Diffeological generalized formal series: an overview

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Contents

Bibliography

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- various examples:
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Diffeologies

Definition (Diffeology)

Let X be a set. A **parametrization** of X is a map of sets $p\colon U \to X$ where U is an open subset of Euclidean space (no fixed dimension). A **diffeology** $\mathcal{P}$ on X is a set of parametrizations satisfying the following three conditions:

1. (Covering) $\forall x \in X$, $\forall n \in \mathbb{N}$, the constant function $p\colon \mathbb{R}^n \to \{x\} \subset X$ is in $\mathcal{P}$.
2. (Locality) Let $p\colon U \to X$ be a parametrisation such that for every $u \in U$ there exists an open neighbourhood $V \subset U$ of u satisfying $p|_V \in \mathcal{P}$. Then $p \in \mathcal{P}$.
3. (Smooth Compatibility) Let $(p\colon U \to X) \in \mathcal{P}$. Then for every n, every open subset $V \subset \mathbb{R}^n$, and every smooth map $F\colon V \to U$, we have $p \circ F \in \mathcal{P}$.

A set X equipped with a diffeology $\mathcal{P}$ is called a **diffeological space**, and the parametrisations $p \in \mathcal{P}$ are called **plots**.

Frölicher spaces, Lie groups...

Let $(X, \mathcal{P})$ and $(Y, \mathcal{P}')$ be two diffeological space. Then $f : X \to Y$ is smooth if $f(\mathcal{P}) \subset \mathcal{P}'$.
Subcategory of Frölicher spaces $(X, \mathcal{P})$ is a **Frölicher space** iff

$$p \in \mathcal{P} \Leftrightarrow \forall f \in C^\infty(X, \mathbb{R}), f \circ p \text{ is smooth.}$$

A group G is a **diffeological group** if multiplication and inversion are smooth for the underlying diffeology.
A diffeological group has a kinematic tangent space at identity, but the bracket may not exist. If there is a smooth bracket, G is called **diffeological Lie group**. If moreover there is an exponential map

$$\exp : C^\infty(\mathbb{R}, \mathfrak{g}) \to C^\infty(\mathbb{R}, G),$$

solving $dg.g^{-1} = v$, then G is called **regular**.
If $\mathfrak{g}$ is regular as an abelian diffeological group (vector space), that is if there exists a smooth integral ($=$ left inverse for derivation) for smooth paths in $\mathfrak{g}$ we say that G is fully regular.

Subsets, quotients and infinite products

Let $(X, \mathcal{P})$ be a diffeological space.

- any subset $S \subset X$ is a diffeological space
- Let $\mathcal{R}$ be a relation of equivalence, then $X/\mathcal{R}$ is a diffeological space
- Let $(X_i, \mathcal{P}_i)_I$ be a family of diffeological or Frölicher spaces indexed by *any* set I, then $\prod_{i \in I} X_i$ is a diffeological space.

These properties make the framework of diffeologies much more flexible in the technical requirements than other more classical settings (manifolds modeled in Banach, Fréchet, complete locally convex topological vector spaces...) in which an underlying diffeology is always defined.

Theorem

In all these settings,

$$\textit{smooth in the classical sense} \Leftrightarrow \textit{smooth in diffeologies}$$

Exponential mapping, regularity

Theorem

Let $(A_n)_{n\in\mathbb{N}}$ *be a sequence of coregular and integral Frölicher vector spaces equipped with a graded smooth multiplication operation on* $\bigoplus_{n\in\mathbb{N}^*} A_n$ *, i.e. a multiplication such that for each* $n, m \in \mathbb{N}^*$*,* $A_n.A_m \subset A_{n+m}$ *is smooth with respect to the corresponding Frölicher structures.*
Let us define the (non unital) algebra of formal series:
$\mathcal{A} = \left\{\sum_{n\in\mathbb{N}^*} a_n | \forall n \in \mathbb{N}^*, a_n \in A_n\right\}$ *, Then, the space*

$$1 + \mathcal{A} = \left\{1 + \sum_{n\in\mathbb{N}^*} a_n | \forall n \in \mathbb{N}^*, a_n \in A_n\right\}$$

is a regular Frölicher Lie group with regular Frölicher Lie algebra $\mathcal{A}$*. Moreover, the exponential map defines a smooth bijection* $\mathcal{A} \to 1 + \mathcal{A}$*.*

Classical formal series

Let $\mathcal{A}$ be a diffeological algebra, and let $\mathcal{A}^*$ be the diffeological group of its invertible elements. Let $(X_k)_{k\in K}$ be a family of formal variables, $K = \mathbb{N}^*$ or $K = \mathbb{N}_n$.

Definition

Let

$$\mathcal{A}[[(X_k)_{k\in K}]] = \sum_{\alpha\in\mathbb{N}} \sum_{\sum_{k\in K}\alpha_k=\alpha} a_{\alpha_1,\ldots\alpha_k} \prod_{\alpha_k\neq 0} X_k^{\alpha_k}$$

be the set of formal series.

The set of formal series is an algebra, with degree

$$\deg\left(\prod_{\alpha_k\neq 0} X_k^{\alpha_k}\right) = \sum_{k\in K}\alpha_k,$$

with *algebraic* exponential map defined on $\mathcal{A}_1 = \{a \,|\, val(a) > 0\}$ and set of invertible elements

$$(\mathcal{A}[[(X_k)_{k\in K}]])^* = \{a \in \mathcal{A}[[(X_k)_{k\in K}]] \,|\, a_0 \in \mathcal{A}^*\}$$

Diffeologies

In order to define a diffeology on $\mathcal{A}[[(X_k)_{k\in K}]]$, there are different ways:

- by infinite product diffeology
- by ultrametric completion along the lines of [Eslami-Rad, M, Reyes]
- if $\mathcal{A}$ is the diffeological dual of a vector space $\mathcal{V}$, by dual diffeology of $\mathcal{V}[(X_k)_{k\in K}]$.

In these settings we get an algebraic smooth exponential, but $(\mathcal{A}[[(X_k)_{k\in K}]])^*$ is not a priori regular.

Theorem

- *If $\mathcal{A}$ is regular and coregular, then $1+\mathcal{A}_1$ is regular with Lie algebra $\mathcal{A}_1$.*
- *If moreover $\mathcal{A}^*$ is a regular diffeological Lie group, then $(\mathcal{A}[[(X_k)_{k\in K}]])^*$ is a regular diffeological Lie group.*

Among the fields of application: Mulase groups for the construction of the dressing operator for the Kadomtsev-Petviashvili hierarchy [M,Reyes].

Groups of tensor products

let V, W be a diffeological vector spaces over a diffeological field $\mathbb{K}$ that are separated by their diffeological dual. Let us denote by $B(V, W)$ the space of smooth bilinear forms on V and W. We define, generalizing [Bogachev,Smolyanov, 2012] the tensor product $V \otimes W$ as the (algebraic) linear span of the image of $V \times W$ in the diffeological dual $(B(V, W))'$ of $B(V, W)$ by the evaluation mapping.

Theorem

The power series tensor algebra $(T_0((V)), +, \otimes)$ *defined by*

$$T_0((V)) = \mathbb{K} \oplus \prod_{k \in \mathbb{N}^*} V^{\otimes k}$$

is a diffeological $\mathbb{K}$*-algebra for the infinite product diffeology, whose group of the units* $T_0((V))^*$ *is* $T_0((V)) - \{0\}$*, which is a diffeological Lie group for the subset diffeology.*

The same holds for *symmetric* tensors.

Series over a small category

Let $(I, *)$ be a small category with neutral elements e. By small category, we require that $*$ is associative, with neutral element(s) and that it is not necessarily total. Let $\{\mathcal{A}_i; i \in I\}$ be a family of regular and coregular Frölicher vector spaces indexed by I, equipped with a multiplication, associative and distributive with respect to addition in the vector spaces $\mathcal{A}_i$, such that $A_i.A_j \begin{cases} \subset A_{i*j} & \text{if } i*j \text{ exists} \\ = 0 & \text{otherwise.} \end{cases}$ We also assume that, forall $k \in I$, there is a finite number of indexes i, j such that $i * j = k$, and that $*$ is $\mathbb{N}-$graded.

Theorem

If $\mathcal{A}_e^*$ is a diffeological Lie group with Lie algebra $\mathcal{A}_e$ then $\mathcal{A} = \sum_{i \in I} \mathcal{A}_i$ is the Lie algebra of the diffeological Lie group $\mathcal{A}^*$.

Here, the diffeology is the infinite product diffeology, but under mild assumptions, a diffeology on I can be also taken under consideration.
Applications: series indexed by cobordisms (link with TQFT) and series of stochastic cosurfaces [M,2022]

Outlook

Yet other possible developments:

- Semantics
- Morphogenesis

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