0 \end{pmatrix}, \qquad U_2 = \begin{pmatrix} -p_2 & \lambda + q_1 \\ \lambda^2 - q_1\lambda + q_1^2 - 2q_2 & p_2 \end{pmatrix}.$$

Lecture 4

Non-autonomous restrictions of KdV hierarchy

Non-autonomous restrictions of KdV hierarchy

Definition

For a given $m \geq -1$, the n-th non-autonomous restricted system of the KdV hierarchy is the set of the first n systems from the KdV hierarchy, together with the non-autonomous constraint $\mathcal{K}_m = 0$:

$$u_{t_1} = K_1[u], \quad \dots, \quad u_{t_n} = K_n[u], \qquad \mathcal{K}_m = 0, \tag{31}$$

$$\mathcal{K}_m = \sigma_m + \sum_{j=1}^{N} \tau_j(t_1, \dots, t_n) K_j, \qquad N = \max\{n+1, n+m\}$$

with $\tau_j = \kappa_{j-m} t_{j-m} + \alpha_j, \quad j = 1, \dots, N.$

Frobenius condition

The above choice of τ_j is necessary and sufficient for the Frobenius condition for $K_1, \dots, K_n, \mathcal{K}$ to hold:

$$\frac{\partial \mathcal{K}}{\partial t_i} + [K_i, \mathcal{K}] = 0, \qquad i = 1, \dots, n,$$

The non-autonomous invariant manifold $\mathcal{S}$

By the Frobenius condition, the finite dimensional non-autonomous (i.e. times-dependent) submanifold $\mathcal{S} \subset \mathcal{M}$ defined by our times-dependent constraint

$$\mathcal{K}_m(t_1, \ldots, t_n, [u]) = 0,$$

is invariant with respect to all the vector fields K_i, $i = 1, \ldots, n$ in the sense that the trajectories of all K_i that start in $\mathcal{S}$ remain in $\mathcal{S}$, in spite of the fact that K_i are not tangent to $\mathcal{S}$.

Important remark

Using the constraint $\mathcal{K}_m = 0$ and its differential consequences we can eliminate higher-order jet coordinates from the fields K_i, which turns the first n equations $u_{t_i} = K_i[u]$ of the KdV hierarchy into a *non-autonomous* dynamical system

$$\xi_{t_1} = \mathcal{X}_1(\xi, \boldsymbol{t}), \quad \ldots, \quad \xi_{t_n} = \mathcal{X}_n(\xi, \boldsymbol{t}) \tag{32}$$

where $\boldsymbol{t} := (t_1, \ldots, t_n)$, defined on $\mathcal{S}$. Moreover, (32) is Frobenius integrable,

$$\frac{\partial \mathcal{X}_i}{\partial t_j} - \frac{\partial \mathcal{X}_j}{\partial t_i} + [\mathcal{X}_i, \mathcal{X}_j] = 0, \qquad i, j = 1, \ldots, n.$$

Lax formulation for non-autonomous constrained KdV systems

Lax pair for $u_\tau = \mathcal{K}_m$

$$\Psi_\tau = \mathcal{W}_m \Psi - \lambda^{m+1}\Psi_\lambda, \qquad \Psi_x = L\Psi, \qquad L \equiv U_1,$$

where

$$\mathcal{W}_m = V_m + \sum_{i=1}^{N} \tau_i U_i.$$

Compatibility condition

$$\left[\partial_\tau - \mathcal{W}_m + \lambda^{m+1}\partial_\lambda, L - \partial_x\right] = L_\tau - [\mathcal{W}_m, L] - (\mathcal{W}_m)_x + \lambda^{m+1} L_\lambda = 0$$

Lax equation for $u_\tau = \mathcal{K}_m$

$$L_\tau = [\mathcal{W}_m, L] + (\mathcal{W}_m)_x - \lambda^{m+1} L_\lambda \equiv L'[\mathcal{K}_m].$$

Lax representation for non-autonomous constrained KdV systems

Spectral representation for $\mathcal{K}_m = 0$

$$\mathcal{W}_m \Psi = \lambda^{m+1} \Psi_\lambda .$$

Lax pairs for the system (31)

$$\Psi_{t_i} = U_i \Psi, \qquad i = 1, ..., n,$$
$$\lambda^{m+1} \Psi_\lambda = \mathcal{W}_m \Psi .$$

Lax representation for non-autonomous constrained KdV systems

Isomonodromic Lax formulation for non-autonomous constrained system

The compatibility conditions $(\Psi_{t_i})_\lambda = (\Psi_\lambda)_{t_i}$ yield

$$\left[\partial_{t_i} - U_i, \mathcal{W}_m - \lambda^{m+1}\partial_\lambda\right] = (\mathcal{W}_m)_{t_i} - [U_i, \mathcal{W}_m] - \lambda^{m+1}(U_i)_\lambda = 0$$

and thus the isomonodromic Lax formulation for our non-autonomous constrained system on $\mathcal{S}$ is

$$(\mathcal{W}_m)_{t_i} = [U_i, \mathcal{W}_m] + \lambda^{m+1}(U_i)_\lambda, \quad i = 1, \ldots, n. \tag{33}$$

Remark

Note that equation i in (33) contains not only the corresponding equation $u_{t_i} = K_i$ of the reduced system but also the constraint $\mathcal{K}_m = 0$ itself.

The case σ_{-1}

- In this lecture I will only take up the case $m = -1$. Remind: $\sigma_{-1} = 1$.
- Then

$$\mathcal{K}_{-1} = \sigma_{-1} + \tau_1 K_1 + \ldots + \tau_{n+1} K_{n+1},$$

- $\mathcal{K}_{-1}$ is Hamiltonian with $\mathcal{K}_{-1} = \pi_0 \Gamma_{-1}$, with

$$\Gamma_{-1} = x + \tau_0 \gamma_0 + \ldots + \tau_{n+1} \gamma_{n+1} \equiv x + \tilde{\gamma}_{n+1}.$$

- τ_j specify to

$$\tau_0 = \frac{1}{2} t_1 + \alpha_0, \quad \tau_1 = \frac{3}{2} t_2 + \alpha_1, \quad \ldots, \quad \tau_{n-1} = \left(n - \frac{1}{2}\right) t_n + \alpha_{n-1}, \quad \tau_n = \alpha_n,$$

Since $\pi_0 \equiv \partial_x$, the constraint

$$\Gamma_{-1} + c \equiv x + \tilde{\gamma}_{n+1} + c = 0 \tag{34}$$

yields a co-rank one Hamiltonian foliation of $\mathcal{S}$ into the leaves $\mathcal{S}_c$ given by (34), so that

$$\mathcal{S} = \bigcup_c \mathcal{S}_c,$$

where each leaf $\mathcal{S}_c$ is invariant with respect to all K_i.

We will now attempt to restrict the first n flows of the KdV hierarchy to leaves $\mathcal{S}_c$.

The integrated non-autonomous KdV system

Definition

For any given $c \in \boldsymbol{R}$, the n-th non-autonomous system of the KdV hierarchy restricted to the leaf $\mathcal{S}_c$ is the set of the first n systems from the KdV together with the non-autonomous constraint (34), that is, the system of the form

$$u_{t_1} = K_1[u], \quad \ldots, \quad u_{t_n} = K_n[u], \qquad \Gamma_{-1} + c \equiv x + \tilde{\gamma}_{n+1} + c = 0. \qquad (35)$$

Remark

Using the constraint above and its differential consequences we can eliminate higher order jet coordinates from $K_i[u]$ which turns the first n equations in (35) into a non-autonomous Frobenius inegrable finite dimensional system on $\mathcal{S}_c$:

$$\xi_{t_1} = \mathbb{X}_1(\xi), \quad \ldots, \quad \xi_{t_n} = \mathbb{X}_n(\xi), \quad \xi \in \mathcal{S}_c \qquad (36)$$

$$\frac{\partial \mathbb{X}_i}{\partial t_j} - \frac{\partial \mathbb{X}_j}{\partial t_i} + [\mathbb{X}_i, \mathbb{X}_j] = 0, \qquad i, j = 1, \ldots, n.$$

We will sometimes denote the vector fields $\mathbb{X}_i$ on $\mathcal{S}_c$ obtained by this elimination procedure as $K_i|_{\mathcal{S}_c}$, so that $\mathbb{X}_i = K_i|_{\mathcal{S}_c}$.

Lax formulation of the restricted KdV system on $\mathcal{S}_c$

$$\mathbb{L}_{t_i} = [\mathbb{U}_i, \mathbb{L}] + \lambda^{m+1}(\mathbb{U}_i)_\lambda, \quad i = 1, \ldots, n. \tag{37}$$

where

$$\mathbb{L} \equiv \mathcal{W}_m|_{\mathcal{S}_c}, \qquad \mathbb{U}_k \equiv U_k|_{\mathcal{S}_c} \qquad k = 1, \ldots, n.$$

Towards the non-autonomous version of the beautiful theorem

We will now parametrize the system (36) as a Painlevé-type system. We will need:

Auxiliary Stäckel system

$$\sum_{k=0}^{n+1} a_k \lambda^{n+k} + \sum_{k=1}^{n} H_k \lambda^{n-k} = \mu^2. \tag{38}$$

$$a_0 = \sum_{i=0}^{n+1} \tau_i \tau_{n+1-i} + \tau_{n+1}(x+c), \qquad a_k = \sum_{i=k}^{n+1} \tau_i \tau_{n+1+k-i}, \qquad 1 \le k \le n+1,$$

The Hamiltonians of this system, given canonical Viète coordinates

$$H_k = \frac{1}{2} \boldsymbol{p}^T A_k G_0 \boldsymbol{p} + \sum_{i=0}^{n+1} a_i V_k^{(n+i)}, \qquad k = 1, \ldots, n.$$

Here G_0 is a Stäckel metric, A_k are its Killing tensors and $V_k^{(r)}$, $r \in \boldsymbol{Z}$, are basic separable potentials. The geodesic term in H_k will be further denoted by E_k.

The non-autonomous beautiful theorem

On the leaf $\mathcal{S}_c$, the transformation given by

$$q_i = \frac{1}{2}\tilde{\gamma}_i, \qquad p_i = \frac{1}{2}\sum_{j=1}^{n}\left(G_0^{-1}\right)_{ij}(\tilde{\gamma}_j)_x, \qquad i = 1, \ldots, n,$$

between the jet variables and the canonical Viète coordinates, transforms the Frobenius integrable system (36) into the Painlevé-type system with $\mathbb{H}_k$ given by:

$$\mathbb{H}_k = \sum_{i=0}^{k-1} c_i(\boldsymbol{t})\left(H_{k-i} + w_{k-i}\right), \qquad k = 1, \ldots, n, \tag{39}$$

where H_k are the Hamiltonians of the auxiliary Stäckel systems (38),

$$c_0(\boldsymbol{t}) = 1, \qquad c_k(\boldsymbol{t}) = \sum_{r=0}^{k-1}(-1)^{r+1}\sum_{i_1+\ldots+i_r=k}\tau_{n+1-i_1}\tau_{n+1-i_2}\cdots\tau_{n+1-i_r},$$

and

$$w_k = \sum_{i=n-k+2}^{n}(n-i+1)q_{i+k-n-2}p_i \equiv \boldsymbol{Y}_k^T\boldsymbol{p},$$

with the vector fields $\boldsymbol{Y}_k$ being particular Killing vectors of the metric G_0.

The non-autonomous beautiful theorem continued

The isonomonodromic Lax representation of this Painlevé-type system is

$$\frac{d\mathbb{L}}{dt_k} = [\mathbb{U}_k, \mathbb{L}] + (\mathbb{U}_k)_\lambda, \quad k = 1, \ldots, n.$$

where $\mathbb{L}$ and $\mathbb{U}_k$ are reductions of the isomonodromic Lax representation of the restricted KdV system to $\mathcal{S}_c$ (see (37)).

Remark

The theorem above means that the system (36) is Hamiltonian, with the Hamiltonians $\mathbb{H}_i$ given by (39), and also Frobenius integrable, meaning

$$\frac{\partial X_{\mathbb{H}_i}}{\partial t_j} - \frac{\partial X_{\mathbb{H}_j}}{\partial t_i} + \left[X_{\mathbb{H}_i}, X_{\mathbb{H}_j}\right] = 0, \qquad i, j = 1, \ldots, n.$$

where $X_{\mathbb{H}_i} = \mathcal{X}_i|_{\mathcal{S}_c} = \mathbb{X}_i$, which also implies

$$\frac{\partial \mathbb{H}_i}{\partial t_j} - \frac{\partial \mathbb{H}_j}{\partial t_i} + \{\mathbb{H}_i, \mathbb{H}_j\} = f_{ij}(\boldsymbol{t}, x), \qquad i, j = 1, \ldots, n. \tag{40}$$

Note that f_{ij} do not depend on the space variables.

Example 2

The second $(n = 2)$ constrained KdV system is: $u_{t_1} = u_x, u_{t_2} = \frac{1}{4} u_{3x} + \frac{3}{2} u\, u_x$ and

$$0 = \Gamma_{-1} + c = x + t_1 + c + \frac{3}{2} t_2 u_x + \frac{1}{16} u_{4x} + \frac{5}{8} u u_{xx} + \frac{5}{16} u_x^2 + \frac{5}{8} u^3.$$

In the (q, p) representation

$$q_1 = \frac{1}{2} u, \quad q_2 = \frac{1}{8} u_{xx} + \frac{3}{8} u^2 + \frac{3}{2} t_2, \;\; p_1 = \frac{1}{16} u_{3x} + \frac{1}{4} u\, u_x, \quad p_2 = \frac{1}{4} u_x,$$

the isomonodromic Lax representation of this system takes the form

$$(\mathbb{L})_{t_k} = [\mathbb{U}_k, \mathbb{L}] + \frac{\partial}{\partial \lambda} \mathbb{U}_k, \qquad k = 1, 2,$$

where

$$\mathbb{U}_1 = \begin{pmatrix} 0 & 1 \\ \lambda - 2q_1 & 0 \end{pmatrix}, \quad \mathbb{U}_2 = \begin{pmatrix} -p_2 & \lambda + q_1 \\ \lambda^2 - q_1 \lambda - q_1^2 - 2q_2 + 3t_2 & p_2 \end{pmatrix}$$

$$\mathbb{L} = \begin{pmatrix} -p_2 \lambda - q_1 p_2 - p_1 & \lambda^2 + q_1 \lambda + q_2 \\ \begin{matrix} \lambda^3 - q_1 \lambda^2 + (q_1^2 - 2q_2 + 3t_2)\lambda \\ -p_2^2 - q_1^3 + 2q_1 q_2 - 3t_2 q_1 + x + t_1 + c \end{matrix} & p_2 \lambda + q_1 p_2 + p_1 \end{pmatrix}.$$

The auxiliary Stäckel system takes the form

$$\det(\mathbb{L} - \mu I) = 0 \Longleftrightarrow \ \lambda^5 + 3t_2\lambda^3 + (t + c)\lambda^2 + H_1\lambda + H_2 = \mu^2,$$

so that

$$H_1 = 2p_1p_2 + q_1p_2^2 - q_1^4 + 3q_1^2q_2 - q_2^2 + 3t_2(q_2 - q_1^2) + (x + c + t_1)q_1,$$
$$H_2 = p_1^2 + 2q_1p_1p_2 + (q_1^2 - q_2)p_2^2 - q_1^3q_2 + 2q_1q_2^2 - 3t_2q_1q_2 + (x + c + t_1)q_2.$$

The Hamiltonian $\mathbb{H}_i$ are

$$\mathbb{H}_1 = H_1, \quad \mathbb{H}_2 = H_2 + w_2, \quad w_2 = p_2.$$

The Frobenius equation (40) becomes

$$\frac{\partial \mathbb{H}_1}{\partial t_2} - \frac{\partial \mathbb{H}_2}{\partial t_1} + \{\mathbb{H}_1, \mathbb{H}_2\} = 3t_2.$$

The system above can be considered as the second system from the P_I hierarchy. Taking higher n we obtain higher flows of P_I hierarchy.

Painlevé-type hierarchies constructed by two different methods

One can ask the question whether Painlevé-type hierarchies, such as $P_{34}, P_I - P_{IV}$ constructed by non-autonomous restrictions of soliton systems and by deformation of Stäckel systems coincide?

The answer is positive but both constructions use different evolution parameters. If we denote evolution parameters of the n-th Painlevé system, constructed by deformation, by $(\bar{t}_1, ..., \bar{t}_n)$ and evolution parameters of the same Painlevé system, constructed by appropriate non-autonomous restriction of soliton hierarchy, by $(t_1, ..., t_n)$, then both sets of evolution parameters are related by the non-linear but invertible transformation

$$t_1 = \bar{t}_1 + f_{n,1}(\bar{t}_2, ..., \bar{t}_n), \ldots, t_i = \bar{t}_i + f_{n,i}(\bar{t}_{i+1}, ..., \bar{t}_n), \ldots, t_n = \bar{t}_n$$

for the case with σ_0. For the case σ_{-1} additionally $f_{n,n-1}(\bar{t}_n) = 0$.

We have reached our conclusion:

The following relations can be demonstrated

soliton hierarchies

autonomous constraints ↙ ↘ non-autonomous constraints

Stäckel systems $\xrightarrow{\text{deformation}}$ *Painlevé-type systems*

Moreover, this diagram commutes

Conclusions and perspectives

- During these lectures I attempted to present a new emerging perspective on the relationships between three major classes of integrable systems: classical separable (in the sense of Hamilton–Jacobi theory) Stäckel systems, soliton hierarchies, and Painlevé type systems.
- We have constructed explicit (and non-autonomous i.e. times-dependent) maps between these three types of systems
- We have created these maps using Lax formulations of the systems considered, in order to get unique (one to one) maps (comparing spectral curves only would give too few conditions on the maps)
- This research hopefully opens a new perspective on Painlevé-type systems, especially in the context of their hierarchies.
- Taking higher σ_m is an open problem. At the moment we have succeeded with constructing our maps for $m = 0$ as well.
- A nest step would be to perform similar program for multi-component soliton hierarchies (cKdV, cHD, AKNS,...)

Thank you for your attention