CONJUGATE POINTS IN THE GRASSMANN MANIFOLD OF A C^*-ALGEBRA

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Contents

1. Introduction and context

Remark 1.1. Program: develop a systematic approach to the geometry in the Frechet-manifold setting, emphasis in homogenous manifolds of Frechet-Lie groups. Not just "existence" theorems but explicit formulas, equalities and inequalities.

It is classical (in oppposition to noncommutative) differential geometry, but in the Banach setting; if your are new to the subject, see the books

- H. Upmeier's book "Jordan Algebras in Analysis, Operator Theory, and Quantum Mechanics" 1985/1987,
- S. Lang "Fundamentals of Differential Geometry" 1999 and newer editions.
- D. Beltita "Smooth Homogeneous Structures in Operator Theory" 2005.
- B. Khesin, R. Wendt "The geometry of infinite dimensional groups" 2009.

However, the setting the first three is of Banach manifolds. In Upmeier's book there are Finsler norms, but no connections, and in Lang's book everything is seudo-Riemannian (i.e. everything is in the setting of a smooth non-degenerate bilinear form).

Remark 1.2. By *geometry* we mean additional structures in M:

- Linear connections ∇ in M, geodesics and their exponential map, paralell transport.
- Metric structures: a Finsler metric (a continuous selection of tangent norms) permits to compute the length of paths, and induces a pseudo-distance in M by taking the infima of the lengths of paths joining given endpoints.
- Compatibility: is paralell transport an isometry for the distance?
- Geodesics, two definitions: solutions of Euler's equation $\nabla_{\gamma'}\gamma' = 0$ and on the other hand $L(\gamma) = dist(\gamma(0), \gamma(1))$. In the second case, there can be none (no Hopf-Rinow theorem).
- Curvature: if there is ∇, as a tensor R. But if we are interested in the geometry, another suitable definitions should follow (no inner products in the tangent spaces).

Remark 1.3. In Argentina a school of geometry and operator theory began with Porta and Recht in the 1980s; then Corach and his students Andruchow and Stojanoff, and their students (me for instance :) and so on.

Examples come first! We learn from them

- $G = GL(H)$ or $GL(X)$ or $GL(\mathcal{A})$ or
- congruence groups

$$GL_{congr}(H) = \{g \in GL(H) : g - 1 \text{ is a Hilbert-Schmidt operator}\}$$

- replace "Hilbert-Schmidt" with your favorite ideal $\mathcal{I}_\phi$ of compact operators in $B(H)$.

- restricted groups
$$GL_{res}(H) = \{g \in GL(H) : [g, \varepsilon] \text{ is a Hilbert-Schmidt operator}\}$$
where ε is the symmetry corresponding to the polarization $H = H_+ \oplus H_-$,
- likewise
$$GL^{\phi}_{res}(H) = \{g \in GL(H) : [g, \varepsilon] \in \mathcal{I}_\phi\}$$
- On the other hand, one can consider groups of isometries $G = U(H)$ or $Iso(X)$ or $\mathcal{U}_\mathcal{A}$ or $U_{congr}(H)$, $U_{res}(H)$ etc.
- their homogeneous spaces G/K: of "noncompact" type for the action of the whole $G = GL$ etc., and of "compact type" for the action of a group of isometries $G = U(H)$ etc.
- Loop groups
- Groups of diffeomorphisms, symplectomorphisms, Hamiltonian symplectomorphisms, etc.

See the 1986/1998 [39] and [40] by Presley, Segal and Wilson for the connection bewtween loop groups and restricted Grassmannians.
See [26] L-2019, Section 5 for a list of examples of this type, and also [27] L-Miglioli 2023 for a mix of classical groups and the groups of Hamiltonian symplectomorphisms.

2. Connections, geodesics, metrics and curvature in the infinite dimensional setting

We will revisit the notions briefly described in the previous section, but with one particular example as guideline. Let $\mathcal{A}$ be a C^*-algebra i.e. a subalgebra of $B(H)$ closed in the operator norm, for instance $\mathcal{A} = \mathcal{K}(H)$. Let $\mathcal{U}_\mathcal{A}$ be the unitary group of $\mathcal{A}$ i.e.
$$\mathcal{U}_\mathcal{A} = \{U \in \mathcal{A} : U^{-1} = U^*\}$$
where X^* denotes the operator adjoint uniquely defined by the equalities
$$\langle X^*\xi, \eta\rangle = \langle \xi, X\eta\rangle.$$
Fix a projection $P_0 = P_0^2 = P_0^* \in \mathcal{A}$, the following set is a component of the Grassmann manifold of $\mathcal{A}$
$$Gr = \{UPU^* : U \in \mathcal{U}_\mathcal{A}\}.$$

Remark 2.1 (Subspaces and rotations)**.** Let $S_0 = \mathrm{Ran}(P_0)$ be the range of P_0, then the range of P is $S = U(S_0)$. So there is a correspondence between elements of the orbit and subspaces that are obtained from S_0 by a "rotation" by certain U. All subspaces of the same dimension and codimension as S_0 can be reached with such rotation: if e_i f_i are respective orthonormal basis of S_0 and $S_0^\perp$, and likewise E_i, F_i of S, then $Ue_i = E_i$, $Uf_i = F_i$ does the job.

In the finite dimensional case $dim(H) = n$, it suffices to count the dimension of S_0, say it is $0 \leq k \leq n$ and then we obtain a different presentation of the Grassmannian of k dimensional subspaces $Gr \simeq Gr_{k,n}(\mathbb{K})$.
If the dimension of P is one, we obtain projective spaces $\mathbb{P}^n(\mathcal{K})$.

3. Differentiable structure

Our set Gr has a natural structure of Banach-homogeneous space: let

$$\mathcal{D}_0 = \{k \in \mathcal{U}_{\mathcal{A}} : UP_0U^* = P_0\}$$

be the isotropy group for the action. Then $\mathcal{D}_0$ is the "diagonal" subgroup of unitary operators and it is not hard to see that it is a split, embedded Banach-Lie subgroup of $\mathcal{U}_{\mathcal{A}}$. Thus by very general considerations

$$Gr \simeq \mathcal{U}_{\mathcal{A}} / \mathcal{D}_0$$

has a structure of Banach-manifolds that makes of the quotient map $q : \mathcal{U}_{\mathcal{A}} \to Gr$ a smooth submersion and the action $\alpha : \mathcal{U}_{\mathcal{A}} \times Gr \to Gr$ given by $(U, P) \mapsto UPU^*$ a smooth map.
However, for our purposes we will be necessary to give an explicit description of the atlas for the manifold Gr. In this presentation there are natural charts better described by means of the exponential map of $\mathcal{B}(H)$. We discuss first the tangent spaces.
Fix $P \in Gr$, operators $A \in \mathcal{A}$ as 2×2 block matrices:

$$A = \begin{pmatrix} PAP & PAP^{\perp} \\ P^{\perp}AP & P^{\perp}AP^{\perp} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

and the algebra $\mathcal{A}$ decomposed as

$$A = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} + \begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix} = A_d + A_c \in \mathcal{D}_P \oplus \mathcal{C}_P = \mathcal{A}$$

$\mathcal{D}_P$ is the P-*diagonal part* of $\mathcal{A}$,
$\mathcal{C}_P$ is the P-*co-diagonal part* of $\mathcal{A}$.

We have

$$[\mathcal{D}_P, \mathcal{D}_P] \subset \mathcal{D}_P \quad [\mathcal{D}_P, \mathcal{C}_P] \subset \mathcal{C}_P \quad [\mathcal{C}_P, \mathcal{C}_P] \subset \mathcal{D}_P$$

a "reductive algebra" or reductive decomposition.

We denote with $s_P = 2P - 1$ around the range of P, we have $s_P = s_P^* = s_P^{-1}$, in matrix notation

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad s_P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We have

(1) $X \in \mathcal{D}_P$ iff it commutes with P

(2) $X \in \mathcal{C}_P$ iff $X = XP + PX$ iff $s_P X = -X s_P$, here $[P,[P,X]] = X$.
(3) $\mathcal{C}_{UPU^*} = U\,\mathcal{C}_P\,U^*$, $U \in \mathcal{U}_{\mathcal{A}}$
(4) $Gr \subset \mathcal{A}_h$, tangent space $T_P Gr = \mathcal{C}_P \cap \mathcal{A}_h$. Typical tangent vector at P: $X_P = [x,P]$ with $x^* = -x \in \mathcal{C}_P$. Correspondence

$$\begin{array}{ccc} \mathcal{C}_p \cap \mathcal{A}_{sk} & \longleftrightarrow & \mathcal{C}_P \cap \mathcal{A}_h \\ x = \begin{pmatrix} 0 & -\lambda \\ \lambda^* & 0 \end{pmatrix} & \longleftrightarrow & \begin{pmatrix} 0 & \lambda \\ \lambda^* & 0 \end{pmatrix} = X \end{array}$$

Definition 3.1 (Charts)**.** $\|z\| < \pi/2$, $z \in \mathcal{C}_P$ then

$$\{Q \in Gr : \|Q - P\| < 1\} \longleftrightarrow \{z^* = -z : \|z\| < \pi/2, z \in \mathcal{C}_P\}$$

by means of $z \mapsto e^z P e^{-z}$ (a parametrization of the orbit), is a real analytic chart of Gr around P.
Sketch of proof: let $s_P = 2P - 1$, $s_Q = 2Q - 1$ be the induced symmetries, then

$$\|s_Q s_P - 1\| = \|s_Q - s_P\| = 2\|Q - P\| < 2$$

hence there exists an analytic logarithm of $u = s_Q s_P \in \mathcal{U}_{\mathcal{A}}$, let's say

$$z = z(Q) = \frac{1}{2}\log(s_Q s_P).$$

It is clear that $z^* = -z$ and it is also not hard to see that $z \in \mathcal{C}_P$, i.e. $z s_P = -s_P z$. This in turn implies

$$e^z P e^{-z} = e^z(\frac{s_P + 1}{2})e^{-z} = \frac{e^{2z} s_P + 1}{2} = \frac{s_Q s_P s_P + 1}{2} = \frac{s_Q + 1}{2} = Q$$

hence $z = z(Q)$ is the inverse map of $z \mapsto e^z P e^{-z}$.

3.1. **Other Grassmann manifolds.** Fix $P_0 = P_0^2 = P^* \in \mathcal{B}(H)$, and consider some group of Hilbert space operators say $\mathcal{U}^{\phi}_{res}(H)$, or $\mathcal{U}^{\phi}_{congr}(H)$. The respective Grassmann manifold will be then for instance

$$Gr^{\phi}_{res}(H) = \{UP_0U^* : U \in \mathcal{U}^{\phi}_{res}(H)\}$$

and likewise with the congruence groups. In particular for the restricted Hilbert-Schmidt group, you get the Sato Grassmannian. Then you can get away with the previous construction of charts verbatim, since all the operations are real analytic and those are Banach-Lie groups.

4. Linear connection, geodesics and exponential map

Remark 4.1. Project onto the tangent spaces

$$\mathcal{A}_h \ni V^* = V = \begin{pmatrix} a & \lambda \\ \lambda^* & b \end{pmatrix} \mapsto \begin{pmatrix} 0 & \lambda \\ \lambda^* & 0 \end{pmatrix} = \Pi_P(V) \in T_P Gr.$$

From previous remarks we see that

$$\Pi_P(V) = [P,[P,V]] = [[V,P],P].$$

Remark 4.2. If $\mathcal{A}$ has a faithful trace, Π_P are the orthogonal projections for the Riemannian metric

$$\langle X,Y\rangle = Tr(XY) \qquad X,Y \in \mathcal{A}_h$$

If $\mu : [0,1] \to Gr$ is a *vector field along a path* $\gamma \subset Gr$ i.e. $\mu(t) \in T_{\gamma(t)}Gr = \mathcal{C}_{\gamma(t)}$ for each $t \in [0,1]$.

Remark 4.3. Connection ∇: when there is a suitable projection for each subspace (tangent space), then we can differentiate and project:

$$D_t\mu := \Pi_{\gamma(t)}(\mu'(t)) \qquad \text{covariant derivative of } \mu$$

$\mathcal{A}$ with trace: D_t is the Levi-Civita connection.

Fix $P \in Gr$, $Z = [z,P] \in T_PGr$, then $\delta(t) = e^{tz}Pe^{-tz}$ is the unique geodesic of the connection ∇ i.e

$$D_t\delta' = 0 \qquad \text{(Euler's equation)}$$

with

$$\delta(0) = P, \qquad \delta'(0) = Z = [z,P].$$

We have

$$\delta'(t) = e^{tz}[z,P]e^{-tz} = e^{tz}Ze^{-tz} = [z,\delta(t)] \in T_{\delta(t)}Gr$$

where the initial conditions are verified, and

$$\delta''(t) = [z,\delta'(t)] = [z,[z,\delta(t)]] = e^{tz}[z,[z,P]]e^{-tz} = e^{tz}[z,Z]e^{-tz}.$$

Since $[z,Z] \in \mathcal{D}_P$, $\delta''(t) \in D_{\delta(t)}$, hence

$$D_t\delta' = \Pi_\delta(\delta'') = 0.$$

Thus geodesics are defined for all t, the exponential map of the connection ∇ is defined in the whole tangent space and it is

$$\mathrm{Exp}_P\,(Z) = \delta(1) = e^zPe^{-z} = e^{[Z,P]}Pe^{-[Z,P]}$$

Notice that our charts back then are the exponential charts!

Remark 4.4. The paralell transport equation along γ

$$D_t\mu = 0,\ \mu(0) = W \in T_PGr \qquad \Longrightarrow \qquad P_0^t(\gamma)W = \mu(t)$$

is solved explicitly when γ is a geodesic $t \mapsto e^{tz}Pe^{-tz}$: it is

$$\mu(t) = e^{tz}We^{-tz}$$

5. Length and distance; short paths

Definition 5.1 (L and $dist$)**.** Length of paths $\gamma : [0,1] \to Gr$ is $L(\gamma) = \int_0^1 \|\gamma'(t)\| dt$, here $\|X\|$ is the C^*-algebra norm of X.

$$dist(P, Q) = \inf\{L(\gamma) : \gamma(0) = P, \gamma(1) = Q\}.$$

Compatibility of the connection with the metric:

$$\|P_0^t(\gamma)W\| = \|W\| \qquad dist(UPU^*, UQU^*) = dist(P, Q).$$

Remark 5.2. Triangular inequality straight from the definitions, but

(1) $dist(p, q) = 0$ implies $p = q$. In this case yes, we will see why soon.
(2) Is the topology of (Gr, d) equivalent to the manifold topology of Gr? In this case again, yes.

Again: by general methods, since in this example we measure with a norm that gives the ambient topology, and the space is homogeneous, we could get away with it without getting our hands dirty. But we want more detail, so we will give a finer description of the distance function.

Definition 5.3. We say that $\gamma : [0,1] \to Gr$ is *short*, or *minimizing* if

$$L(\gamma) = \mathrm{dist}(\gamma(0), \gamma(1)).$$

Then it is easy to see that

$$dist(\gamma(s), \gamma(t)) = L_s^t(\gamma) = \int_s^t \|\gamma'\|$$

Theorem 5.4. *If $z^* = -z \in \mathcal{C}_P$ and $\|z\| \leq \frac{\pi}{2}$, then $\delta(t) = e^{tz} P e^{-tz}$ is minimizing in $[0,1]$.*

Proof. (Porta-Recht 1987 [36]): by means of the be the GNS representation of $-z^2 = z^*z \geq 0$, we can assume that $-z^2$ has a unit norm, norming eigenvector $\xi \in H$. Then if $\gamma \subset Gr$ joining $P, e^z P e^{-z}$ we consider the respective symmetries $s_\delta, s_\gamma \in \mathcal{U}_{\mathcal{A}}$ and we push them to the unit sphere S of H evaluating at ξ. This map decreases distances, but for δ it preserves it. And the fun fact: $g = s_\delta \xi$ is a geodesic of the Riemannian sphere S! This is apparent from $g = s_\delta \xi = s_P e^{-2tz} \xi$, hence

$$\|g'\|^2 = 4\langle s_P e^{-2tz} z\xi, s_P e^{-2tz} z\xi\rangle = 4\langle z\xi, z\xi\rangle = 4\langle -z^2\xi, \xi\rangle = 4\|z\|^2 = (2\|\delta'\|)^2$$

and

$$g'' = 4 s_P e^{-2tz} z^2 \xi = -4\|z\|^2 s_P e^{-2tz} \xi = -4\|z\|^2 s_\delta \xi = -4\|z\|^2 g = -L_S(g)^2 g.$$

Thus

$$2L_{Gr}(\delta) = L_S(g) \leq L_S(s_\gamma \xi) \leq L(s_\gamma) = 2L_{Gr}(\gamma).$$

□

Recall

$$\{Q \in Gr : \|Q - P\| < 1\} = \{e^z P e^{-z} : z^* = -z, \|z\| < \pi/2, z \in \mathcal{C}_P\}$$

With this we obtain

$$dist(Q, P) = \|z\| = \|z(P, Q)\| = \frac{1}{2}\| \log(s_Q s_P)\|.$$

Corollary 5.5. *d is a distance in Gr that gives the manifold topology.*

5.1. **Direct rotations and principal angles.** What are we measuring? Recall that if $\operatorname{Ran} P = S$, then $Q = UPU^* = e^z P e^{-z}$ is the projection corresponding to the subspace $S' = U(S) = e^z(S)$.

So $U = e^z$ for a good z is a "direct rotation", a notion introduced by Dixmier in 1948 [15], though I would recommend the 1958 paper by Chandler Davis [14] which is available electronically and much more clear, revisiting Dixmier's results. There is however no notion of paths or geodesics in Dixmier's paper, and the closest to this is Kovaric's 1979 paper [23], in the setting of Banach algebras's idempotents (but then there there is no notion of length in general, it starts discussing lengths when he restricts to the Banach algebra of Hilbert-Schmidt operators, which is a Hilbert space hence the metric is Riemannian).

Since

$$Z = [z, P] = \begin{pmatrix} 0 & \lambda \\ \lambda^* & 0 \end{pmatrix}$$

then

$$-z^2 = Z^2 = \begin{pmatrix} \lambda\lambda^* & 0 \\ 0 & \lambda^*\lambda \end{pmatrix}$$

and

$$|Z| = |z| = \begin{pmatrix} |\lambda^*| & 0 \\ 0 & |\lambda| \end{pmatrix}.$$

The non-zero spectrum of $|\lambda|$ and $|\lambda^*|$ coincide (exercise on functional calculus), so for the non-zero spectrum (since $z^* = -z$) we have $\sigma(z) = \pm i\sigma|\lambda|$. In particular

$$dist(P, Q) = \|z\| = \|\lambda\| = \text{ maximum angle among subspaces.}$$

To clarify: when the spectrum is discrete (for instance, if Z is compact), we have a set of numbers called *principal angles* of the direct rotation.

When $\mathcal{A}$ is finite dimensional (classical case, $\mathcal{A} = M_n(\mathbb{R})$ or $\mathcal{A} = M_n(\mathbb{C})$, we have, counting with multiplicity

$$\sigma(z) = \{i\theta_1, i\theta_2, \ldots, i\theta_k\}$$

with $-\frac{\pi}{2} \leq \theta_k \leq \frac{\pi}{2}$ and $k = dim \operatorname{Ran} P$.

Note that in this case we can use the Riemannian metric induced by the trace, and then

$$dist(P,Q) = L(\delta) = \|z\| = \sqrt{Z^2} = \sqrt{\sum_{k=1}^{n} \theta_k^2}$$

This is the *angular distance* among subspaces of the Grassmann manifold $Gr_{k,n}(\mathbb{R})$ coming from the Riemannian metric considered by Wong in a classical paper [44] back in 1967.

The angles can be obtained in a different fashion that gives another insight: let $\{e_i\}, \{f_i\}$ be orthonormal basis of $S = \operatorname{Ran} P$ and $S' = \operatorname{Ran} Q$ respectively, consider the matrix

$$M = \{\langle e_i, f_j\rangle\} \in \mathbb{R}^{k\times k}.$$

Then the numbers $\cos\theta_i$ are the eigenvalues of $|M| = \sqrt{M^*M}$.

In particular by choosing adequate basis one obtains $M = PQ$ or QP since by a result of Halmos [18] from 1969 "two subspaces" in generic position can be described as

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix}$$

where $c = \cos|\lambda^*|$ and $s = \sin|\lambda^*|$. This can also be shown by direct computation of $Q = \delta(1) = e^z P e^{-z}$. In particular

$$dist(P,Q) = \|z\| = \||\lambda|\| = \arccos\|PQ\|.$$

6. Cut locus and conjugate locus

Definition 6.1. The *cut locus* C_P at $P \in Gr$ is the set of points $Q \in Gr$ such that geodesics from P to Q are not minimizing past P. The *tangent cut locus* $TC_P \subset T_PM$ is the pre-image of C_P at P by means of the exponential map i.e.

$$TC_P = \{V \in T_PM : \operatorname{Exp}_P(V) \in C_P\}$$

.

Remark 6.2 (Monoconjugate and epiconjugate points)**.** Those vectors V_0 where our operator $D(\operatorname{Exp}_P)_{V_0}$ is not invertible, will be called *tangent conjugate points* to P. If the operator is not injective, it is customary to call the point *monoconjugate*, and the (real) dimension of its kernel is the *order of nullity* or just *nullity* of the conjugate point. If the operator is not surjective the point is called *epiconjugate*. This phenomena on conjugate points was first observed in the Riemann-Hilbert setting by Grosmann [17] and McAlpin [33]. The conjugate locus is the image of the tangent conjugate points through the exponential map at P.

6.1. Motivation.

We state here a couple of results to show how are these related, and their relevance:

Theorem 6.3. *If (M, g) is (finite-dimensional) Riemanian, the cut locus of P consists exactly of the points Q that are either conjugate to P, or the points such that there exist two minimizing geodesics from P arriving at the point.*

The order of nullity of a minimizing γ joining P to its first conjugate point $Q = \mathrm{Exp}_P(V)$ (the dimension of the kernel of the $D(\mathrm{Exp}_P)_V$) is also called Morse index of γ in Morse theory.

Definition 6.4. Let $P, Q \in M$ a smooth manifold. Let

$$\Omega_P = \{\gamma : [0,1] \to M : \gamma(0) = P\} \subset H^1([0,1] : M)$$

$$\Omega_{PQ} = \{\gamma \subset M : \gamma(0) = P, \gamma(1) = Q\} \subset H^1([0,1] : M)$$

Where H^1 is the Sobolev space of fuctions (i.e. γ and γ' are square integrable in a chart).

Then $\pi : \Omega_P \to M$ given by $\gamma \mapsto \gamma(1)$ is a *fibration*, and the fiber over Q is exactaly Ω_{PQ}. What interested Morse is that if (M, d) Riemannian is connected and complete, then *all the fibers are homotopically equivalent*, so you see $\Omega_{PQ} \simeq \Omega_{PP}$ hence $\pi_1(M)$ is in the picture.

Theorem 6.5. *Let (M, g) be a (finite-dimensional, Riemannian and complete) manifold. Let Q be the first conjugate point along $\gamma \in \Omega_{PQ}$ with dist$(P, Q) = L(\gamma)$. Then Ω_{PQ} has the homotopy type of a countable CW-complex which contains one cell of dimension d for each minimizing γ from P to Q of order d.*

The main tools in that setting to study these problems are the variational formulas: let $\nu(s,t) : (-\varepsilon, \varepsilon) \times [0,1]$ be a variation of the path $\gamma(t) = \mu(0,t)$, which we denote $\nu_s(t)$. Then

$$f(s) = E(\nu_s) = \int_0^1 \| \frac{d}{dt} \nu_s(t) \|^2_{\nu_s(t)} dt.$$

The *first variation formula* is just a clever rewriting of $f'(0)$, which involves the *variation field* μ along γ

$$X(t) = \frac{d}{ds}\Big|_{s=0} \nu(s,t).$$

Now let $\nu(u,s,t) : (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \times [0,1]$ be a two-parameter variation of the path $\gamma(t) = \mu(0,0,t)$, with

$$X(t) = \frac{d}{du}\Big|_{u=0} \nu(u,0,t), \quad Y(t) = \frac{d}{ds}\Big|_{s=0} \nu(0,s,t)$$

the corresponding variation fields along γ. Then the *second variation formula* is another clever rewriting of

$$B_\gamma(X,Y) = \frac{d^2}{ds\, du}\Big|_{u=s=0} E(\nu_{u,s}),$$

now involving *Jacobi fields* along γ. The *index* of B is the number of negative eigenvalues of B (a well-defined quantity due to Sylvester's inertia law).
Then:

(1) The path γ is a geodesic if and only if it is a critical point of the first variation, i.e. $f'(0)=0$.
(2) If γ is minimizing then B has index 0 i.e. $B_\gamma(X,X)\geq 0$ along γ.
(3) (The fundamental theorem of Morse theory) The index of B is the number of conjugate points to $\gamma(0)$ along γ, counted with multiplicity.

We won't get into details, so now we return to our setting, knowing that these theorems are not available.

6.2. **Cut locus, existence and uniqueness of geodesics.**

Definition 6.6. $\mathcal{A}$ has *real rank zero* when the set of self-adjoint elements with finite spectrum is dense in norm in the space of self-adjoint elements of $\mathcal{A}$. The first examples of such C^*-algebras are the von Neumann algebras or the compact operators on a separable Hilbert space.
$\mathcal{A}$ is *purely infinite simple* if for every non-zero $a\geq 0$ there exists $x\in\mathcal{A}$ such that $x^*ax=1$. The first examples are the Cuntz algebras $\mathcal{O}_n$, $n\geq 2$, the algebra generated by n isometries $S_i^*S_i=1$ such that $\sum_{i=1}^n S_iS_i^*=1$. Cuntz algebras are also of real rank zero.

Remark 6.7 (Diameter of Gr)**.** By the results of the previous section, the geodesic diameter of the Grassmannian of a C^*-algebra is greater or equal than $\pi/2$ (we assumed throughout that P_0 is non-central).
Exponential diameter is a relevant invariant in the classification program of $C^*-algebras$; in N.C. Phillips paper [35] it is shown that

(1) If $\mathcal{A}$ has real rank zero and the cancellation property (see [42] for the definition), then the diameter of Gr is exactly $\pi/2$.
(2) If $\mathcal{A}$ is purely infinite simple then this rectifiable diameter is exactly π. Cuntz algebras are purely infinite simple, and also real rank zero! So the geodesics diameter is π, but no geodesics is minimizing past $\pi/2$ (see the next theorem). In particular this shows that Cuntz algebras do not have the cancellation property.

The proofs of the above facts are very indirect, we can be more precise:

Theorem 6.8. *Assume that $\mathcal{A}$ has real rank zero. Then unit speed geodesics of Gr are not minimizing past $|t|=\frac{\pi}{2}$.*

Proof. Let $\gamma(t)=e^{tv}Pe^{-tv}$ with $\|v\|=1$, assumme that $t_0>\pi/2$, let $v_0=t_0v$, then $\|v_0\|>\pi/2$. Let $Q=\gamma(t_0)=e^{v_0}Pe^{-v_0}$. Let $v_n^*=-v_n\in\mathcal{A}$ be such that v_n has finite spectrum and $\|v_n - v_0\| < \frac{1}{n}$, let $Q_n = e^{v_n} P e^{-v_n}$. Consider the truncation z_n of v_n into the interval $i[-\frac{\pi}{2}, -\frac{\pi}{2}]$, i.e. for $k \in \mathbb{N}_0$ let

$$f(x) = \begin{cases} x + (k+1)\frac{\pi}{2} & -2(k+2)\frac{\pi}{2} \le x < -(k+1)\frac{\pi}{2} \\ x & -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ x - (k+1)\frac{\pi}{2} & (k+1)\frac{\pi}{2} < x \le (k+2)\frac{\pi}{2} \end{cases},$$

and let $z_n = if(-iv_n) = -z_n^*$. Since the spectrum of v_n is finite, $z_n \in \mathcal{A}$, and moreover $\|z_n\| \le \pi/2$. It is plain also that $e^{z_n} = e^{v_n}$, thus $Q_n = e^{z_n} P e^{-z_n}$ and if we let $\beta(t) = e^{tz_n} P e^{-tz_n}$ then

$$\begin{aligned} \operatorname{dist}_\infty(P, Q_n) &\le L_0^1(\beta) = \|z_n P - P z_n\| \le \max\{\|P z_n (1-P)\|, \|(1-P) z_n P\|\} \\ &\le \|z_n\| \le \pi/2 < \|v_0\| = L_0^{t_0}(\gamma). \end{aligned}$$

Since $Q_n \to Q$, we are done. $\square$

Now we consider the enveloping von Neumann algebra of $\mathcal{A}$, let $P \wedge P'$ denote the infimum of the projections (again a projection) and let $\sim$ denote the Murray-von Neumann equivalence of projections. Then a full characterization of points that can be joined with a geodesic was obtained by Andruchow in [1] following ideas in Dixmier's paper [15] as follows:

Theorem 6.9. *Let $P, Q \in Gr(P_0)$ with $\mathcal{A}$ a von Neumann algebra. Then there exists a geodesic joining P, Q if and only if*

$$P \wedge (1-Q) \sim Q \wedge (1-P).$$

In this case there exist a minimizing geodesic joining them. Moreover, the geodesic is unique if and only if $P \wedge (1-Q) = 0$.

In finite dimensional algebras, or in algebras of compact operators, the condition is automatically fulfilled, see [1]. If $\mathcal{A}$ is not a von Neuman algebra, and the condition is fullfilled, the geodesic might not have speed in $\mathcal{A}$, so some caution is required. But by considering the enveloping von Neumann algebra of the C^*-algebra $\mathcal{A}$, it follows that if there exist a geodesic joining P, Q, then the condition must be fullfilled.
We now discuss uniqueness of geodesics, and show that before the first cut locus they are unique as in the Riemannian setting.

Theorem 6.10. *Let $V \in T_P Gr$, let γ be the unique geodesic from P with initial speed V. Then*

(1) *If $t\|V\| < \pi/2$, then the only minimizing geodesic joining $P, Q = \gamma(t)$ is γ.*
(2) *If $t_0\|V\| = \frac{\pi}{2}$ and either $\pm\frac{\pi}{2}$ is an eigenvalue of $t_0 V$, then there is another minimizing geodesic $\gamma_1 \subset \mathcal{A}''$ joining P to $Q = \gamma(t_0)$. Moreover, if the eigenvalue is isolated, then $\gamma_1 \subset \mathcal{A}$ and γ is not minimizing past t_0.*

Remark 6.11 (Real case)**.** In the case of the real Grassmannians the proof needs some adaptation: let $e_1 = \mathfrak{Re}(\xi_1)$ and $e_2 = \mathfrak{Im}(\xi_1)$, then $ve_1 = -e_2$ and $ve_2 = e_1$ so

$ue_1 = -e_2$ and $ue_2 = e_1$ and we can write

$$v = v_\perp + \frac{\pi}{2}(-e_1 \otimes e_2 + e_2 \otimes e_1).$$

Then we change the sign of the second term to obtain v_1 and the rest of the proof follows in the same fashion.

Corollary 6.12. *Assume P_0 has finite rank or co-rank (in particular, any finite dimensional Grassmannian). If $P, Q \in Gr(P_0)$ and $\operatorname{dist}(P,Q) = \pi/2$, then there exist at least two minimizing geodesics joining them, and unit speed geodesics are not minimizing past $\pi/2$.*

7. Tangent conjugate points

One of the first problems of interest is the charaterization of the conjugate points Q to P along γ, and the tangent conjugate locus of P, which as we said before, are those V such that $D(\mathrm{Exp}_P)_V$ is non-invertible.
This was solved for the classical Grassmannians $Gr_k(n)$ in a wonderful paper by Sakai in the seventies [41] 1977. It follows the ideas in the paper of Crittenden [13] 1962, where the method of proof is based on the presentation of the Grassmanian as a symmetric space of the compact type, and using the machinery of Cartan subalgebras and real root decompositions. See the paper by Berceanu [9] for further explanation and history of these developments.

The results of this section with full detail and proofs can be found in [3].

We begin by recalling the well-known formulas for the differential of the exponential map in a Banach-Lie group:

Lemma 7.1. *Let $v, w \in Lie(G)$ where G is a Banach-Lie group, let $\exp$ be its exponential map, let $F(\lambda) = \frac{1-e^{-\lambda}}{\lambda}$ extended by 1 at $\lambda = 0$, let $G(\lambda) = e^\lambda F(\lambda)$. Then*

$$D\exp_v(w) = e^v F(\operatorname{ad} v)w = [G(\operatorname{ad} v)w]e^w.$$

Remark 7.2 (Differential of the exponential map of the connection)**.** Since the exponential map of the reductive connection is $Exp_P(V) = e^{[V,P]}Pe^{-[V,P]}$, one can compute its differential explicitly,

$$D(Exp_P)_V(W) = e^v[\operatorname{sinhc}(\operatorname{ad} v)w, P]e^{-v}$$

where as before, $V = [v,P]$ and $W = [w,P]$ with $v, w \in \widetilde{\mathcal{C}}_P = \mathcal{A}_{sk} \cap \mathcal{C}_P$. This formula was obtained for any symmetric Banach space in [34, Lemma 3.10].

Remark 7.3 (Factorization of the differential of the exponential map)**.** For $v \in \widetilde{\mathcal{C}}_P$, the operator $\operatorname{ad} v$ does not preserve that space. However the operator $\operatorname{ad}^2 v$ does. Thus if we pair the roots of the entire function sinhc and their opposites, the Weierstrass factorization of that function allows us to write

$$\operatorname{sinhc}(z) = \prod_{k\neq 0}\left(1+\frac{z}{ik\pi}\right) = \prod_{k\geq 1}\left(1+\frac{z^2}{k^2\pi^2}\right) \qquad z\in\mathbb{C} \tag{1}$$

By means of the holomorphic functional calculus we obtain

$$\operatorname{sinhc}(t\operatorname{ad} v) = \prod_{k\neq 0}\left(1+\frac{t\operatorname{ad} v}{ik\pi}\right) = \prod_{k\geq 1}\left(1+\frac{t^2\operatorname{ad}^2 v}{k^2\pi^2}\right),$$

and in the last expression we have as building blocks linear operators from $\widetilde{C}_P$ into itself.

Remark 7.4 (Building blocks)**.** For each $T = T(k,s,s')$ the point $Q = Exp_P(TV)$ is conjugate to P when $\operatorname{sinhc}(T\operatorname{ad} v)$ is not invertible, and this happens if and only if any of the operators

$$x \mapsto (\operatorname{ad}^2 v + \frac{j^2}{k^2}|s-s'|^2 1)x, \quad j\in\mathbb{Z}^*$$

is not invertible in $\mathcal{C}_P$. Equivalently, naming $\mu_j = \frac{|j|}{|k|}|s-s'| > 0$, when any of the operators $\operatorname{ad}^2 v + \mu_j^2 1$ is not invertible. This will happen only if there exists $s_1 \neq s_2 \in \sigma(V)$ such that $|s_1 - s_2| = \mu_j$.

Theorem (A)**.** *Let $P\in Gr(P_0)$ and let $V\in T_PGr(P_0)$ of unit speed. If Q is conjugate to P along γ then $Q=\gamma(T)$ with*

$$T = T(k,s,s') = \frac{k\pi}{|s-s'|} \qquad k\in\mathbb{Z}^*, \qquad s\neq s'\in\sigma(V).$$

For a complete proof see [3], were it is done studying the spectrum of $\operatorname{ad} v = L_v - R_v$ (left and right multiplication, which can be non-trivial).

8. First tangent conjugate point

The smallest $T>0$ is obtained for $k=1$, and $s=1, s'=-1$ and it is then $T=\frac{\pi}{2}$ (note that this is the cut locus and also the geodesic diameter for real rank zero, which includes the classical setting!). At this points then the unique (possible non-invertible) operator is

$$1 + 4\operatorname{ad}^2 v.$$

Is it invertible? Is it injective? To answer this We consider $V = U|V|$ the polar decomposition of V in the enveloping von Neumann algebra of $\mathcal{A}$, with $|V| = \sqrt{V^*V}$ and U the partial isometry that maps the range of $|V|$ in the range of V, Let

$$\lambda = (1-P)VP, \quad \Omega = (1-P)UP.$$

Let $P_{|\lambda|}$ stand for the projection onto the closure of the range of λ, $P_{|\lambda|}\leq 1-P$, and consider the C^*-algebra $\mathcal{A}_0 = P_{|\lambda|}\mathcal{A}P_{|\lambda|}$. In simpler terms, using block-operator notations with respect to P:

$$V = \begin{pmatrix} 0 & \lambda \\ \lambda^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & \Omega \\ \Omega^* & 0 \end{pmatrix} \begin{pmatrix} |\lambda^*| & 0 \\ 0 & |\lambda| \end{pmatrix}$$

and

$$P_{|\lambda|} \in \mathcal{A}_0 = \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix}$$

For the first conjugate point we obtain:

Theorem (B)**.** *Let V be a unit length tangent vector at $P \in Gr(P_0)$. Then the kernel of $D(\mathrm{Exp}_P)_{\frac{\pi}{2}V}$ at the first tangent conjugate point $Q = \gamma(\frac{\pi}{2})$ is*

$$\mathcal{S} = \{\Omega z - z\Omega^* : z^* = -z \in \mathcal{A}_0 \text{ and } |\lambda| z = z\}.$$

If Q is not monoconjugate to P, then it is epiconjugate to P.

9. Further conjugate points

For other candidates $Q = \gamma(T)$, the situation is different, since they might not be conjugated to P.

Definition 9.1. For given $V \in T_P Gr$, we now consider the projection P_v onto the range of V. We will decompose $\mathcal{A}_{sk}$ in a direct sum of three subspaces, but now *relative to the projection* P_v:

$$\mathcal{A}_{sk} = \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix},$$

or in other terms

$$\mathcal{A}_{sk} = P_v \mathcal{A}_{sk} P_v \bigoplus \widetilde{\mathcal{C}}_{P_v} \bigoplus (1 - P_v)\mathcal{A}_{sk}(1 - P_v), \tag{2}$$

where

$$\widetilde{\mathcal{C}}_{P_v} = P_v \mathcal{A}_{sk}(1 - P_v) \oplus (1 - P_v)\mathcal{A}_{sk} P_v = P_v \mathcal{A}_{sk} \oplus \mathcal{A}_{sk} P_v.$$

Denoting $P_{|\lambda^*|} \leq P$ the range projection of $\lambda\lambda^*$ and $P_{|\lambda|}$ the range projection of $\lambda^*\lambda$, we have we

$$P_V = P_{|V|} = P_v = P_{|\lambda^*|} + P_{|\lambda|}.$$

We will use $\mathfrak{Re}, \mathfrak{Im} : \mathcal{A} \to \mathcal{A}$ to denote the real linear operators that take the symmetric and skew symmetric part of an operator in $\mathcal{A}$, i.e $\mathfrak{Re}(x) = (x + x^*)/2$ and $\mathfrak{Im}(x) = (x - x^*)/2$.

Proposition 9.2. *Let $V = [v, P] \in T_P Gr(P_0)$. Then each of the three subspaces in (2) are invariant for* $\mathrm{sinhc}(T \,\mathrm{ad}\, V)$. *If V has unit norm, $v = u|v|$ is the polar decomposition of v and $T = T(k, s, s')$, then with respect to this direct sum we have*

$$\mathrm{sinhc}(T \,\mathrm{ad}\, v) = L_u(\Pi_- \oplus \Pi_+)L_{u^*} \bigoplus (L_{\mathrm{sinhc}(Tv)} + R_{\mathrm{sinhc}(Tv)} - 1) \bigoplus 1$$

where $\Pi_-, \Pi_+ \in \mathcal{B}(\mathcal{C}_P)$ *are given by*

$$\Pi_- = \Pi_{j\in\mathbb{N}}\big(1 - \frac{1}{\mu_j^2}(L_{|v|} - R_{|v|})^2\big)\mathfrak{Re} \qquad \Pi_+ = \Pi_{j\in\mathbb{N}}\big(1 - \frac{1}{\mu_j^2}(L_{|v|} + R_{|v|})^2\big)\mathfrak{Im}$$

for $\mu_j = j|k|^{-1}|s - s'|$*, and they preserve self-adjoint (resp. skew-adjoint) operators.*

The proof of this is part of the paper [3], where we also obtain:

Theorem (C)**.** *Let* $T = T(k, s, s')$*, let* $\mu_j = \frac{j}{|k|}|s - s'|$ *and*

$$\Lambda = \{j \in \mathbb{N} : \exists s_1 \neq s_2 \in \sigma(V) \textit{ with } j|s - s'| = |k||s_1 - s_2|\},$$

let

$$\mathfrak{H} = \oplus_{j\in\Lambda} \ker((L - R)^2 - \mu_j^2)\big|_{(\mathcal{A}_0)_h}, \qquad \mathfrak{K} = \oplus_{j\in\Lambda} \ker(L + R - \mu_j)\big|_{(\mathcal{A}_0)_{sk}}.$$

Then $\ker(DExp_P)_{TV} = \mathcal{S} \oplus \mathcal{T}$*, where*

$$\mathcal{S} = \{\Omega(a + b) + (a - b)\Omega^* : \, a \in \mathfrak{H}, \, b \in \mathfrak{K}\}$$

and

$$\mathcal{T} = \{x = P_v x + x P_v \in T_P Gr(P_0) : \operatorname{sinhc}(TV)x = xP_v\}.$$

We remark that from the results of Sakai [41], in the classical Grassmannians $Gr_k(n)$, along a unit speed geodesic the point $Q = \gamma(T)$ for $T = (k, s, s')$ as above is always conjugate to P. In our setting, in [3] we obtained

Theorem 9.3. *If* $\mathcal{A}_0$ *is a prime* C^**-algebra or a von Neumann factor, then each* $Q = \gamma(T)$ *is either monogonjugate or epiconjugate to* P*.*

10. Counting dimensions and examples

To compute the dimension of the kernel at the first tangent conjugate point, by Theorem (B) it then suffices to compute the dimension of $X = \{\, b^* = -b \, : \, |\lambda| b = b\}$. Since $\| |\lambda| \| = 1$, we know that $1 \in \sigma(|\lambda|)$. But it is possible that 1 is not an eigenvalue of $|\lambda|$, hence in that case the space X is null, thus the kernel is null.

On the other hand, if 1 is an eigenvalue of V (equivalently, it is isolated in the spectrum), let $P_1 \in \mathcal{A}_0$ be the associated eigenprojection and let $r = \dim_{\mathbb{R}}(P_1)$. Then it must be

$$r \leq \operatorname{rank}(P_{|\lambda|}) = \operatorname{rank}(|\lambda|) = \operatorname{rank}(\lambda^*\lambda) \leq \min\{\operatorname{rank}(P), \operatorname{rank}(1 - P)\}.$$

Corollary 10.1. *For the complex Grassmannian the order at* $V_0 = \frac{\pi}{2}V$ *is* $d = r^2$*, and for the real Grassmannian it is* $d = \frac{r(r-1)}{2}$

Proof. We have to count the solutions of $b^* = -b$, $P_1 b = b$, or equivalently $b^* = -b$ and the range of b is of dimension r. Now, those numbers are the real dimensions of the spaces of skew-Hermitian (resp. skew-symmetric) matrices acting on a space of real dimension r. □

Lemma 10.2 (Eigenvalues of V produce monoconjugate points). *Assumme $s \neq s' \in \sigma(V)$ are eigenvalues with respective eigenvectors $\xi_s, \xi_{'}$ and assume that both $\xi_s \otimes \xi_{'}$ and $\xi_{'} \otimes \xi_s$ belong to $\mathcal{A}$. Then $\gamma(T)$ for $T = T(k, s, s')$ is monoconjugate to $P = \gamma(0)$ for any $k \in \mathbb{Z}^*$.*

Proof. Consider the equations for $j = k$, $s_1 = s$, $s_2 = s'$

$$|\lambda|^2 a + a|\lambda|^2 - 2|\lambda|a|\lambda| - |s - s'|^2 a = 0 \tag{3}$$

$$|\lambda|^2 b + b|\lambda|^2 + 2|\lambda|b|\lambda| - |s - s'|^2 b = 0. \tag{4}$$

for self-adjoint $a \in \mathcal{A}_0$ (resp. skew-adjoint b). Assumme $\mathcal{A}_0$ represented in some Hilbert space $\mathcal{H}$. We can safely assume that $0 < s \leq 1$. Consider first the case of $0 < s' < s$. Let $\xi_s, \xi_{'} \in \mathcal{H}$ such that $|\lambda|\xi_s = s\xi_s$ and $|\lambda|\xi_{'} = s'\xi_{'}$. Note that $\xi_s, \xi_{'} \in \operatorname{Ran} P_{|\lambda|} \cap \operatorname{Ran}(1 - P)$ since $\xi_s = s^{-1}|\lambda|\xi_s$ and likewise for s'. Consider $a = a^* = \xi_s \otimes \xi_{'} + \xi_{'} \otimes \xi_s \in \mathcal{A}$, then

$$P_{|\lambda|}a = (P_{|\lambda|}\xi_s) \otimes \xi_{'} + (P_{|\lambda|}\xi_{'}) \otimes \xi_s = a$$

and likewise $aP_{|\lambda|} = a$. The same argument shows that $(1 - P)a = a = a(1 - P)$, which shows that $a \in \mathcal{A}_0$. On the other hand, it is easy to check that a is a solution of equation (3), hence the kernel is nontrivial. If $s < s' \leq 1$, the same solution applies since we can exchange s, s'. Now assume $s' \leq 0$, take ξ_s as before and $\xi_{'}$ such that $|\lambda|\xi_{'} = |s'|\xi_{'} = -s'\xi_{'}$. In this case consider $-b^* = b = \xi_s \otimes \xi_{'} - \xi_{'} \otimes \xi_s \in \mathcal{A}_0$. Then it is easy to check that b is a solution of equation (4). If $s' \neq -s$, then $b \neq 0$ and the kernel is nontrivial. If $s' = -s$, one would need to ask that the eigenspace of s has real dimension at least 2 (this is plain for the complex case), for in this case one can take two linearly independent eigenvectors $\xi_s, \xi_{'}$ of the eigenvalue s, and then $b \neq 0$. □

Remark 10.3 (Compact operators and the restricted Grassmannian). Consider the unitary group of a proper ideal $\mathcal{I} \subset \mathcal{K}(\mathcal{H})$ of compact operators (cf. Gohberg and Krein [16, Chapter III]):

$$\mathcal{U}_{\mathcal{I}} = \{u \in \mathcal{U}(H) : u - 1 \in \mathcal{I}\} = \exp\{A : A^* = -A \in \mathcal{I}\}.$$

A relevant case of infinite dimensional Grassmannian occurs when we consider the coadjoint orbit a a projection $P \in \mathcal{B}(\mathcal{H})$ for the action of the group $\mathcal{U}_{\mathcal{I}}$, i.e.

$$Gr_{\mathcal{I}}(P_0) = \{uPu^* : u \in \mathcal{U}_{\mathcal{I}}\}.$$

Then the number d of Corollary 10.1 is finite, and the statement of that corollary holds for the restricted Grassmannian, with the same proof (despite the fact that in general it is not the unitary group of a C^*-algebra).

We also want to mention that the argument in the previous Lemma 10.2 holds for these restricted Grassmannians (since the ideal $\mathcal{I}$ contains all finite rank operators $\xi_s \otimes \xi_s'$), with one exception. Indeed since V is compact then $|\lambda|$ is positive compact, thus for each nonzero eigenvalue we have a finite dimensional nontrivial eigenspace.

The exception is for the case of $s \neq 0, s' = 0$, i.e. the candidate to conjugate point v such that $\operatorname{ad}^2 v + s^2 1 = 0$ with $0 \neq s \in \sigma(|\lambda|)$. This is an exception because the kernel of V (equivalently, of $|\lambda|$) might be trivial thus we cannot build neither a nor b. Thus *all candidates $Q = \gamma(T)$ are monoconjugate to $P = \gamma(0)$*, except perhaps for the case of $T = (k, s, 0)$. Other restricted Grassmannians can be approached with our techniques, for instance those considered in [7] by Ratiu et al.

11. Examples

We now present a series of examples to conclude.

Example 11.1 (First epiconjugate point which is not monoconjugate)**.** Let $\mathcal{H} = L^2[-1, 1]$, and let $\mathcal{A} = \mathcal{B}(\mathcal{H})$. Let P be the orthogonal projection given taking the even part of a function $f \in \mathcal{H}$, i.e.

$$Pf(x) = \frac{1}{2}(f(x) + f(-x)).$$

Let $V^* = V \in \mathcal{A}$ be given by $Vf(x) = xf(x)$. Then $\sigma(V) = [-1, 1]$ and in particular $\|V\| = 1$. Moreover if $v = [V, P]$ then $vf(x) = xf(-x)$. Now

$$PVf(x) = \frac{1}{2}(xf(x) - xf(-x)) = V(1 - P)f(x)$$

hence $V = PV + VP$ thus V is P-codiagonal. Let γ be the geodesic through P with intial speed V. We claim that $Q = \gamma(\frac{\pi}{2})$ *is not monoconjugate but epiconjugate to P*.

To this end, note first that $|V|f(x) = |x|f(x)$ and it is plain that $\sigma(|V|) = [0, 1]$ while $|V|$ has no eigenvalues. This also tells us that $P_V f(x) = f(x)$ i.e. P_V is the identity operator. Therefore $\operatorname{sinhc}(T \operatorname{ad} V)$ is unitary equivalent to $H \oplus K$ for any T. We have $P_{|\lambda|} = 1 - P$ and moreover $|\lambda| = (1 - P)|V|(1 - P)$ is described by $(|\lambda|f)(x) = |x|f(x)$ for odd functions $f \in L^2[-1, 1]$, which is the range of $1 - P$. We use the characterization of first conjugate points obtained in Theorems (A) and (B). The point Q must be conjugate to P. But $|\lambda|b = b$ for $b^* = -b$ has no solutions in $\mathcal{A}_0$, because $|\lambda|$ has no eigenvalues. Hence the point is not monoconjugate but epinconjugate.

By taking the product of algebras, we show an example where

Example 11.2 ($Q = \gamma(\pi/2)$ is monoconjugate and epiconjugate to P)**.** Consider the direct sum $\mathcal{A} = \mathcal{B}(\mathcal{H}) \oplus M_2(\mathbb{C})$, with the maximum norm, with $\mathcal{H} = L^2[-1, 1]$ as above. Let P, V be as in the previous example and let $p = e_1 \otimes e_1 \in M_2(\mathbb{C})$, while $w = e_1 \otimes e_2 - e_2 \otimes e_1$ is p-codiagonal, skew adjoint and of unit norm. consider $P' = (P, p)$, $V' = (V, w)$, then with the product in each coordinate it is plain that P' is a projection and V' is P'-codiagonal and of unit norm. In both cases P_v, P_w is the unit of the respective algebra, hence $P_{V'}$ is the unit of $\mathcal{A}$. Therefore for the first conjugate point we are again dealing only with the right-down corner of the algebra

$\mathcal{A}$, which is the direct sum of both corners. Moreover $|\lambda|(f,\xi) = (|x|f,\xi)$ for odd $f \in L^2[-1,1]$ and $\xi \in \mathbb{C}$, i.e.

$$L_{|\lambda|} = \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}$$

Where $M = M_{|x|}$ is the multiplication operator. In the second coordinate, the kernel is the span of $b = (0, i)$, in particular Q is monoconjugate to P. Now $L_M + R_M - 2$ is not invertible and injective (previous example), therefore it is not surjective and Q is also epiconjugate to P.

11.1. **Projective spaces.** We now characterize the kernel for all conjugate points in projective spaces, presented as the orbit of a one-dimensional projection:

Example 11.3 (Complex projective space)**.** Here P is (complex) one-dimensional projection. In this case $\lambda\mathcal{X}^*$, $\mathcal{X}\lambda^*$ are complex numbers and $\lambda\lambda^*$ is a real non-negative number. The normalization condition implies that $\lambda\lambda^* = \|\lambda\lambda^*\| = 1$, and this also tells us that $p = |\lambda|^2 = \lambda^*\lambda$ is a one-dimensional projection in $\mathcal{A}_0$. It is apparent that $\Omega = \lambda \in \mathcal{A}$ and in particular solving for $z \in \mathcal{A}_0$ solves the problem in $\mathcal{A}$.

Proposition 11.4. *For each tangent V in the complex projective space, there are two kinds of monoconjugate points:*

i) $T_0 = (2k+1)\frac{\pi}{2}$, $k \in \mathbb{Z}$*: the kernel is spanned by $X = iV$ (in particular the order is always* 1*)*

ii) $T_1 = k\pi, k \in \mathbb{Z}_{\neq 0}$*: the kernel is given by all $b^* = -b \in \mathcal{A}_0$, and the whole $\widetilde{\mathcal{C}}_{P_v}$.*

For the case of $Gr(P_0) = \mathbb{C}^n/\mathbb{C}$, the order of the T_1 points is $2(2n-3)$*.*

Example 11.5 (Real projective space)**.** In this case the computations are done in the same fashion as in the previous example, but now the subspaces are real. Therefore the solutions for the points of type T_0 is $\{0\}$, because there are no skew-adjoint matrix in real dimension 1. Therefore these are not monoconjugate points. The points of type T_1 fulfill the same conditions as in the previous example: $b^* = -b \in \mathcal{A}_0$ and the whole space of P_v co-diagonal operators. For the case of $Gr(P_0) = \mathbb{R}^n/\mathbb{R}$, these spaces have dimension $n-2$ and $n-1$ respectively so the order of these points is $2n-3$.

We close with a finite dimensional example where $Q = \gamma(T)$ for $T(k,s,s')$ *is not* conjugate to P, except for the case of the points $T = \frac{k\pi}{2}$ already discussed. This is unlike the classical Grassmannians $Gr_k(n)$ where they all are conjugate, and the main reason of failure is that $\mathcal{A}_0 \simeq \mathbb{C} \oplus \mathbb{C}$ is not a factor.

Example 11.6. Let $\mathcal{A} = M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$, let

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}$$

for some $0<\alpha<1$. Then V is P-codiagonal, $\sigma(V)=\{-1,-\alpha,\alpha,1\}$ and $P_V=1\oplus 1$ is the identity of $\mathcal{A}$. We also have

$$\mathcal{A}_0=\begin{pmatrix} 0 & 0\\ 0 & \mathbb{C}\end{pmatrix}\oplus\begin{pmatrix} 0 & 0\\ 0 & \mathbb{C}\end{pmatrix},\quad |\lambda|=\begin{pmatrix} 0 & 0\\ 0 & 1\end{pmatrix}\oplus\begin{pmatrix} 0 & 0\\ 0 & \alpha\end{pmatrix},$$

and the identity of $\mathcal{A}_0$ is of course $1-P$. Then $\sigma(|\lambda|)=\{1,\alpha\}$ but $L-R=0$ in $\mathcal{A}_0$ hence $\sigma(L-R)\subsetneq\{0,1-\alpha,\alpha-1\}$. On the other hand $L+R=2\oplus 2\alpha$ in $\mathcal{A}_0$ hence $\sigma(L+R)=\{2,2\alpha\}$ again with strict inclusion in $\{2\alpha,1+\alpha,2\}$. There are four family of candidates to conjugate points,

$$T_1=\frac{k\pi}{2},\quad T_2=\frac{k\pi}{1+\alpha},\quad T_3=\frac{k\pi}{1-\alpha},\quad T_4=\frac{k\pi}{2\alpha}.$$

For the first family we know that $\gamma(T_1)$ is conjugate to P, in fact monoconjugate because the algebra is finite dimensional. On the other hand it is easy to see that *none of the other points are conjugate to* P: we only show that for the case of T_2, the other cases being similar. For this case one can check that the only possible value of μ_j is $\mu=1+\alpha>1$. Since $P_V=1$ all conjugate points occur inside $\mathcal{A}_0$. Therefore we are only interested in

$$H=(L-R)^2-(\alpha+1)=-(\alpha+1)$$

which is invertible and

$$K=L+R-(\alpha+1)=(1-\alpha)\oplus(\alpha-1)$$

which is also invertible. Hence $\operatorname{sinhc}(T_2\operatorname{ad}v)$ is invertible and $\gamma(T_2)$ is not conjugate to P along γ.

References

[1] Andruchow, E. *Geodesics of projections in von Neumann algebras.* Proc. Amer. Math. Soc. 149 (2021), no. 10, 4501–4513.

[2] Andruchow, E.; Chiumiento, E.; Larotonda, G. *Homogeneous manifolds from noncommutative measure spaces.* J. Math. Anal. Appl. 365 (2010) 541–558.

[3] Andruchow, E.; Larotonda, G.; Recht. L. *Conjugate points in the Grassmann manifold of a C^*-algebra*, arXiv preprint, submited (2023) https://arxiv.org/abs/2307.09345v1.

[4] Abrams, G.; Tomforde, M. *A class of C^*-algebras that are prime but not primitive.* Münster J. Math. 7 (2014), no. 2, 489–514.

[5] Arias, M. L.; Mbekhta, M. *On partial isometries in C^*-algebras.* Studia Math. 205 (2011), no. 1, 71–82.

[6] Avron, J.; Seiler, R.; Simon, B. *The index of a pair of projections.* J. Funct. Anal. 120 (1994), no. 1, 220–237.

[7] Beltita, D.; Ratiu, T. S.; Tumpach, A. B. *The restricted Grassmannian, Banach Lie-Poisson spaces, and coadjoint orbits.* J. Funct. Anal. 247 (2007), no. 1, 138–168.

[8] Biliotti, L,; Exel, R.; Piccione, P.; Tausk, D.V. *On the singularities of the exponential map in infinite dimensional Riemannian manifolds.* Math. Ann. 336 (2006), no. 2, 247–267.

[9] Berceanu, S. *On the geometry of complex Grassmann manifold, its noncompact dual and coherent states.* Bull. Belg. Math. Soc. Simon Stevin 4 (1997), no. 2, 205–243.

[10] Bonsall, F.F; Duncan, J. NUMERICAL RANGES OF OPERATORS ON NORMED SPACES AND OF ELEMENTS OF NORMED ALGEBRAS. London Mathematical Society Lecture Note Series **2**, Cambridge University Press, London-New York, 1971.

[11] Corach, G.; Porta, H.; Recht, L. *The geometry of spaces of projections in C^*-algebras.* Adv. Math. 101 (1993), no. 1, 59–77.

[12] Chu, C.-H. BOUNDED SYMMETRIC DOMAINS IN BANACH SPACES. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2021.

[13] Crittenden, R. *Minimum and conjugate points in symmetric spaces.* Canad. J. Math. 14 (1962), 320–328.

[14] Davis, C. *Separation of two linear subspaces.* Acta Sci. Math. Szeged 19 (1958) 172–187.

[15] Dixmier, J. *Position relative de deux variétés linéaires fermées dans un espace de Hilbert.* (French) Revue Sci. 86, (1948). 387–399.

[16] Gohberg, I. C.; Krein, M. G. INTRODUCTION TO THE THEORY OF LINEAR NONSELFADJOINT OPERATORS. Translated from the Russian by A. Feinstein Translations of Mathematical Monographs, Vol. 18 American Mathematical Society, Providence, R.I. 1969.

[17] Grossman, N. *Hilbert manifolds without epiconjugate points* Proc. Amer. Math. Soc. 16 (1965), 1365–1371.

[18] Halmos, P. R. *Two subspaces.* Trans. Amer. Math. Soc. 144 (1969) 381–389.

[19] Helgason, S. DIFFERENTIAL GEOMETRY, LIE GROUPS, AND SYMMETRIC SPACES. Corrected reprint of the 1978 original. Graduate Studies in Mathematics, 34. American Mathematical Society, Providence, RI, 2001.

[20] Kato, T. *Perturbation theory for linear operators.* Reprint of the 1980 edition. Classics in Mathematics. Springer-Verlag, Berlin, 1995.

[21] Khesin, B.; Wendt, R. *The geometry of infinite-dimensional groups.* Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], 51. Springer-Verlag, Berlin, 2009.

[22] Kobayashi, S.; Nomizu, K. FOUNDATIONS OF DIFFERENTIAL GEOMETRY. VOL. II. Reprint of the 1969 original. Wiley Classics Library. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1996.

[23] Kovarik, Z. *Manifolds of linear involutions.* Linear Algebra Appl. 24 (1979), 271–287.

[24] Lang, S. FUNDAMENTALS OF DIFFERENTIAL GEOMETRY. Graduate Texts in Mathematics, 191. Springer-Verlag, New York, 1999.

[25] Larotonda, G. ESTRUCTURAS GEOMÉTRICAS PARA LAS VARIEDADES DE BANACH. 2da Edición, book preprint 2022.

[26] Larotonda, G. *Metric geometry of infinite-dimensional Lie groups and their homogeneous spaces.* Forum Math. 31 (2019), no. 6, 1567–1605.

[27] Larotonda, G.; Miglioli, M. *Hofer's metric in compact Lie groups.* Groups Geom. Dyn. 17 (2023), no. 3, 839–898.

[28] Li, C.-K; Qiu, L; Zhang, Y.; *Unitarily invariant metrics on the Grassmann space.* SIAM J. Matrix Anal. Appl. 27 (2005), no. 2, 507–531.

[29] Lumer, G.; Rosenblum, M. *Linear operator equations.* Proc. Amer. Math. Soc. 10 (1959), 32–41.

[30] Mata-Lorenzo, L. E.; Recht, L. *Infinite-dimensional homogeneous reductive spaces.* Acta Cient. Venezolana 43 (1992), no. 2, 76–90.

[31] Mathieu, M. *Spectral theory for multiplication operators on C^*-algebras.* Proc. Roy. Irish Acad. Sect. A 83 (1983), no. 2, 231–249.

[32] Mathieu, M. *Elementary operators on prime C^*-algebras. I.* Math. Ann. 284 (1989), no. 2, 223–244.

[33] Mcalpin, J. H. INFINITE DIMENSIONAL MANIFOLDS AND MORSE THEORY, ProQuest LLC, Ann Arbor, MI, 1965, Thesis (Ph.D.) Columbia University.

[34] Neeb, K.-H. *A Cartan-Hadamard theorem for Banach-Finsler manifolds.* Proceedings of the Conference on Geometric and Combinatorial Group Theory, Part II (Haifa, 2000). Geom. Dedicata 95 (2002), 115–156.

[35] N.C. Phillips. *The rectifiable metric on the space of projections in a C^*-algebra.* Internat. J. Math. 3 (1992), no. 5, 679–698.

[36] Porta, H.; Recht, L. *Minimality of geodesics in Grassmann manifolds.* Proc. Amer. Math. Soc. 100 (1987), no. 3, 464–466.

[37] Porta, H.; Recht, L. *Spaces of projections in a Banach algebra.* Acta Cient. Venezolana 38 (1987), no. 4, 408–426 (1988).

[38] Rosenberg, J. *Structure and applications of real C^*-algebras.* Operator algebras and their applications, 235–258, Contemp. Math., 671, Amer. Math. Soc., Providence, RI, 2016.

[39] Pressley, A.; S. Graeme. *Loop groups.* Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1986

[40] Segal, G.; Wilson, G. *Loop groups and equations of KdV type.* Iintegrable systems, 403–466, Surv. Differ. Geom., 4, Int. Press, Boston, MA, 1998.

[41] Sakai, T. *On cut loci on compact symmetric spaces*, Hokkaido Math. J. 6 (1977), no. 1, 136–161.

[42] Toms, A. S. *Cancellation does not imply stable rank one.* Bull. London Math. Soc. 38 (2006), no. 6, 1005–1008.

[43] Tumpach, A. B. *Infinite-dimensional hyperkähler manifolds associated with Hermitian-symmetric affine coadjoint orbits.* Ann. Inst. Fourier (Grenoble) 59 (2009), no. 1, 167–197.

[44] Wong, Y.-c. *Differential geometry of Grassmann manifolds.* Proc. Nat. Acad. Sci. U.S.A. 57 (1967), 589–594.

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