Regular C^*-irreducible inclusions and Galois correspondence

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2026.07.01

based on almost finished work (probably with Ralf Meyer)

Let $A \subseteq B$ be a C^*-inclusion. Call C **intermediate** if $A \subseteq C \subseteq B$.

Def. (Rørdam 2021)

$A \subseteq B$ is C^*-**irreducible** if every intermediate C^*-algebra C is simple.

$A \subseteq B$ is **irreducible** if $A' \cap \mathcal{M}(B) = \mathbb{C} \cdot 1$.

Examples coming from group actions (known so far)

Let $G \overset{\alpha}{\curvearrowright} A$ and $K \overset{\beta}{\curvearrowright} A$ (faithful) be actions where G discrete, K compact, A unital

1. $A \subseteq A \rtimes_{\mathrm{r}} G$ **(crossed product inclusion)**
   Condition: A is simple and α is **outer**, i.e. $\alpha_g(\cdot) \neq u(\cdot)u^*$ for $u \in A$.
   Intermediates: $A \rtimes_{\mathrm{r}} H$ for $H \leq G$.
2. $A^K \subseteq A$ **(fixed point algebra inclusion)**
   Condition: A^K is simple and **irreducible** in A
   Intermediates: A^L for closed $L \leq K$.
   (A is separable, K second count.)
3. $A^K \subseteq A \rtimes_r G$ **(mixed inclusion)**
   Condition: A is simple and (α, β) **jointly outer**
   Intermediates: $A^L \rtimes_{\mathrm{r}} H$ for $L \times H \leq K \times G$.
   (K is abelian finite, α and β commute)

Def. (Kumjian 1986)

$A \subseteq B$ is **regular** if $N_A(B) := \{b \in B : bAb^* \subseteq A, b^*Ab \subseteq A\}$ generates B

Rem. Let $B = \overline{\bigoplus_{g\in G} B_g}$ be a G-**graded** C^*-algebra, that is B_g is a closed linear space and $B_g B_h \subseteq B_{gh}$ and $B_g^* = B_{g^{-1}}$ for all $g, h \in G$.

Then $A := B_1$ is a C^*-subalgebra of B and $A \subseteq B$ is **regular**, as

$$B_g A B_g^* = B_g A B_{g^{-1}} \subseteq B_{g1g^{-1}} = B_1 = A.$$

Moreover, $\mathcal{B} := (\{B_g\}_{g\in G}, \cdot, ^*)$ is a **Fell bundle** over G. If there is a faithful conditional expectation $E : B \to A$ s.t. $E(B_g) = 0$ for $g \in G \setminus \{1\}$

$$B \cong C_r^*(\mathcal{B}) \qquad \textbf{reduced } C^*\textbf{-algebra} \text{ of } \mathcal{B}.$$

Ex1. $A \rtimes_{\mathrm{r}}^{\alpha} G \cong C_r^*(\mathcal{B}^{\alpha})$ where $B_g^{\alpha} = Au_g$ for $g \in G$.

K is abelian compact

Ex2. $A \cong C_r^*(\mathcal{B}^{\beta})$ where $B_{\chi}^{\beta} := \{a \in A : \beta_k(a) = \langle \chi, k\rangle a,\ k \in K\}$, $\chi \in \widehat{K}$.

Def. (K-Meyer 2018, Muhly-Solel 2000) Let M be a normed A-bimodule

M is **aperiodic** if every $m \in M$ satisfies

$$\forall_{0 \neq D \subseteq A \text{ hereditary}} \ \forall_{\varepsilon > 0} \ \exists_{a \in D^+, \|a\|=1} \qquad a \cdot m \cdot a \approx_\varepsilon 0$$

Ex. Let $\alpha : A \to A$ automorphism and $M_\alpha := A$ with $a \cdot m = \alpha(a)m$, $m \cdot a = ma$

If A is simple, then M_α is aperiodic $\iff \alpha$ is outer

Def. (K-Meyer 2021)

$A \subseteq B$ is **aperiodic** if the Banach A-bimodule B/A is aperiodic.

Rem. If $E : B \to A$ a conditional expectation, then $B/A \cong \ker E$.

Thm. (K-Meyer 2022)

If $A \subseteq B$ is aperiodic it admits a **unique pseudo-expectation** $E : B \to I(A)$ and

1. $I \triangleleft B$ and $I \cap A = 0 \implies I \subseteq \ker E$.
2. E is faithful $\iff A$ detects ideals in all intermediate $A \subseteq C \subseteq B$.

Cor. $A \subseteq B$ is aperiodic, A simple and E faithful $\implies A \subseteq B$ is C^*-irreducible.

Thm. Let $A \subseteq B$ be a regular C^*-inclusion with A **simple**. TFAE:

1. $A \subseteq B$ is a **C^*-irreducible**, i.e. all intermediate $A \subseteq C \subseteq B$ are simple
2. $A \subseteq B$ is **irreducible** and B is simple
3. $A \subseteq B$ is an **aperiodic inclusion** and the necessarily unique pseudo-expectation $E : B \to I(A)$ is faithful
4. there is a **unique** faithful conditional expectation $E : B \to A$
5. $B \cong \mathrm{C}^*_\mathrm{r}(\mathcal{B})$ for an **outer Fell bundle** $\mathcal{B} = \{B_g\}_{g \in G}$ over a **discrete group** G with the unit fiber $B_1 = A$.

If the above hold, then $\mathcal{B}$ and G are uniquely determined by $A \subseteq B$, and

$$G \geq H \longmapsto \mathrm{C}^*_\mathrm{r}(\mathcal{B}|_H) \subseteq \mathrm{C}^*_\mathrm{r}(\mathcal{B}) = B$$

is bijection between subgroups of G and intermediate C^*-subalgebras of B.

Cor1. (Non-unital Cameron-Smith)

Let (α, σ) be a twisted action of a discrete group G on a simple A. Then $A \subseteq A \rtimes^{\mathrm{r}}_{\alpha,\sigma} G$ is C*-irreducible if and only if α is outer, and then

$$G \geq H \longmapsto A \rtimes^{\mathrm{r}}_{\alpha,\sigma} H \subseteq A \rtimes^{\mathrm{r}}_{\alpha,\sigma} G$$

is a bijection between subgroups of G and intermediate C*-subalgebras.

The inclusion $A \subseteq A \rtimes^{\mathrm{r}}_{\alpha,\sigma} G$ determines (α, σ) up to exterior equivalence.

Cor2. (Non-separable, extended, but abelian, Mukohara)

Let β be a faithful action of a compact abelian group K on a C*-algebra A and assume that its fixed point algebra A^β is simple.

1. $A^\beta \subseteq A$ is C*-irreducible;
2. $A^\beta \subseteq A$ is irreducible;
3. the action β is outer and A is simple.

If the above holds, then β and K are uniquely determined by $A^\beta \subseteq A$, and

$$K \geq L \longmapsto A^{\beta|_L} \subseteq A$$

is a bijection between closed subgroups of K and intermediates for $A^\beta \subseteq A$.

Ex. (Cuntz algebras) Let $\mathcal{O}_n := C^*(s_1, ..., s_n : s_i^* s_i = 1, \sum_{i=1}^n s_i s_i^* = 1)$. The core inclusion $\mathcal{F}_n = UHF(n^\infty) \subseteq \mathcal{O}_n$ is C^*-irreducible.

Cor3. (generalized Echterhoff-Rørdam)

Let A be equipped with twisted action (α, σ) of a **discrete group** G and a faithful action β of a **compact abel**[illegible] K. Assume that α and β commute. TFAE:

1. $A^\beta \subseteq A \rtimes^{\mathrm{r}}_{\alpha,\sigma} G$ is C*-irreducible;
2. A^β, A are simple and α, β are jointly outer.

If the above hold, then we have a bijection between closed subgroups of $K \times G$ and intermediate C^*-subalgebras for $A^\beta \subseteq A \rtimes^{\mathrm{r}}_{\alpha,\sigma} G$ where

$$K \times G \geq L \times H \longmapsto A^{\beta|_L} \rtimes^{\mathrm{r}}_{\alpha,\sigma} H \subseteq A \rtimes^{\mathrm{r}}_{\alpha,\sigma} G.$$