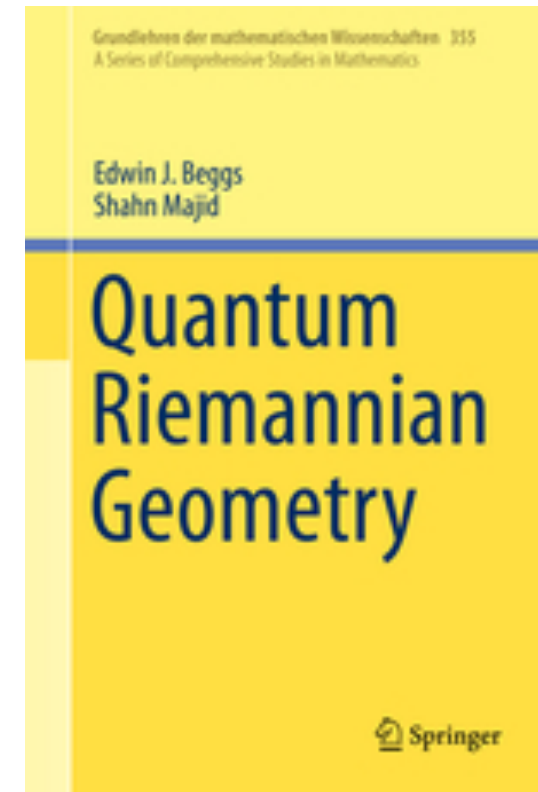


Quantum and classical geodesic flows on a black-hole background

SHAHN MAJID (QMUL)

BIALYSTOK 26

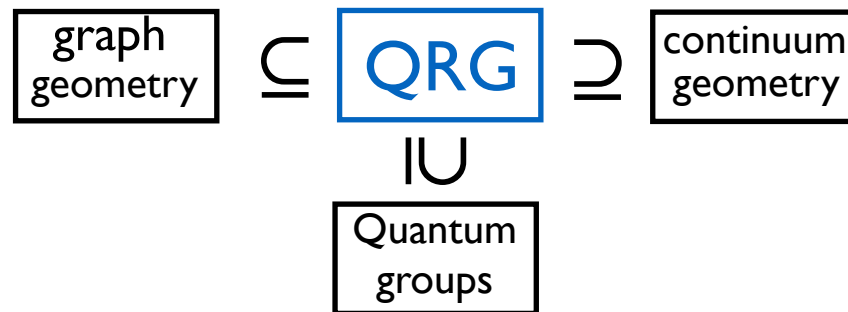


w/ Beggs LMP 2023

- What is quantum Riemannian geometry?

$$g \in \Omega^1 \otimes_A \Omega^1, \quad \nabla : \Omega^1 \rightarrow \Omega^1 \otimes_A \Omega^1$$

Leibniz rule $\rightarrow \sigma : \Omega^1 \otimes_A \Omega^1 \rightarrow \Omega^1 \otimes_A \Omega^1$



- Classical geodesics as velocity fields and Ricci tensor

$\rightarrow$ Amplitude flows around a black hole w/ Kumar arXiv:2603.03222, arXiv:2605.28582

- Quantum geodesics

Beggs JGP 2020

$\rightarrow$ Generally covariant quantum mechanics w/ Beggs LMP 2026

- A-B-bimodule connections and the master equation $\nabla(\sigma_E) = 0$

I Classical geodesic flows

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- dust particles moving on geodesics $\longrightarrow$ tangents define vector field X_s obeying *geodesic velocity equation*

$$\dot{X}_s + \nabla_{X_s} X_s = 0$$

density ρ obeys *continuity equation*

we reverse usual concept
and first solve for this X_s
for flow to time s

$$\dot{\rho} = -X_s(d\rho) - \rho \operatorname{div}(X_s)$$

- Let $\rho = |\psi|^2$ for a wave function ψ obeying the *amplitude flow equation*

$$\dot{\psi} = -X_s(d\psi) - \frac{1}{2}\psi \operatorname{div}(X_s)$$

$\longrightarrow$ *Convective derivative* $\frac{D}{Ds} := \frac{\partial}{\partial s} + X_s$ of the divergence is the Ricci tensor

$$\frac{D \operatorname{div}(X_s)}{Ds} = -X^j_{;i} X^i_{;j} - X^i X^j \operatorname{Ricci}_{ij}$$

- Wave function $\psi(x, t)$ on spacetime, s is external geodesic proper time

Classical geodesic flows around a Schwarzschild black-hole

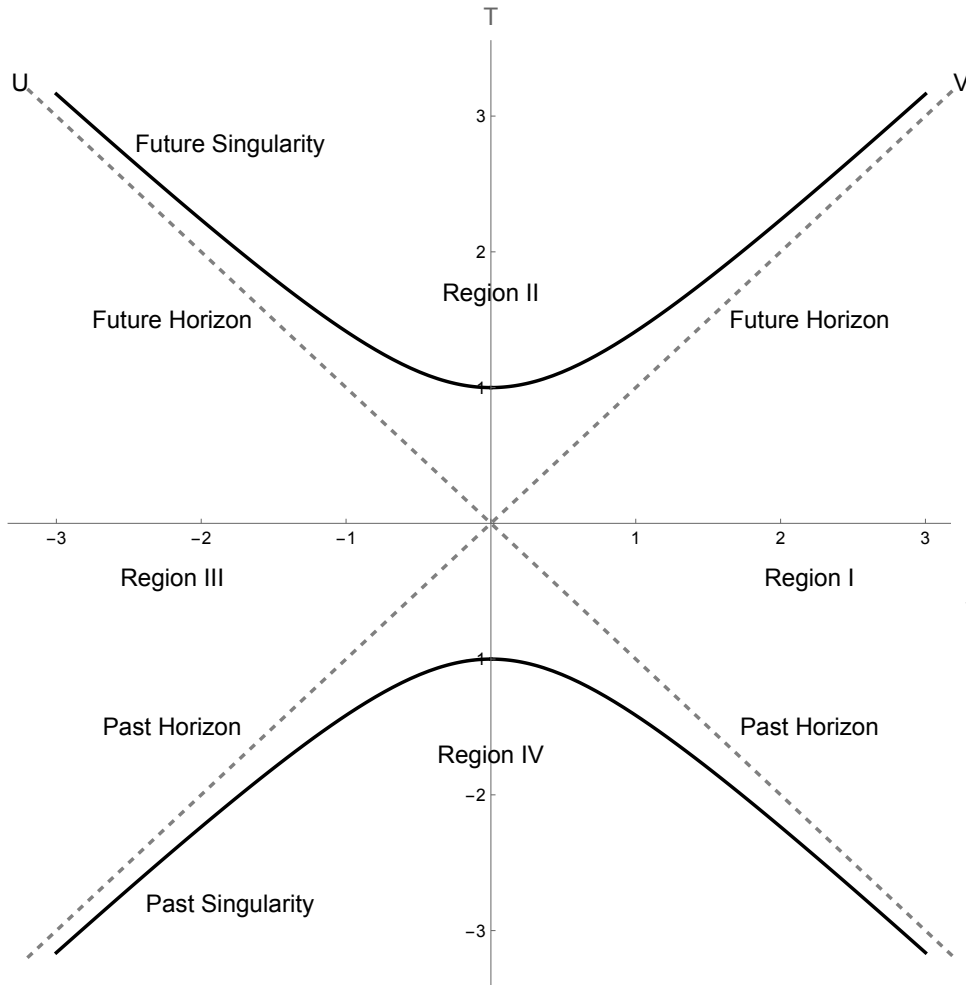
w/ Kumar arXiv:2603.03222

$$ds^2 = - \left(1 - \frac{r_s}{r}\right) dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 d\Omega^2,$$

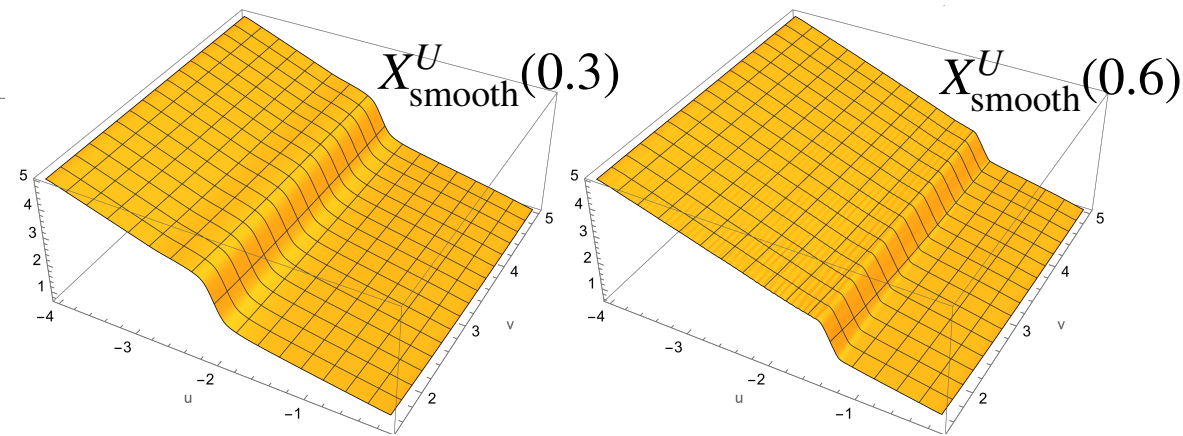
● Kruskal-Szekeres UV coordinates

$$UV = - \left(\frac{r}{r_s} - 1 \right) e^{r/r_s}, \quad \frac{U}{V} = - \exp \left(- \frac{t}{r_s} \right)$$

$$ds^2 = \frac{4r_s^3}{r} e^{-r/r_s} (-dUdV) + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$



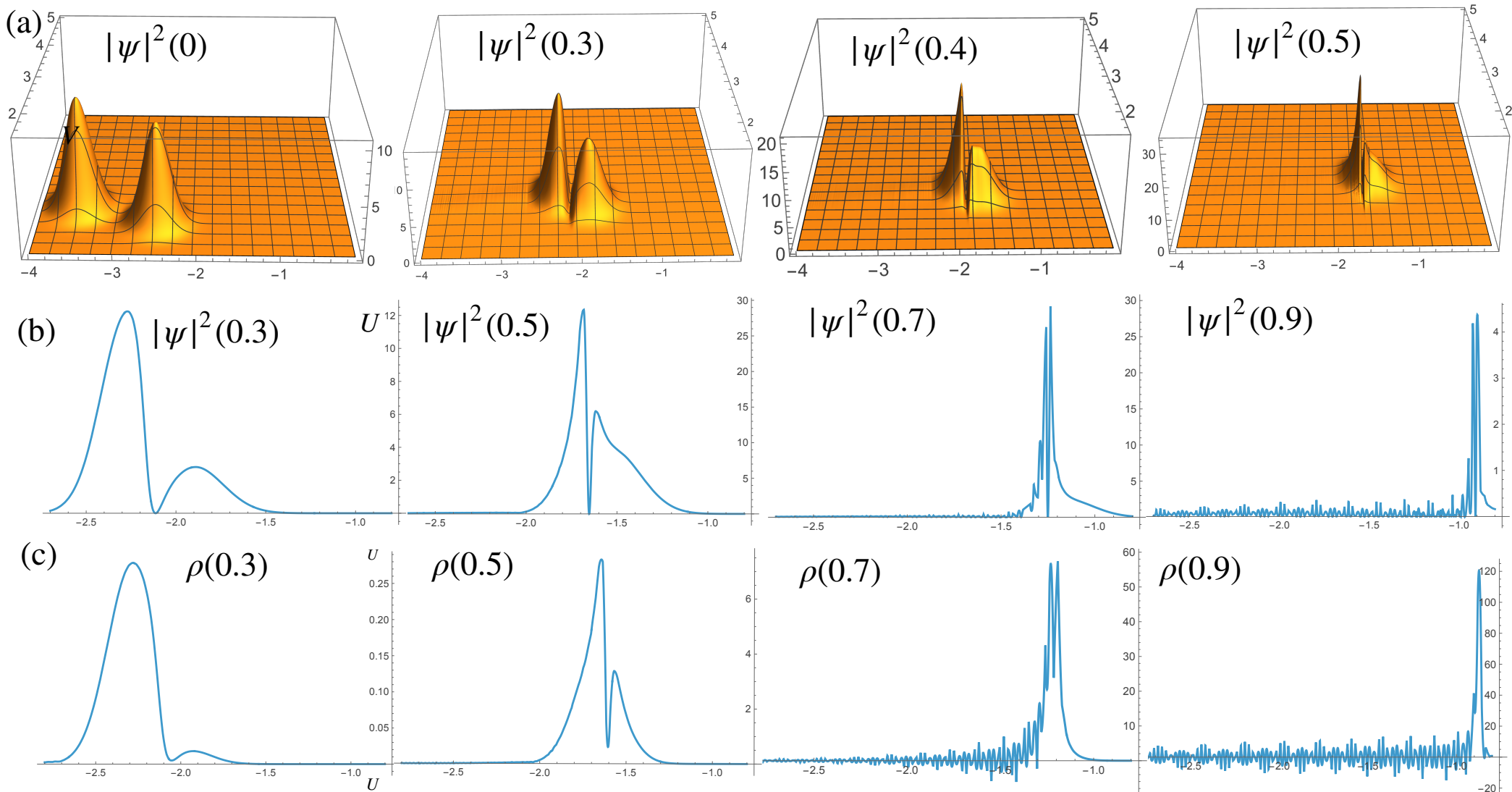
● Choose $X(0) = X^U \partial_U + X^V \partial_V$ and evolve



This will be used to sweep one Gaussian bump onto another so they collide

● ψ_1, ψ_2 initial Gaussian bumps in the UV plane

● $\psi(0) = \psi_1 - \psi_2$ evolves to a dipole in (a) and cross-section (b)



● $\rho(0) = |\psi_1|^2 + |\psi_2|^2$ evolves in (c) to a merged bump

- Ω^1 space of differential 1-forms for algebra A

drop the (graded) commutativity but keep:

$$\Omega^1$$

$$a((db)c) = (a(db))c$$

bimodule

$$d : A \rightarrow \Omega^1$$

$$d(ab) = (da)b + a(db)$$

Leibniz rule

$$\left\{ \sum a db \right\} = \Omega^1$$

surjectivity

- Extend to DGA of differential forms (generated by A, dA)

$$\Omega = T_A \Omega^1 / \mathcal{I} = \bigoplus_n \Omega^n, \quad d^2 = 0$$

- *-DGA antilinear graded-involution commutes with d

- $\int : A \rightarrow \mathbb{C}$ non-deg $\rightarrow \text{div}_f : \chi = {}_A \text{Hom}(\Omega^1, A) \rightarrow A$

positive linear functional

$$\int (a \text{div}_f(X) + X(da)) = 0 \quad \forall a \in A,$$

Metric

$$g \in \Omega^1 \otimes_A \Omega^1$$

$$ds^2 = \sum_{\mu, \nu} g_{\mu\nu} dx^\mu dx^\nu$$

w/ Beggs book

with bimodule map inverse

$$(\ , \) : \Omega^1 \otimes_A \Omega^1 \rightarrow A$$

Connections

Dubois-Violette & Michor JGP 1996

$$\nabla : \Omega^1 \rightarrow \Omega^1 \otimes_A \Omega^1$$

$$\nabla(f\omega) = df \otimes \omega + f\nabla\omega$$

$$\forall f \in A$$

$$\omega \in \Omega^1$$

$$\sigma : \Omega^1 \otimes_A \Omega^1 \rightarrow \Omega^1 \otimes_A \Omega^1$$

$$\nabla(\omega f) = \sigma(\omega \otimes df) + (\nabla\omega)f$$

bimodule connections extend to tensor products

$$\nabla(\omega \otimes \eta) = \nabla\omega \otimes \eta + (\sigma \otimes \text{id})(\omega \otimes \nabla\eta)$$

→ metric compatibility

● $T_\nabla : \Omega^1 \rightarrow \Omega^2$ $T_\nabla = \wedge \nabla - d$ torsion tensor

Quantum Levi-Civita connection $T_\nabla = \nabla g = 0$

$$\Rightarrow \nabla_\chi : \chi \rightarrow \chi \otimes_A \Omega^1$$

right conn on vect fields

$$\text{Ex } A = M_2(\mathbb{C})$$

$$\Omega = M_2(\mathbb{C})[s, t] / \langle s^2, t^2 \rangle$$

$$ds = 2E_{21}st, \quad dt = 2E_{12}st$$

$$g = s \otimes s + t \otimes t$$

$$s^* = -t, \quad s, t \in \Omega^1$$

$$f_s, f_t \text{ dual basis of } \chi$$

$$\nabla_{\mathfrak{X}} f_s = \rho E_{21} f_s \otimes s - (\rho E_{12} + 2 E_{21}) f_s \otimes t + \rho E_{12} f_t \otimes s - \rho E_{21} f_t \otimes t$$

$$\sigma_{\mathfrak{X}}(s \otimes f_s) = f_s \otimes s + \rho(f_t \otimes s - f_s \otimes t)$$

Example of 1-param QRG $\rho \in i\mathbb{R}$

etc.

Quantum geodesic equations

$$\kappa_s = \frac{1}{2} \text{div}_{\mathfrak{f}}(X_s)$$

flow divergence

$$\dot{X}_s + [X_s, \kappa_s] + (\text{id} \otimes X_s) \nabla_{\chi}(X_s) = 0$$

velocity flow

$$\dot{\psi}_s = -\psi_s \kappa_s - X_s(d\psi_s)$$

amplitude flow

$$X_s \in \chi$$

$$\kappa_s, \psi_s \in A$$

$$s \in \mathbb{R}$$

proper time

Beggs JGP 2020

w/ Beggs JMP 2024

● Need $\int X_s(\omega^*) - X_s(\omega)^* = 0 \quad \forall \omega \in \Omega^1$

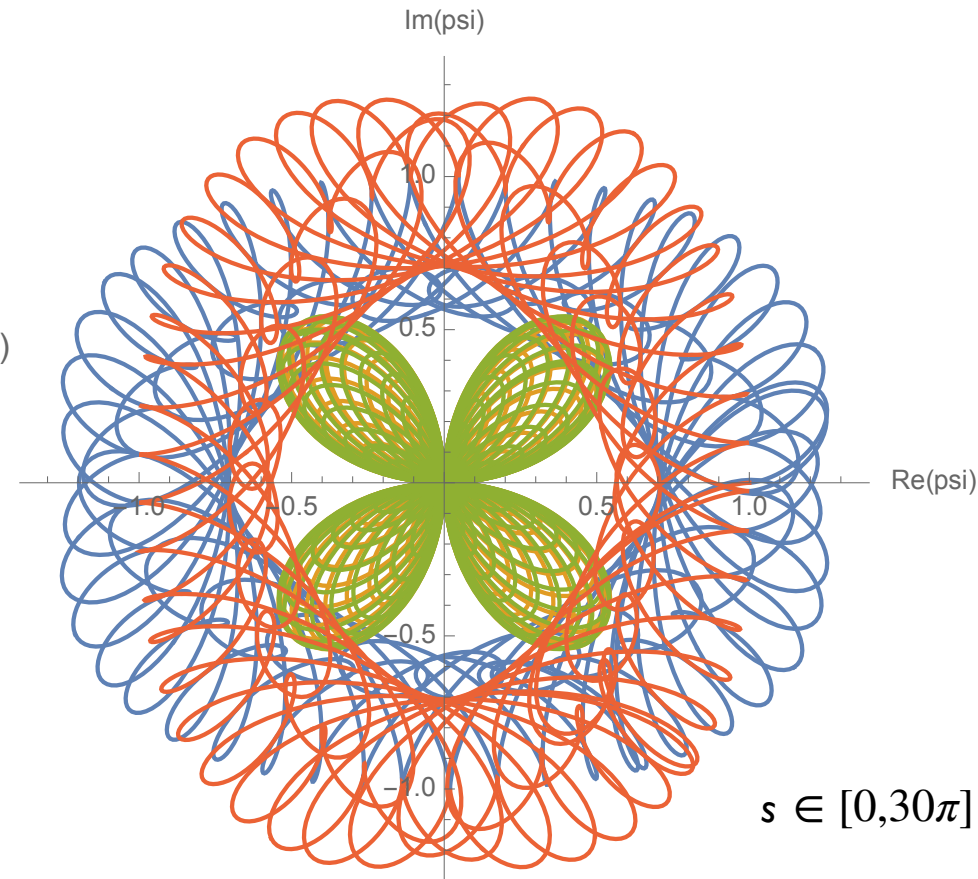
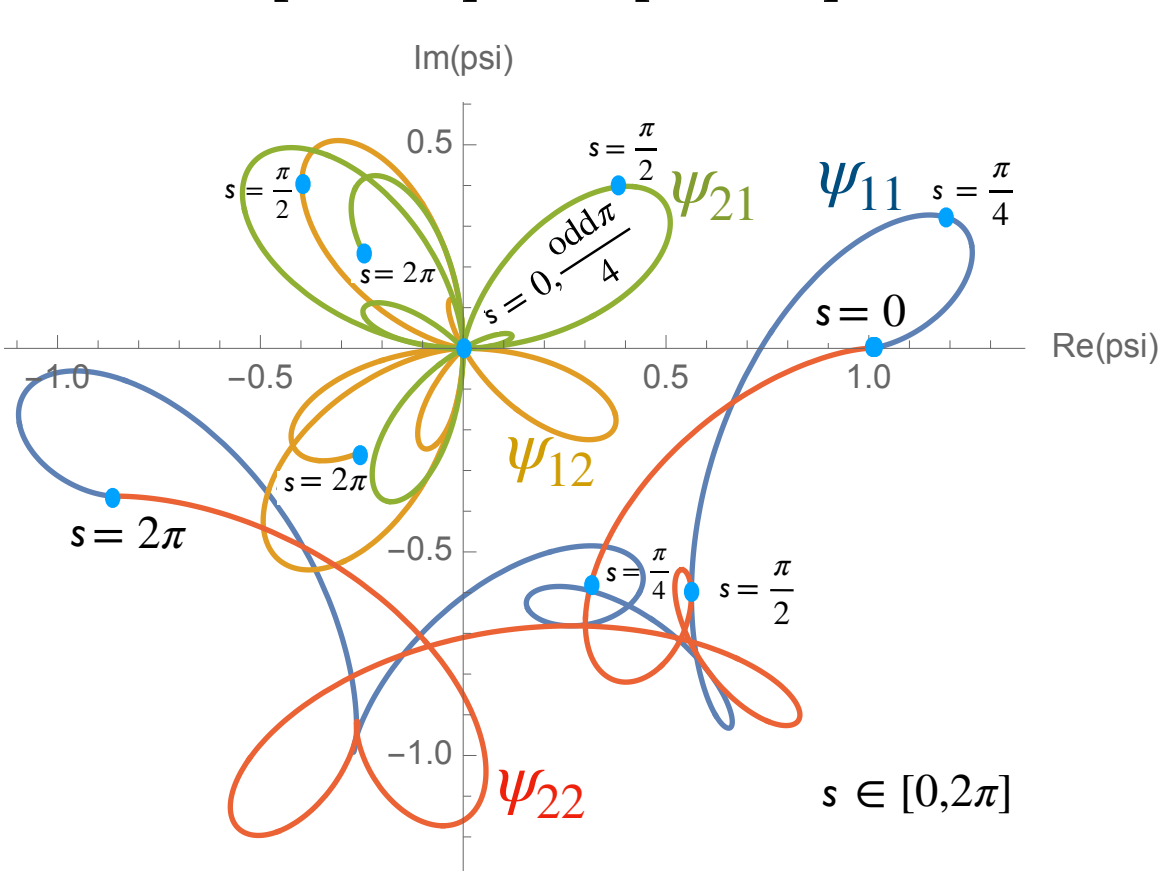
X_s 'real' with respect to $\int$

Lemma Then $\int \psi_s^* \psi_s$ constant in s

➡ probabilistic picture

$$\langle \mathcal{O} \rangle_{\psi_s} = \frac{\int \psi_s^* \mathcal{O} \psi_s}{\int \psi_s^* \psi_s}$$

- $\dot{X}^s = \frac{1}{2} \left[([E_{12}, X^s] + [E_{21}, X^t]), X^s \right] - [E_{12}, X^s] X^s - [E_{21}, X^s] X^t$
 $\dot{X}^t = \dots - (E_{21} X^s X^s \rho - (2E_{21} + \rho E_{12}) X^s X^t + E_{12} X^t X^s \rho - E_{21} X^t X^t \rho)$
- $\dot{\psi} = -[E_{12}, \psi] X^s - [E_{21}, \psi] X^t - \psi \kappa.$



$\psi(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ does not stay real.

- (M, g) a (pseudo) Riemannian manifold

$A=D(M)$ algebra of differential operators gen by $C^\infty(M)$, $\text{Vect}(M)$

Local coordinates $\{x^\mu, p_\nu = \partial_\nu\} \longrightarrow$ Heisenberg algebra

$$[x^\mu, p_\nu] = \lambda \delta^\mu_\nu, \lambda = -i\hbar$$

- $\mathfrak{h} = \frac{1}{2m}(g^{\mu\nu} p_\mu p_\nu - \lambda \Gamma^\mu p_\mu) + V(x) \in D(M)$

$\longrightarrow \Omega_{D(M)}$ nonassociative at order λ^2 in presence of curvature

$$[X, \widehat{\xi}] = \lambda (\widehat{\nabla_X \xi}) - \frac{\lambda}{m} \theta' (g^{\mu\nu} \xi_\mu \nabla_\nu X) - \frac{\lambda^2}{2m} \theta' (X^\rho \xi_\mu g^{\mu\nu} R_{\nu\rho} + g^{\mu\nu} X^\rho{}_{;\nu} \xi_{\mu;\rho})$$

$\widehat{\xi} \in \Omega^1_{D(M)}$ corresp to $\xi \in \Omega^1(M)$

- $\rho : \Omega_{D(M)} \rightarrow \text{Lin}(L^2(M))$ Schrödinger repn

$$dx^\mu - \frac{\theta'}{m} \left(g^{\mu\nu} p_\nu - \frac{\lambda}{2} \Gamma^\mu \right),$$

$$dp_\mu - \frac{\theta'}{m} \left(\Gamma^\nu{}_{\mu\sigma} g^{\sigma\rho} (p_\nu p_\rho - \lambda \Gamma^\tau{}_{\nu\rho} p_\tau) + \frac{\lambda}{2} g^{\alpha\beta} \Gamma^\nu{}_{\beta\alpha,\mu} p_\nu - V_{,\mu} \right) \in \ker \rho$$

Jacobiator factors through $\ker \rho$

Generally covariant quantum mechanics (ctd)

- $\theta' = ds$
 $V = 0 \quad \rightarrow \quad \frac{d^2 x^\mu}{ds^2} = -\Gamma^\mu_{\nu\rho} \frac{dx^\nu}{ds} \frac{dx^\rho}{ds} + \frac{\lambda}{2m} C^\mu_\nu \frac{dx^\nu}{ds} + O(\lambda^2)$

$$C^{\mu\nu} = -g^{\alpha\beta} (g^{\mu\gamma} \Gamma^\nu_{\gamma\alpha,\beta} + g^{\nu\gamma} \Gamma^\mu_{\gamma\alpha,\beta}) + g^{\mu j} \Gamma^\nu_{;\beta} - g^{\nu\beta} \Gamma^\mu_{;\beta} + \Gamma^{\alpha\beta\mu} \Gamma^\nu_{\alpha\beta} - \Gamma^{\alpha\beta\nu} \Gamma^\mu_{\alpha\beta}$$

‘Heisenberg picture’ geodesic equations arise to maintain associativity

- quantum geodesic flow on $D(M)$ at least in flat spacetime case

(w/ QRG in the curved case partly known)

w/ Beggs JMP 2024

→ ‘Schrödinger picture’ Klein-Gordon flow with potential $V \in C^\infty(M)$

$$-\lambda \frac{\partial \varphi}{\partial s} = \left(-\frac{\hbar^2}{2m} \square + V \right) \varphi \quad \varphi \in L^2(M) \text{ again ‘wave fn’ on spacetime}$$

(or smooth version in practice)

Klein-Gordon fields = stationary modes for KG flow



$$-\lambda \frac{\partial \psi}{\partial s} = \left(-\frac{\hbar^2}{2m} \Delta + V_{eff}\right) \psi$$

$$\varphi_s = e^{\frac{p_t t}{\lambda}} \psi(r, \theta, \phi)$$

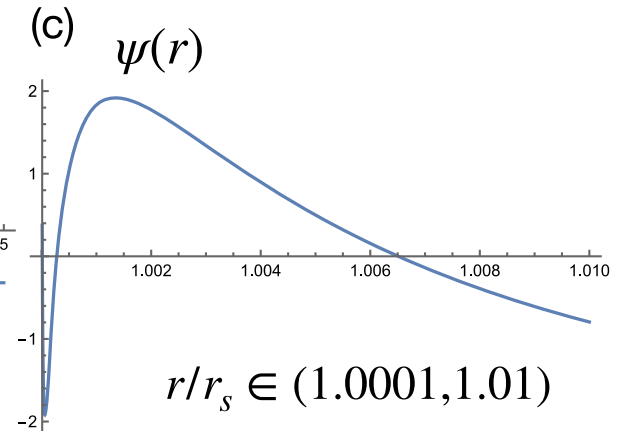
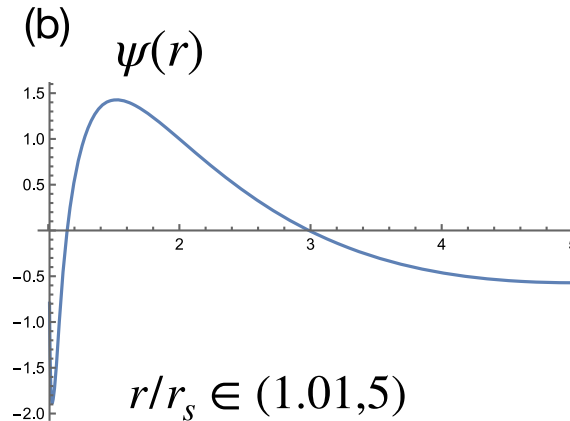
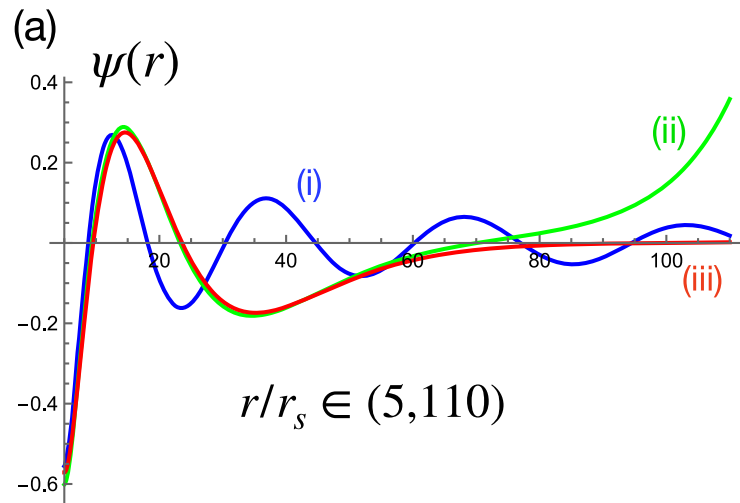
$$V_{eff}(r) := -\left(1 - \frac{r_s}{r}\right)^{-1} \frac{p_t^2}{2m}$$

- **atom-like** stationary states = certain solutions of the KG equations

$$\left(-\frac{\hbar^2}{2m} \Delta + V_{eff}\right) \psi = E_{KG} \psi$$

(i) For $E_{KG} \geq -\frac{p_t^2}{2m}$, $\psi(r) \sim \frac{\sin(\alpha r + \beta)}{r}$
 (ii) For $E_{KG} < -\frac{p_t^2}{2m}$, $\psi(r) \sim \frac{e^{\alpha r}}{r}$

(iii) like (ii) but bounded



fractal behaviour approaching the horizon

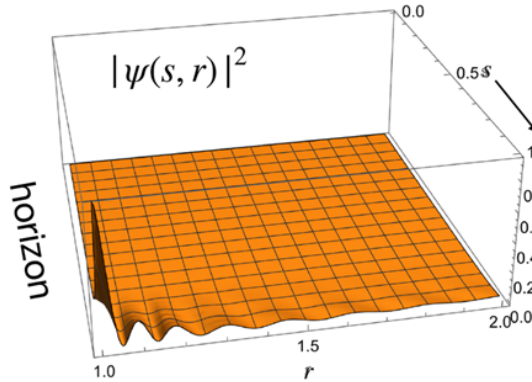
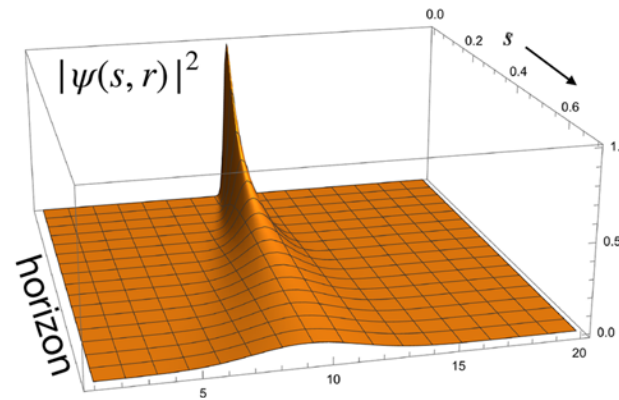
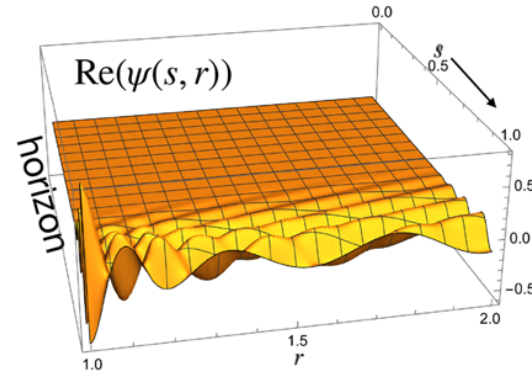
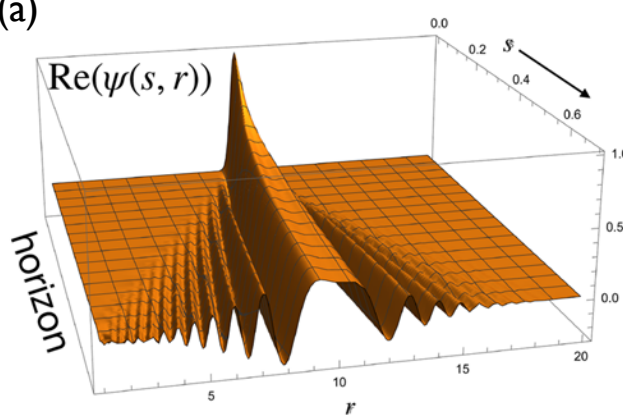
- Also atomic modes inside, with discrete spectrum for Neumann at $r = 0$

Black-hole KG flow

horizon located at $r_S = 1$

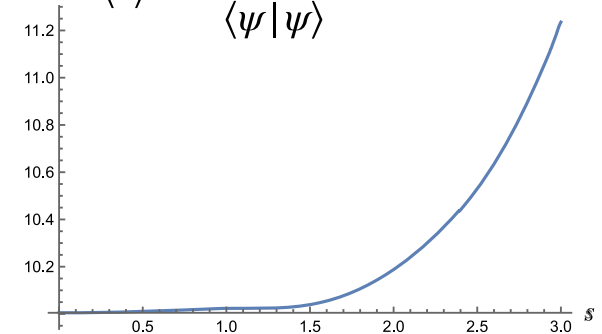
Initial Gaussian bump at $r = 10$ gets swallowed by black-hole releasing 'horizon modes'

(a)

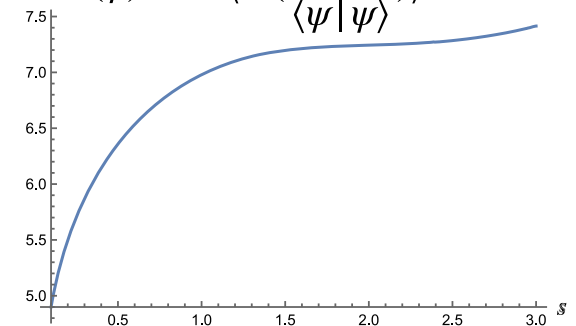


During this process,
 $\langle r \rangle$ and entropy increase

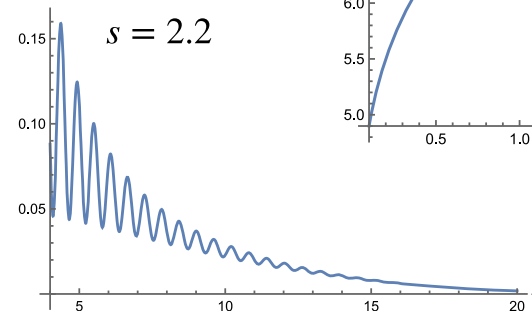
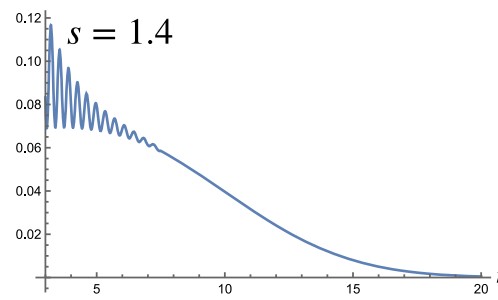
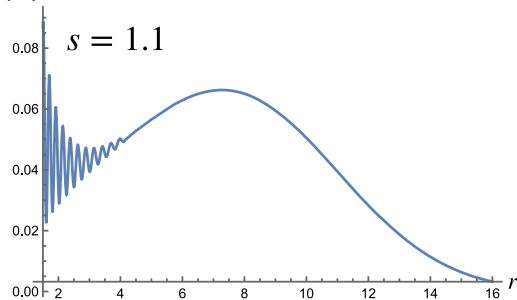
$$(c) \quad \langle r \rangle = \frac{\langle \psi | r | \psi \rangle}{\langle \psi | \psi \rangle}$$



$$(d) \quad S(\psi) = -\langle \ln\left(\frac{|\psi|^2}{\langle \psi | \psi \rangle}\right) \rangle$$



(b) $|\psi(s, r)|^2$



- Eikonal/WKB ansatz $\varphi(s, x) = e^{\frac{iS(s, x)}{\hbar}} \psi(s, x)$

ψ cx, slowly varying
 S real, fast varying

$$\lambda \frac{\partial \varphi}{\partial s} = \frac{\hbar^2}{2m} \square \varphi \quad \longrightarrow \quad \psi \left(\dot{S} + \frac{1}{2m} (\nabla S)^2 \right) = \cancel{\frac{\hbar^2}{2m} \square \psi} + i\hbar \left(\dot{\psi} + \frac{1}{m} \nabla^\mu S \nabla_\mu \psi + \frac{1}{2m} (\square S) \psi \right)$$

$$(1) \quad \dot{S} + \frac{1}{2m} (\nabla S)^2 = 0 \quad \longrightarrow \quad \dot{X}^\nu + X^\mu \nabla_\mu X^\nu = 0 \quad \quad X^\mu := \frac{1}{m} \nabla^\mu S$$

$$(2) \quad \dot{\psi} + X^\mu \nabla_\mu \psi + \frac{1}{2} (\nabla_\mu X^\mu) \psi = 0$$

So KG flow $\longrightarrow$ classical quantum geodesic flow

- $\frac{D \text{div}(X)}{Ds} = \dots$ in context reduces to the Raychaudhuri eqns
- Aharonov-Bohm phase effect for motion past a black-hole
- s arises as the proper time of any 'particle' in the flow

III Master equation

A-B bimodule connections

Beggs JGP 2020
w/ Beggs book

$$\nabla_E : E \rightarrow E \otimes_B \Omega_B^1$$

$$\sigma_E : \Omega^1 \otimes_A E \rightarrow E \otimes_B \Omega_B^1$$

$$\nabla_E(eb) = e \otimes db + (\nabla_E a)b$$

$$\nabla_E(ae) = \sigma_E(da \otimes e) + a\nabla_E e$$

$$\text{if } \phi : (E, \nabla_E) \rightarrow (F, \nabla_F) \quad \nabla(\phi) := \nabla_F \phi - (\phi \otimes \text{id}) \nabla_E : E \rightarrow F \otimes_B \Omega_B^1$$

2-category of 'differentiable' bimodules

Objects: diff algebras (A, Ω^1, d)

Mor(A,B): category of A-B-bimodules w/ con (E, ∇_E, σ_E) with 2-Mor bimodule maps ϕ with $\nabla(\phi) = 0$ and usual (vertical) composition

$$\otimes_B : \text{Mor}(B, C) \times \text{Mor}(A, B) \rightarrow \text{Mor}(A, C)$$

$$F \quad E \mapsto E \otimes_B F$$

(horizontal) composition

$$\nabla_{E \otimes F} = \begin{array}{c} E \quad F \\ | \quad \nabla_F \curvearrowright \\ E \quad F \quad \Omega_C^1 \end{array} + \begin{array}{c} E \quad F \\ \nabla_E \curvearrowleft \quad \sigma_F \\ E \quad F \quad \Omega_C^1 \end{array}$$

$$\sigma_{E \otimes F} = \begin{array}{c} \Omega^1 \quad E \quad F \\ \sigma_E \quad \diagdown \quad \sigma_F \\ E \quad F \quad \Omega_C^1 \end{array}$$

- If Ω^1 and Ω_B^1 have right bimodule connections then both sides of σ_E have tensor product right A - B -bimodule connections.

Propn $\nabla(\sigma_E)$ a bimodule map (eg 0) iff
Yang-Baxter or **braid relations** hold

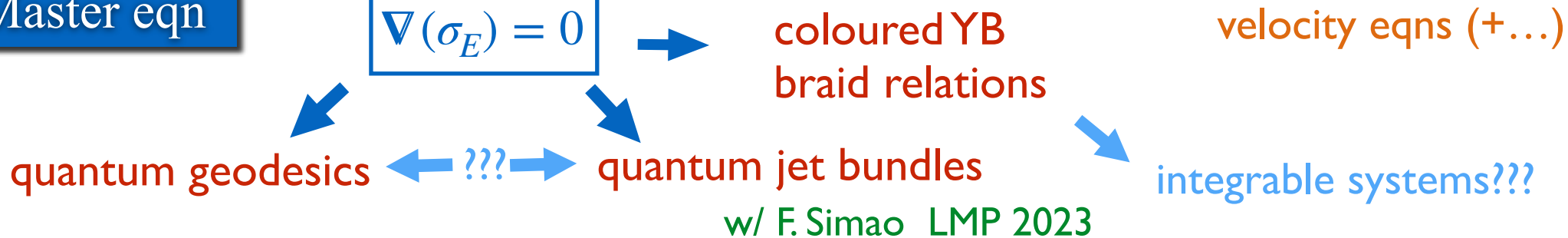
Example

$B = C^\infty(\mathbb{R}), \quad \Omega_B^1 \text{ classical} \quad \nabla_B ds = 0, \quad \sigma_B(ds \otimes ds) = ds \otimes ds$

$E = A \otimes B \ni \psi \rightarrow \begin{aligned} \nabla_E \psi &= (\dot{\psi} + \psi \kappa_s + X_s(d\psi) \otimes_B ds \\ \sigma_E(da \otimes_A \psi) &= X_s((da) \cdot \psi) \otimes_B ds \end{aligned} \rightarrow \begin{aligned} \nabla_E \psi &= 0 \\ \text{amplitude flow} \\ \nabla(\sigma_E) &= 0 \end{aligned}$

Master eqn

$\nabla(\sigma_E) = 0$



$\rightarrow$ Noncommutative var calculus on lattice w/ F. Simao arXiv:2508.02628

$\rightarrow$ exactly conserved lattice energy-momentum

Thank You