Kepler problem in the harmonic oscillator setting: recursion operator and relevant properties

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Outline

On Integrable Hamiltonian systems

- In the last few decades, there was a renewed interest in completely integrable Hamiltonian systems (IHS), the concept of which goes back to Liouville in 1897 and Poincaré in 1899.
- Many of these systems obey Hamiltonian dynamics with respect to two compatible symplectic structures [F. Magri, 1978 and 1980; I. M. Gelfand, I. Ya. Dorfman, 1980; G. Vilasi, 1980]. (This permits a geometrical interpretation of the so-called recursion operator [P. D. Lax, 1968]).
- Since the work by Magri [F. Magri, 1978], the integrability using bi-Hamiltonian structures became one of the most powerful methods used for the integrability of evolution equations in both finite and infinite dimensional dynamical systems [P. D. Lax,(1968); R. G. Smirnov, 1997].
- When a completely integrable Hamiltonian system admits a bi-Hamiltonian construction, one can generate infinite hierarchies of conserved quantities using the construction by Oevel [W. Ovel, 1986] based on scaling invariances and master symmetries [R. Fernandes, 1994; R.G. Smirnov, 1999].

On bi-Hamiltonian system and master symmetries

- In this same spirit, we constructed a hierarchy of bi-Hamiltonian structures for some relevant physical systems and computed conserved quantities using related master symmetries [M. J. Landalidji *et al.*, 2021, 2022, 2025].
- In addition, many works in nanotechnology create a remarkable opportunity to trace their dynamical interplay in mesoscopic systems such as quantum dots, which confine a few electrons to a space of a few hundred nanometers [N.S. Simonović *et al.* (2003)].
- In 2013, P. M. Zhang *et al.* showed that the separability and Runge-Lenz-type dynamical symmetry of the internal dynamics of certain two-electron Quantum Dots, found by Simonović *et al.* (2003), are traced back to that of the perturbed Kepler problem.
- In that work, they started with the pure Kepler problem and then inquire what potential can be added such that separability in parabolic coordinates is preserved.
- They also viewed the Runge-Lenz-type conserved quantity K_Z as the Keplerian expression corrected by a term due to the oscillator.

In this talk, we address recursion operator and some relevant properties for the Kepler problem in the harmonic oscillator setting.

Recursion operator

Let's quickly recall main results and facts from previous works.

Theorem 1 [De Filippo *et al.*, 1982]

Let X be a vector field on a $2n$-dimensional manifold $\mathcal{M}$ which admits a diagonalizable mixed $(1,1)$-tensor field T satisfying the following conditions:

1. T is invariant under X;
2. T has a vanishing Nijenhuis torsion;
3. T has doubly degenerate eigenvalues with nowhere vanishing differentials.

Then, there exists a symplectic structure, and the vector field X is a separable Hamiltonian vector field of some Hamiltonian function which is completely integrable with respect to the symplectic struture.

The $(1,1)$-tensor field T is called a **recursion operator of X**.

In the particular case of $\mathbb{R}^{2n}$, Takeuchi (2015) gave the following construction:

Recursion operator

Lemma 1 [T. Takeuchi, 2015]

Let us consider vector fields

$$X_l = -\frac{\partial}{\partial x_{n+l}}, l = 1, ..., n \tag{1}$$

on $\mathbb{R}^{2n}$ and let U be a $(1,1)$-tensor field on $\mathbb{R}^{2n}$ given by

$$U = \sum_{i=1}^{n} x_i \left(\frac{\partial}{\partial x_i} \otimes dx_i + \frac{\partial}{\partial x_{n+i}} \otimes dx_{n+i} \right), \tag{2}$$

then we have that the Nijenhuis torsion of U is vanishing, *i.e.*,

$$U_i^k \frac{\partial U_j^h}{\partial x^k} - U_j^k \frac{\partial U_i^h}{\partial x^k} + U_k^h \frac{\partial U_i^k}{\partial x^j} - U_k^h \frac{\partial U_j^k}{\partial x^i} = 0, \tag{3}$$

and $\mathcal{L}_{X_l} U = 0$, *i.e.*, the $(1,1)$-tensor field U is a recursion operator of X_l.

Schouten-Nijenhuis bracket (1/2)

- Let Λ^k be the space of multivectors on $\mathcal{Q}$ and $\mathfrak{F}(\mathcal{Q})$ the set of mappings from $\mathcal{Q}$ into $\mathbb{R}$ that are of class C^∞ .
- The Schouten-Nijenhuis bracket is a bilinear pairing [B. Dubrovin, 2005]

$$[.,.]_{NS} : \Lambda^k \times \Lambda^l \to \Lambda^{k+l-1}, \quad a, b \mapsto [a, b]_{NS} \tag{4}$$

uniquely determined by the properties of supersymmetry

$$[a, b]_{NS} = (-1)^{kl}[b, a]_{NS}, \quad a \in \Lambda^k, b \in \Lambda^l, \tag{5}$$

the graded Leibnitz rule

$$[c, a \wedge b]_{NS} = [c, a]_{NS} \wedge b + (-1)^{kl+k} a \wedge [c, b]_{NS}, \quad a \in \Lambda^k, \ c \in \Lambda^l, \tag{6}$$

and the conditions:

$$(1) \ \ [f, g]_{NS} = 0, \quad f, g \in \Lambda^0 = \mathfrak{F}(\mathcal{Q});$$

Schouten-Nijenhuis bracket (2/2)

(2) $[v, f]_{NS} = v^i \dfrac{\partial f}{\partial x^i}, \quad v \in \Lambda^1;$

(3) $[v_1, v_2]_{NS} =$ commutator of vector fields for $v_1, v_2 \in \Lambda^1$.

- It also satisfies the graded Jacobi identity

$$(-1)^{km}[[a,b]_{NS},c]_{NS} + (-1)^{lm}[[c,a]_{NS},b]_{NS} + (-1)^{kl}[[b,c]_{NS},a]_{NS} = 0. \quad (7)$$

- **Example:** For two bivectors $\sigma = (\sigma^{ij})$ and (τ^{ij}), their Schouten-Nijenhuis bracket is the following trivector:

$$[\sigma,\tau]^{ijk}_{NS} = \frac{\partial \sigma^{ij}}{\partial x^s}\tau^{sk} + \frac{\partial \tau^{ij}}{\partial x^s}\sigma^{sk} + \frac{\partial \sigma^{ki}}{\partial x^s}\tau^{sj} + \frac{\partial \tau^{ki}}{\partial x^s}\sigma^{sj} + \frac{\partial \sigma^{jk}}{\partial x^s}\tau^{si} + \frac{\partial \tau^{jk}}{\partial x^s}\sigma^{si}. \quad (8)$$

Bi-Hamiltonian system

- Given a general dynamical system defined on a $2n$-dimensional manifold $\mathcal{Q}$ [R. G. Smirnov, 1997],

$$\dot{x}(t) = X(x), \quad x \in \mathcal{Q}, \quad X \in \mathcal{T}\mathcal{Q}. \tag{9}$$

(i) If this system (9) admits two different Hamiltonian representations:

$$\dot{x}(t) = X_{H_1,H_2} = \mathcal{P}_1 dH_1 = \mathcal{P}_2 dH_2, \tag{10}$$

its integrability as well as many other properties are subject to Magri's approach. *i.e.*, the bi-Hamiltonian vector field X_{H_1,H_2} is defined by two pairs of Poisson bivectors $\mathcal{P}_1, \mathcal{P}_2$ and Hamiltonian functions H_1, H_2.

- $\mathcal{P}_1$ and $\mathcal{P}_2$ are compatible Poisson bivectors with vanishing Schouten-Nijenhuis bracket: $[\mathcal{P}_1, \mathcal{P}_2]_{NS} = 0$.

- Such a manifold $\mathcal{Q}$ equipped with two Poisson bivectors is called a double Poisson manifold and the quadruple $(\mathcal{Q}, \mathcal{P}_1, \mathcal{P}_2, X_{H_1,H_2})$ is called bi-Hamiltonian system.

Perturbed Kepler problem

- We consider the following Hamiltonian function describing the Kepler problem perturbed by a harmonic (but not necessarily isotropic) oscillator [P. M. Zhang *et al.*, (2013),]

$$H = H_{Kepler} + V_{osc}, \tag{11}$$

where $H_{Kepler} = \frac{1}{2M}(p_x^2 + p_y^2 + p_z^2) - \frac{k}{r}$, $V_{osc} = \frac{1}{2}(\omega_\rho^2 \rho^2 + \omega_z^2 z^2)$, $\omega_\rho = \sqrt{\omega_L^2 + \omega_0^2}$, $\omega_L = \frac{eB}{M}$.

- In the sequel, we put $M = 1$

Hamiltonian function in parabolic coordinates

- In parabolic coordinates, given by $x = \xi\eta\sin\varphi,\ y = \xi\eta\cos\varphi,\ z = \frac{1}{2}\left(\xi^2 - \eta^2\right),$ $(0 \leq \xi < \infty, 0 \leq \eta < \infty,\ 0 \leq \varphi < 2\pi)$, the Hamiltonian function H takes the form

$$H = \frac{1}{2(\xi^2+\eta^2)}\left[p_\xi^2 + p_\eta^2 + \left(\frac{1}{\xi^2} + \frac{1}{\eta^2}\right)p_\varphi^2\right] - \frac{2k}{\xi^2+\eta^2} + \frac{1}{2}\left((\xi\eta)^2\omega_\rho^2 + \frac{1}{4}\omega_z^2(\xi^4 - 2\xi^2\eta^2 + \eta^4)\right), \quad (12)$$

  where $p_\xi = (\xi^2+\eta^2)\dot{\xi},\ p_\eta = (\xi^2+\eta^2)\dot{\eta}$, and $p_\varphi = \xi^2\eta^2\dot{\varphi}$.
- Separability of H refering to that of the Hamilton-Jacobi equation $\frac{\partial V}{\partial t} + H\left(\frac{\partial V}{\partial q}, q, t\right) = 0,\ (q = \xi, \eta, \varphi)$ is achieved when $\omega_\rho = 2\omega_z$.
- In this case, H becomes

Hamiltonian function in parabolic coordinates

$$H = \frac{1}{2(\xi^2+\eta^2)}\left[p_\xi^2 + p_\eta^2 + \left(\frac{1}{\xi^2}+\frac{1}{\eta^2}\right)p_\varphi^2\right] - \frac{2k}{\xi^2+\eta^2} + \frac{\omega_\rho^2}{2}\left(\xi^4 - \xi^2\eta^2 + \eta^4\right). \quad (13)$$

- For computational reasons, we restrict the work to a two-dimensional coordinate system by putting $\varphi = 0$, in particular, parabolic coordinates in the $x - z$ plane,

$$x = \xi\eta, \quad z = \frac{1}{2}\left(\xi^2 - \eta^2\right), \quad (14)$$

- This condition constrains the motion into a vertical plane through the z axis and in fact reduces the problem to the perturbed Kepler problem in $2D$.
- Since $\xi\eta > 0$, we notice that (14) is in fact only half of a coordinate system, and should therefore just be supplemented with $-x = -\xi\eta$ to cover the whole vertical plane.

Hamiltonian function in parabolic coordinates

- This subtlety is not present in $3D$, since the first coordinate is indeed $\rho > 0$, and the angular variable φ takes care of the $x < 0$ half plane, namely for $\varphi = \pi$.
- Furthermore, we notice the separability condition $\omega_\rho = 2\omega_z$ does not change in this two-dimensional coordinate system.
- Then, the Hamiltonian function is reduced to

$$H = \frac{1}{2(\xi^2+\eta^2)}\left[p_\xi^2 + p_\eta^2\right] - \frac{2k}{\xi^2+\eta^2} + \frac{\omega_\rho^2}{2(\xi^2+\eta^2)}\left(\xi^6+\eta^6\right). \qquad (15)$$

Hamiltonian system in parabolic coordinates

- The configuration space of the Kepler problem is a manifold $\mathcal{Q} = \mathbb{R}^2 \backslash \{0\}$ that is, a two-dimensional real euclidean vector space with the origin, the so-called collision point, removed.
- Let $\mathcal{T}^*\mathcal{Q} = \mathcal{Q} \times \mathbb{R}^2$ be the cotangent bundle with the local coordinates $(\eta, \xi, p_\eta, p_\xi)$.
- The cotangent bundle $\mathcal{T}^*\mathcal{Q}$ has a natural symplectic structure $\omega : \mathcal{T}\mathcal{Q} \longrightarrow \mathcal{T}^*\mathcal{Q}$ which, in local coordinates, is given by

$$\omega = dp_\eta \wedge d\eta + dp_\xi \wedge d\xi. \tag{16}$$

- Since ω is non-degenerate, it induces the map $\mathcal{P}$: $\mathcal{T}^*\mathcal{Q} \longrightarrow \mathcal{T}\mathcal{Q}$ defined by

$$\mathcal{P} = \frac{\partial}{\partial p_\eta} \wedge \frac{\partial}{\partial \eta} + \frac{\partial}{\partial p_\xi} \wedge \frac{\partial}{\partial \xi}, \tag{17}$$

 where $\mathcal{T}\mathcal{Q}$ is the tangent bundle. The map $\mathcal{P}$ is called the bivector field and $\mathcal{P}$ is the inverse map of ω, $\omega \circ \mathcal{P} = \mathcal{P} \circ \omega = 1$. We used it to construct the Hamiltonian vector field X_f of a Hamiltonian function f by the relation $X_f = \mathcal{P}df$.
- We consider the Hamiltonian function H and its corresponding 1-form $dH \in \mathcal{T}^*\mathcal{Q}$:

Hamiltonian system in parabolic coordinates

$$H = \frac{1}{2(\xi^2+\eta^2)}\left[p_\xi^2 + p_\eta^2\right] - \frac{2k}{\xi^2+\eta^2} + \frac{\omega_\rho^2}{2(\xi^2+\eta^2)}\left(\xi^6+\eta^6\right). \tag{18}$$

$$\begin{aligned} dH = \frac{p_\eta}{\eta^2+\xi^2}dp_\eta + \Bigg\{ -\frac{\eta}{(\eta^2+\xi^2)}\left[(p_\eta^2+p_\xi^2-4k)+\omega_\rho^2(\eta^6+\xi^6)\right] + \frac{3\eta^5\omega_\rho^2}{\eta^2+\xi^2}\Bigg\}d\eta \\ + \frac{p_\xi}{\eta^2+\xi^2}dp_\xi + \Bigg\{ -\frac{\xi}{(\eta^2+\xi^2)}\left[(p_\eta^2+p_\xi^2-4k)+\omega_\rho^2(\eta^6+\xi^6)\right] + \frac{3\xi^5\omega_\rho^2}{\eta^2+\xi^2}\Bigg\}d\xi \end{aligned} \tag{19}$$

- Then, the Hamiltonian vector field of H with respect to the symplectic structure ω is derived as

$$X_H := \{H, .\} = \left(\frac{\partial H}{\partial p_\eta}\frac{\partial}{\partial \eta} - \frac{\partial H}{\partial \eta}\frac{\partial}{\partial p_\eta}\right) + \left(\frac{\partial H}{\partial p_\xi}\frac{\partial}{\partial \xi} - \frac{\partial H}{\partial \xi}\frac{\partial}{\partial p_\xi}\right) \tag{20}$$

Hamiltonian system in parabolic coordinates

$$X_H = \frac{p_\eta}{\eta^2+\xi^2}\frac{\partial}{\partial\eta} + \left\{\frac{\eta}{(\eta^2+\xi^2)}\left[(p_\eta^2+p_\xi^2-4k)+\omega_\rho^2(\eta^6+\xi^6)\right] - \frac{3\eta^5\omega_\rho^2}{\eta^2+\xi^2}\right\}\frac{\partial}{\partial p_\eta}$$
$$\frac{p_\xi}{\eta^2+\xi^2}\frac{\partial}{\partial\xi} + \left\{\frac{\xi}{(\eta^2+\xi^2)}\left[(p_\eta^2+p_\xi^2-4k)+\omega_\rho^2(\eta^6+\xi^6)\right] - \frac{3\xi^5\omega_\rho^2}{\eta^2+\xi^2}\right\}\frac{\partial}{\partial p_\xi} \tag{21}$$

leading to the form

$$X_H = \frac{1}{\eta^2+\xi^2}\left[p_\eta\frac{\partial}{\partial\eta} + \eta(2H-3\eta^4\omega_\rho^2)\frac{\partial}{\partial p_\eta} + p_\xi\frac{\partial}{\partial\xi} + \xi(2H-3\xi^4\omega_\rho^2)\frac{\partial}{\partial p_\eta}\right] \tag{22}$$

- This Hamiltonian vector field satisfies the required condition for a Hamiltonian system, *i.e.*, $\iota_{X_H}\omega = -dH$.
- Hence, the triplet $(\mathcal{T}^*\mathcal{Q}, \omega, H)$ is a Hamiltonian system.

Recursion operator and constants of motion

- We consider the Hamilton-Jacobi equation with respect to the Hamiltonian function H, and introduce a generating function W satisfying the following canonical transformations [V. Arnold, 1978]:
$$p_\xi = \frac{\partial W}{\partial \xi},\ p_\eta = \frac{\partial W}{\partial \eta} \quad \text{and} \quad P_1 = -\frac{\partial W}{\partial Q^1},\ P_2 = -\frac{\partial W}{\partial Q^2}.$$
- Since the Hamiltonian function H, does not explicitly depend on the time, then, setting $V = W - Et$, it is possible to find an additive separable solution:
$$W = W_1(\eta) + W_2(\xi). \tag{23}$$
- The Hamilton-Jacobi equation $\frac{\partial V}{\partial t} + \mathcal{H}\left(\frac{\partial V}{\partial q}\Big/ q/t\right) = 0$ is then reduced to the nonlinear equation
$$E = \frac{1}{2(\xi^2+\eta^2)}\left[\left(\frac{\partial W}{\partial \eta}\right)^2 + \left(\frac{\partial W}{\partial \xi}\right)^2\right] - \frac{2k}{\xi^2+\eta^2} + \frac{\omega_\rho^2}{2(\xi^2+\eta^2)}\left(\xi^6+\eta^6\right), \tag{24}$$
where E is a constant.

Recursion operator and constants of motion

- To separate the variables, multiply the above equation by $\eta^2 + \xi^2$ and reorder the terms, obtaining

$$\left[-2E\eta^2 + \left(\frac{\partial W_\eta}{\partial \eta}\right)^2 + \omega_\rho^2\eta^6 + 2k\right] + \left[-2E\xi^2 + \left(\frac{\partial W_\xi}{\partial \xi}\right)^2 + \omega_\rho^2\xi^6 + 2k\right] = 0 \quad (25)$$

- Since the first term in the square bracket is a function of η, p_η and the second term of ξ, p_ξ only, and remembering that E is a constant, the above equation can be satisfied only if

$$\begin{cases} -2(E+k)\eta^2 + \left(\frac{\partial W_\eta}{\partial \eta}\right)^2 + \omega_\rho^2\eta^6 & = \Upsilon \quad (26) \\ 2(E+k)\xi^2 - \left(\frac{\partial W_\xi}{\partial \xi}\right)^2 - \omega_\rho^2\xi^6 & = \Upsilon, \quad (27) \end{cases}$$

where

$$\Upsilon = \frac{1}{2(\eta^2+\xi^2)}[\xi^2 p_\eta^2 - \eta^2 p_\xi^2] - k\frac{\xi^2-\eta^2}{\eta^2+\eta^2} - \omega_\rho^2(\xi^2\eta^2)\left(\frac{\xi^2-\eta^2}{2}\right) \quad (28)$$

is a constant of motion $i, e.,$ $\mathcal{L}_{X_H}\Upsilon = 0.$

Recursion operator and constants of motion

- We notice that Υ represents the z component of the Runge-Lenz vector in this perturbed Kepler problem [P. M. Zhang *et al.*, (2013)].
- Now, setting

$$\frac{\omega_\rho^2(\eta^4 - \Upsilon^2)}{2(E+k) - \Upsilon^2\omega^2} = 1 + \frac{E^2 - \Upsilon}{\eta^2(\Upsilon^2\omega_\rho^2 - 2(E+k))} \quad (29)$$

$$\frac{\omega_\rho^2(\xi^4 - \Upsilon^2)}{2(E+k) - \Upsilon^2\omega^2} = 2 + \frac{\Upsilon}{\xi^2(\Upsilon^2\omega^2 - 2(E+k))}, \quad (30)$$

with $2(E+k) << \Upsilon^2\omega_\rho^2$, we get

$$W \backsimeq \frac{1}{2}\eta^2 E + \frac{1}{2}\xi^2\omega_\rho\Upsilon, \quad (31)$$

where we considered the case $k < 0$ (the repulsive case).

- Putting $E = Q^1$ and $\Upsilon = Q^2$, W takes the form

$$W(\eta, \xi, Q^1, Q^2) \backsimeq \frac{1}{2}\eta^2 Q^1 + \frac{1}{2}\xi^2\omega_\rho Q^2 \quad (32)$$

leading to the following relationship between the canonical coordinate systems $(\eta, \xi, p_\eta, p_\xi)$ and (Q^1, Q^2, P_1, P_2):

Recursion operator and constants of motion

$$\begin{cases} P_1 = -\frac{1}{2}\eta^2 \\ P_2 = -\frac{1}{2}\xi^2\omega_\rho \end{cases} ; \begin{cases} Q^1 = E = H \\ Q^2 = \Upsilon \end{cases} ; \begin{cases} p_\eta = Q^1\sqrt{-2P_1}, \\ p_\xi = Q^2\sqrt{-2\omega_\rho P_2}, \end{cases} ; \begin{cases} \eta = \sqrt{-2P_1} \\ \xi = \sqrt{\frac{-2P_2}{\omega_\rho}} \end{cases} \tag{33}$$

Define in the coordinate system (Q, P):

- The symplectic form and vector field as:

$$\omega = \sum_{\nu=1}^{2} dP_i \wedge dQ^i, \quad X_H := \{H, .\} = -\frac{\partial}{\partial P_1}, \text{ and} \tag{34}$$

- The related tensor field T of $(1,1)$-type as:

$$T = \sum_{i=1}^{2} Q^i\left(\frac{\partial}{\partial P_i} \otimes dP_i + \frac{\partial}{\partial Q^i} \otimes dQ^i\right). \tag{35}$$

- Then, by Lemma 1, T satisfies $\mathcal{L}_{X_H}T = 0$, $\mathcal{N}_T = 0$ and $degQ^i = 2$ proving that T is a recursion operator of X_H. The constants of motion are:

$$Tr(T^h) = 2^h((Q^1)^h + (Q^2)^h), \; h \in \mathbb{N}. \tag{36}$$

Recursion operator and constants of motion

- Reverting back to the initial canonical coordinate system $(\eta, \xi, p_\eta, p_\xi)$, the solutions W leads to the following result for the perturbed Kepler Hamiltonian H.

Proposition 1

- Provided the conditions

(1) $p_\xi^2 + p_\eta^2 - 4k = \omega_\rho\left((\xi^6 + \eta^2) - \frac{6\eta^6(\xi^2 + \eta^2)}{\xi^2 - \eta^2}\right);$

(2) $\left(\frac{\xi^2}{\eta^2} + (1 - 2\omega_\rho)\right)p_\eta^2 - \left(\frac{\eta^2}{\xi^2} + (1 + 2\omega_\rho)\right)p_\xi^2 =$
$2\left(\frac{\xi^2}{\eta^2} - \frac{\eta^2}{\xi^2} + 2\omega_\rho\right) + \omega_\rho^2(\xi^2 + \eta^2)^2(\xi^2(1 + 4\omega_\rho) - \eta^2(1 + 2\omega_\rho)),$

(3) $\eta p_\eta = \xi p_\xi \eta^2 \omega_\rho = \xi^2 + \eta^2,$

the Hamiltonian vector field X_H has a recursion operator T given by

$$T = \sum_{i,j=1}^{2}\left(\tilde{V}_j^i \frac{\partial}{\partial q^i} \otimes dq^j + \tilde{U}_j^i \frac{\partial}{\partial p_i} \otimes dq^j + \tilde{Y}_j^i \frac{\partial}{\partial p_i} \otimes dp_j\right), \tag{37}$$

Recursion operator and constants of motion

with the corresponding constants of motion

$$Tr(T^h) = 2^h(H^h + \Upsilon^h)$$

$h \in \mathbb{N}$, where the coordinate dependent quantities $\tilde{V}_j^i$, $\tilde{U}_j^i$ and $\tilde{Y}_j^i$, are defined as follows:

$$\begin{cases} \tilde{V}_1^1 = H \\ \tilde{V}_2^2 = \Upsilon \\ \tilde{V}_j^i = 0, \quad \text{otherwise}, \end{cases} ; \quad \begin{cases} \tilde{Y}_1^1 = H \\ \tilde{Y}_2^2 = \Upsilon \\ \tilde{Y}_1^2 = \dfrac{\eta p_\xi H}{\xi^2 + \xi^2}, \\ \tilde{Y}_2^1 = \dfrac{\omega_\rho \Upsilon \xi^3 p_\eta}{\xi^2 + \xi^2} \end{cases} \tag{38}$$

$$\begin{cases} \tilde{U}_1^1 = \tilde{U}_2^2 = 0 \\ \tilde{U}_1^2 = H\left(-2\dfrac{\eta\xi H}{\xi^2 + \eta^2} + 3\dfrac{\eta\xi^5\omega_\rho^2}{\xi^2 + \eta^2}\right) \\ \tilde{U}_2^1 = \dfrac{2\eta\Upsilon\xi\omega_\rho}{\xi^2 + \eta^2}\left(\Upsilon + k + \dfrac{p_\xi^2}{2} + \dfrac{\xi^2\omega_\rho^2}{2}[((\xi)^2 + (\eta)^2) + 2\eta^4]\right), \end{cases} \tag{39}$$

Master symmetries

- Let us consider the Hamiltonian system $(\mathcal{T}^*\mathcal{Q}, \omega, Q^1)$, where the Hamiltonian function H, the vector field X, the symplectic form ω, and the bivector field are given f by:

$$H = Q^1;\ X = \{Q^1, .\} = -\frac{\partial}{\partial P_1};\ \omega = \sum_{i=1}^{2} dP_i \wedge dQ^i;\ \mathcal{P} = \sum_{i=1}^{2} \frac{\partial}{\partial P_i} \wedge \frac{\partial}{\partial Q^i}. \quad (40)$$

and introduce the following function

$$H_0 = \frac{1}{2}(Q^1)^2 + \frac{1}{2}(Q^2)^2 \quad (41)$$

satisfying the relation

$$\iota_{X_0}\omega = -dH_0, \quad (42)$$

with

$$X_0 = -\sum_{i=1}^{2} Q^i \frac{\partial}{\partial P_i} \quad (43)$$

- Relation (42) means that H_0 is a Hamiltonian function and X_0 its associated Hamiltonian vector field.

Master symmetries

- The recursion operator T:

$$T = \sum_{i=1}^{2} Q^i \left(\frac{\partial}{\partial P_i} \otimes dP_i + \frac{\partial}{\partial Q^i} \otimes dQ^i \right), \qquad (44)$$

can be written as: $T = \mathcal{P}_1 \circ \mathcal{P}^{-1}$, where

$$\mathcal{P}_1 = \sum_{i=1}^{2} Q^i \frac{\partial}{\partial P_i} \wedge \frac{\partial}{\partial Q^i} \qquad (45)$$

and $\mathcal{P}$ are two compatible Poisson bivectors with vanishing Schouten-Nijenhuis bracket: $[\mathcal{P}, \mathcal{P}_1]_{NS} = 0$.

- The corresponding symplectic form ω_1 is given by

$$\omega_1 = \sum_{i=1}^{2} (Q^i)^{-1} dP_i \wedge dQ^i. \qquad (46)$$

- Now, introducing the following Poisson bracket $\{.,.\}_1$

$$\{f,g\}_1 := \sum_{i=1}^{2} Q^i \left(\frac{\partial f}{\partial P_i} \frac{\partial g}{\partial Q^i} - \frac{\partial f}{\partial Q^i} \frac{\partial g}{\partial P_i} \right), \qquad (47)$$

Master symmetries

with respect to this symplectic form ω_1, we remark that

$$X_0 = \{H_0, .\} = \{H_1, .\}_1, \quad H_1 = Q^1 + Q^2 \tag{48}$$

proving that X_0 is a bi-Hamiltonian vector fields defined by the two Poisson bivectors $\mathcal{P}$ and $\mathcal{P}_1$.

- Then, the quadruple $(\mathcal{Q}, \mathcal{P}, \mathcal{P}_1, X_0)$ is a bi-Hamiltonian system.

Definition 2 [M. F. Rañada, 2005]

In differential geometric terms, a vector field Γ on $\mathcal{T}^*\mathcal{Q}$ that satisfies

$$[X_H, \Gamma] \neq 0, \quad [X_H, X] = 0, \quad [X_H, \Gamma] = X, \tag{49}$$

is called a master symmetry or a generator of symmetries of degree $m = 1$ for X_H.

- Defining the sequence $Y_k = T^k R X_0 = \sum_{i=1}^{2} (Q^i)^k P_i \frac{\partial}{\partial P_i}$, where $k \in \mathbb{N}$, and the $(1,1)$-tensor R

Master symmetries

$$R = \sum_{i=1}^{2} \Psi^i \left(\frac{\partial}{\partial P_i} \otimes dP_i + \frac{\partial}{\partial Q^i} \otimes dQ^i \right), \quad \Psi^i = -\frac{P_i}{Q^i}, \tag{50}$$

we obtain

$$[Y_i, Y_j] = 0, \quad [Y_i, X_j] = \sum_{i=1}^{2} (Q^i)^{i+j} \frac{\partial}{\partial P_i} = -X_{i+j}, \tag{51}$$

meaning that Y_k are master symmetries for X_0 and also for X.

Master symmetries

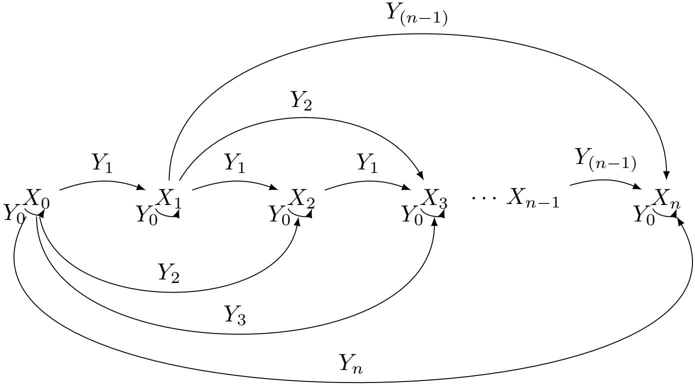

Figure 1: Diagrammatical illustration of equation (51).

Conserved quantities

- In addition, we have

$$\mathcal{L}_{Y_0}(\mathcal{P}) = -\sum_{\nu=1}^{2} \frac{\partial}{\partial P_i} \wedge \frac{\partial}{\partial Q^i},\ (\tilde{\alpha} = -1), \quad \mathcal{L}_{Y_0}(\mathcal{P}_1) = -\sum_{i=1}^{2} Q^i \frac{\partial}{\partial P_i} \wedge \frac{\partial}{\partial Q^i},$$
$$(\tilde{\beta} = -1), \quad \mathcal{L}_{Y_0}(H_0) = 0,\ (\tilde{\gamma} = 0), \tag{52}$$

  i.e., the vector field

$$Y_0 = \sum_{i=1}^{2} P_i \frac{\partial}{\partial P_i} \tag{53}$$

  is a conformal symmetry for $\mathcal{P}, \mathcal{P}_1$, and H_0.

- Defining now the families of quantities $X'_h,\ Y'_h,\ \mathcal{P}'_h,\ \omega'_h$ and dH'_h by:

$$X'_h := T^h X_0,\ \mathcal{P}'_h := T^h \mathcal{P},\ \omega'_h := (T^*)^h \omega',\ Y_h := T^h Y_0,\ dH'_h := (T^*)^h dH_0,\ h \in \mathbb{N} \tag{54}$$

  where $T^* := \mathcal{P}^{-1} \circ \mathcal{P}_1$ is the adjoint of T, we get the following plethora of conserved quantities:

Conserved quantities

$$\mathcal{L}_{Y'_h}(Y'_l) = (h-l)Y'_{l+h},\ \mathcal{L}_{Y'_h}(X'_l) = -(l+1)X'_{l+h},\ \mathcal{L}_{Y'_h}(\mathcal{P}'_l) = (h-l)\mathcal{P}'_{l+h},$$
$$\mathcal{L}_{Y'_h}(\omega'_l) = -(l+h)\omega'_{l+h},\ \mathcal{L}_{Y'_h}(T) = -T^{1+h},\ \langle dH'_l, Y'_h\rangle = -(h+l+1)H'_{l+h},\ l \in \mathbb{N}$$

satisfying

$$\mathcal{L}_{Y'_h}(Y'_l) = (\tilde{\beta}-\tilde{\alpha})(l-h)Y'_{(l+h)},\ \mathcal{L}_{Y'_h}(X'_l) = (\tilde{\beta}+\tilde{\gamma}+(l-1)(\tilde{\gamma}-\tilde{\alpha}))X'_{l+h},$$
$$\mathcal{L}_{Y'_h}(\mathcal{P}'_l) = (\tilde{\beta}+(l-h-1)(\tilde{\beta}-\tilde{\alpha}))\mathcal{P}'_{l+h},\ \mathcal{L}_{Y'_h}(\omega'_l) = (\tilde{\beta}+(l+h-1)(\tilde{\beta}-\tilde{\alpha}))\omega'_{l+h},$$
$$\mathcal{L}_{Y'_h}(T) = (\tilde{\beta}-\tilde{\alpha})T^{1+h},\ \langle dH'_l, Y'_h\rangle = (\tilde{\gamma}+(l+h)(\tilde{\beta}-\tilde{\alpha}))H'_{l+h},$$

analogue to the Oevel's formulae see [W. Oevel, 1986; R. L. Fernandes, 1993].

Concluding remarks

- In this talk, we have:
  (i) derived the Hamiltonian vector fields governing system evolution for a perturbed Kepler problem,
  (ii) constructed and discussed related recursion operator generating the constants of motion,
  and
  (iii) computed conserved quantities using related master symmetries.

THANK YOU FOR YOUR ATTENTION