Noncommutative Geometry

from the

Perspective of Quantum Physics

— talk at WGMP, July 2025

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Prologue : Quantum Gravity

(Geometro-)

Dynamics of Quantum Spacetime

rather than

Quantum Dynamics of (Classical)

Spacetime

Prologue : Geometrodynamics

Spacetime Geometry

Noncommutative G. $\Leftrightarrow$ Quantum Gravity

Non-Euclidean G. $\Leftrightarrow$ Classical Gravity

Algebra vs Geometry —— Physics :-

Observable Algebra $\mathcal{A}$ $\longleftrightarrow$ Symplectic Geometry $\mathcal{S}$

c-number picture : Commutative —— Classical

- coordinates as basic observables
- states $[\phi]$ as functionals
- $[\phi](f) = f([\phi]) = f(x,p)|_{[\phi]}$

  $[\phi] : \mathcal{A} \longrightarrow I\!R$ — c-number (real number) value $[f]_\phi \equiv [\phi](f)$
- values of observables for a state – classical information
- evaluation map (functionals) $[\phi]$ is an algebraic homomorphism
- dynamics given by $\boldsymbol{\frac{d}{ds} f = \{f, H_s\}}$, ($t$ as s)

q-number picture : **Noncommutative — Quantum +**

Symplectic Geometry – (projective) Hilbert/Krein space

Observable Algebra $\mathcal{A}$ – operators β

- **operator coordinates as basic observables**

- **states $[\phi]$ as evaluation map : algebraic homomorphism**
- $[\beta]_\phi \equiv [\phi](\beta) = [f_\beta(\phi)] = \beta(\hat{x}, \hat{p})|_\phi$

  $[\phi] : \mathcal{A} \longrightarrow \mathbb{Q}$ — **q-numbers noncommutative value**
- **value of observable for a state – quantum information**

- **dynamics given by $\frac{d}{ds}\beta = \{\beta, \hat{H}_s\}_q \equiv \frac{1}{i\hbar}[\beta, \hat{H}_s]$, ($t$ as s)**

Evaluation as an Algebra Homomorphism :-

— real number is *only* an algebraic system

- classical $\quad [\phi] : f(x_i, p_i) \to I\!R \quad$ (observables have real values)

e.g. $\quad E = p^2 + x^2 = pp + xx \quad$ (1-D SHO $m = \frac{1}{2}$, $k = 2$)

$[\phi](x) = 2,\ [\phi](p) = 3 \quad \Longrightarrow$

$$[\phi](E) = [\phi](p^2) + [\phi](x^2) = [\phi](p)[\phi](p) + [\phi](x)[\phi](x) = 13$$

$$[\phi](x_i p_i) = [\phi](x_i)[\phi](p_i) = [\phi](p_i)[\phi](x_i) = [\phi](p_i x_i)$$

- **quantum** $\quad [\phi] : \beta(\hat{x}_i, \hat{p}_i) \to ?$

$$[\phi](\hat{x}_i)[\phi](\hat{p}_i) = [\phi](\hat{p}_i)[\phi](\hat{x}_i) + [\phi](i\hbar\hat{I})$$

$\Longrightarrow [\phi](\beta(\hat{x}_i, \hat{p}_i))$ **has to be a noncommutative algebra**

Heisenberg & Dirac 1925/26 :-

— classical to quantum

only needs a new kinematic

- **H : physical quantities *not* real number variables**
- **Quine : real number as 'convenient fiction'**
- **D : q-number as the new convenient fiction**

Dirac 1925/26 :-

— Hamiltonian formulation ; q-number

- $\hat{x}^i$, $\hat{p}_i$ as canonical variables; P.B. $= \frac{1}{i\hbar}[\cdot, \cdot]$

"To distinguish the *two kinds of numbers*, we shall call the quantum variables q-numbers and the numbers of classical mathematics which satisfy the commutative law c-numbers"

"Owing to the fact that we count the time as a c-number, we are allowed to use the notion of *the value* of the dynamical variable at any instance of time. This value is a q-number, capable of being represented by a generalized 'matrix', . . . "

"At present one can form no picture of what a q-number is like."

Concept of Numbers (in history) :

- $x + 2 = 0 \quad \longrightarrow \quad$ **negative numbers**
- $2x - 1 = 0 \quad \longrightarrow \quad$ **rational numbers**
- $x^2 - 2 = 0 \quad \longrightarrow \quad$ **real numbers**
- $x^2 + 1 = 0 \quad \longrightarrow \quad$ **complex numbers**
- $xy - x - i = 0 \quad \longrightarrow \quad$ (i,2), $(\frac{1}{i-1}$,-i), ...
- $xy - yx - 1 = 0 \quad \longrightarrow \quad$ **noncommutative numbers**

★ $\hat{x}\hat{p} - \hat{p}\hat{x} - i\hbar = 0$

needs NC/q-number values for the variables

The New Convenient Fiction —— q-numbers :-

- $[\beta]_\phi = \{ f_\beta|_\phi, V_{\beta_n}|_\phi \}$; $\quad f_\beta(z_n, \bar{z}_n) = \frac{\langle\phi|\beta|\phi\rangle}{\langle\phi|\phi\rangle}$, $\quad |\phi\rangle = \sum_n z_n |n\rangle$,

$$V_{\beta_n} = \frac{\partial f_\beta}{\partial z_n} = -f_\beta \bar{z}_n + \sum_m \bar{z}_m \langle m|\beta|n\rangle$$

Cirelli et.al 90

- Kähler product : $[\beta\gamma]_\phi = [\beta]_\phi \star_\kappa [\gamma]_\phi$

$$f_{\beta\gamma} = f_\beta f_\gamma + \sum_n V_{\beta_n} V_{\gamma_{\bar{n}}} \;,\quad V_{\beta\gamma_n} = -f_{\beta\gamma} \bar{z}_n + \sum_{m,l} \bar{z}_m \langle m|\beta|l\rangle\langle l|\gamma|n\rangle$$

Deutsch & Hayden 00

- locality of quantum information from/in Heisenberg picture

Galvão & Hardy 03

- Substituting a Qubit for an Arbitrarily Large Number of Classical Bits'

- f_β Kählerian – Hamiltonian flows as isometries

The Symplectic Geometry — NC Vs C :-

- **Heisenberg** — $\frac{d}{ds}\alpha(\hat{P}_\mu, \hat{X}_\mu) = \frac{1}{i\hbar}[\alpha(\hat{P}_\mu, \hat{X}_\mu), \hat{H}_s]$

- **Schrödinger** — $\frac{d}{ds} f_\alpha(z^n, \bar{z}^n) = \{f_\alpha(z^n), f_{H_s}\}$

- $f_\alpha(z^n, \bar{z}^n) \equiv \frac{{}_\eta\langle\phi|\alpha(\hat{P}_\mu, \hat{X}_\mu)|\phi\rangle}{{}_\eta\langle\phi|\phi\rangle}$ $\left(|\phi\rangle = \sum_n z^n |n\rangle\right)$

— as the pull-back of $\hat{\alpha}$ under $(z^n, \bar{z}^n) \longrightarrow (\hat{P}_\mu, \hat{X}_\mu)$

- $\rightarrow$ **bijective homomorphism between NC Poisson algebras**

— **NC Kähler product** $f_\alpha \star_\kappa f_{\alpha'} = f_{\alpha\alpha'}$ *Cirelli et.al 90*

Schrödinger Vs Heisenberg – **coordinate transformation** :-

- coordinate map: $\hat{f} : (q_n, s_n) \longrightarrow (f_{\hat{x}_i}, f_{\hat{p}_i}) \rightarrow (\hat{x}_i, \hat{p}_i)$

- intuitively : $\hat{x}_i$, $\hat{p}_i$ as coordinates of quantum phase space

- **noncommutative differential symplectic geometry** :

$$f_\beta \ , \ df_\beta = f_{d\beta} \ , \ d\beta \ , \ \mathcal{X}_\beta = \frac{1}{i\hbar}[\cdot, \beta] \ , \ \ldots \quad \textbf{as pull-backs}$$

$$\frac{d\cdot}{dt} \longrightarrow d\cdot \qquad (\langle\delta\phi|\beta|\phi\rangle + \langle\phi|\beta|\delta\phi\rangle) = \langle\phi|d\beta|\phi\rangle$$

$$|\delta\phi\rangle = dz_n\,|n\rangle \text{ with constant } \langle\phi|\phi\rangle$$

$$d\beta = (d\hat{x}^i \frac{\partial}{\partial \hat{x}^i} + d\hat{p}^i \frac{\partial}{\partial \hat{p}^i})\beta = [D, \beta] \ , \qquad d\eta = D\eta - (-1)^k \eta D \ ,$$

$$d^2 = 0 \ , \qquad \beta^\dagger = \bar{\beta} \ , \qquad (d\eta)^\dagger = (-1)^k d(\eta^\dagger) \ , \quad (\eta\eta')^\dagger = \bar{\eta}'\bar{\eta}$$

Quantum Physics is simply
a q-number version of classical physics
about the q-number reality

Quantum geometry is then
q-number geometry
— ***a true geometric picture of NCG***

Reference Frame Transformations :-

- $\longleftrightarrow$ **Relativity Symmetry**
  — spacetime symmetry/reference frame

- physical/quantum frame Vs absolute/classical frame
  — relative 'uncertainty' / entanglement

- example of quantum spatial translation

$$\hat{x}_B^{(A)} \longrightarrow -\hat{x}_A^{(B)} , \qquad \hat{x}_C^{(A)} \longrightarrow \hat{x}_C^{(B)} - \hat{x}_A^{(B)} ,$$
$$\hat{p}_B^{(A)} \longrightarrow -(\hat{p}_A^{(B)} + \hat{p}_C^{(B)}) , \quad \hat{p}_C^{(A)} \longrightarrow \hat{p}_C^{(B)} .$$

- quantum model of space(time) — the phase space

Unitary Q Spatial Translation :-

— generated by $\hat{p}_C$

Giacomini et.al. 19

- $e^{ix_B \hat{p}_C} \quad \longrightarrow \quad e^{i\hat{x}_B^{(A)} \hat{p}_C^{(A)}} \quad \longrightarrow \quad \hat{S}_x = \hat{\mathcal{P}}_{AB} e^{i\hat{x}_B^{(A)} \hat{p}_C^{(A)}}$

  — $\hat{\mathcal{P}}_{AB} : |x\rangle_B \otimes |y\rangle_C \longrightarrow |-x\rangle_A \otimes |y\rangle_C \; , \quad \hat{x}_B^{(A)} \longrightarrow -\hat{x}_A^{(B)}$

  $\mathcal{H}_B^{(A)} \otimes \mathcal{H}_C^{(A)} \longrightarrow \mathcal{H}_A^{(B)} \otimes \mathcal{H}_C^{(B)}$

- *but* $\hat{x}_B^{(A)}$ generates $\hat{p}_B^{(A)}$ momentum translation

- Schrödinger picture : on $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$

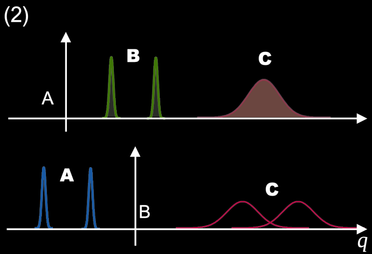
$$\hat{U}_x = \hat{\mathcal{S}}_{AB}^W \hat{I}_A \otimes \int dx' dy' \, |-x'\rangle\langle x'|_B \otimes |y' - x'\rangle\langle y'|_C$$

$$\hat{\mathcal{S}}_{AB}^W : |z\rangle_A \otimes |x\rangle_B \otimes |y\rangle_C \longrightarrow |x\rangle_A \otimes |z\rangle_B \otimes |y\rangle_C$$

for $\quad |\psi\rangle = |\emptyset\rangle_A \otimes \int dx dy \; \psi(x,y) \, |x\rangle_B \otimes |y\rangle_C$

$$\hat{U}_x |\psi\rangle = \int dx dy \; \psi(x, y + x) \, |-x\rangle_A \otimes |\emptyset\rangle_B \otimes |y\rangle_C$$

The 4 scenarios: Case 2



$$|\emptyset\rangle_A \otimes \frac{1}{\sqrt{2}}\left(|x_1\rangle + |x_2\rangle\right)_B \otimes \int \mathrm{d}y\, \psi(y) |y\rangle_C$$

$$\longrightarrow \frac{1}{\sqrt{2}}\Bigg(|{-x_1}\rangle_A \otimes |\emptyset\rangle_B \otimes \int \mathrm{d}y\, \psi(y) |y - x_1\rangle_C$$

$$+ |{-x_2}\rangle_A \otimes |\emptyset\rangle_B \otimes \int \mathrm{d}y\, \psi(y) |y - x_2\rangle_C \Bigg)$$

Distance Translated (A-frame to B-frame) ?

an Noncommutative Value :-

- classical $e^{i x_B \hat{p}_C}$: translation by x_B

  — say $x^i_B = 2$, $x^f_C = x^i_C - 2$, $x^f_A = -2$

- quantum : $[\hat{x}_B]^i_\phi$ as the value

  $$[\hat{x}_C]^f_{\phi'} = [\hat{x}_C]^i_\phi - [\hat{x}_B]^i_\phi \,, \quad [\hat{x}_A]^f_{\phi'} = -[\hat{x}_B]^i_\phi$$

  — $[\hat{x}_B]^i_\phi$ contains full quantum information of position

- evaluation $\hat{\mathcal{O}} \rightarrow [\hat{\mathcal{O}}]_\phi$: *algebraic homomorphism*

Quantum Relativity Principle :-

- **Penrose : Relativity Principle** $\longrightarrow\otimes\longleftarrow$ **Quantum**

- **Heisenberg picture** – $\hat{x}$ **and** $\hat{p}$ **as coordinates**

— **Noncommutative Geometry for Spacetime**

- **Rel. Sym.** $\longleftarrow$ **Quantum Ref. Frame Transformations**

— ***e.g.*** **translation by the NC value of** $\hat{x}_A - \hat{x}_B$ **(ans. Penrose)**

- **Quantum Gravity as General Quantum Relativity**

Lorentz Covariant Quantum Physics :-

• Schrödinger wavefunction $\phi(x^\mu)$

— basic operators x_μ and $-i\hbar\partial_\mu$

• abstract operators as Minkowski four-vectors

— $\hat{X}_i \longrightarrow \hat{X}_\mu$ and $\hat{P}_i \longrightarrow \hat{P}_\mu$

— $[\hat{X}_\mu, \hat{P}_\nu] = i\hbar\eta_{\mu\nu}$

• Heisenberg-Weyl symmetry — $[Y_\mu, E_\nu] = i\hbar\, c\, \eta_{\mu\nu}\, M$

— **M is an effective Casimir element** → **Newtonian mass m**

— $m\hat{X}_\mu \longleftarrow Y_\mu$, different m for different irr. representations

— $\hat{P}_\mu \longleftarrow \frac{1}{c}E_\mu$, constant c (… $c \to \infty$ limit)

Minkowski Metric Operator $\hat{\eta}$ on Krein Space :-

• **Minkowski** nature of proper invariant **inner product**

— effectively, bra as ${}_{\eta}\langle\cdot| = \langle\cdot|\,\hat{\eta}$

— naive $|\phi(x^\mu)|^2$ **integral** cannot avoid **divergence**

• **observables (pseudo-)Hermitian,** ${}_{\eta}\Big\langle \cdot|\hat{A}^{\dagger^\eta}\cdot\Big\rangle = {}_{\eta}\Big\langle \hat{A}\cdot|\cdot\Big\rangle$

— $\hat{X}_\mu = \hat{\eta}\hat{X}^\mu\hat{\eta}^{-1}$ and $\hat{P}_\mu = \hat{\eta}\hat{P}^\mu\hat{\eta}^{-1}$

• **noncommutative geometric picture**

— $\hat{X}^\mu$ **and** $\hat{P}^\mu$ **as coordinates**

Towards Gravity : on a particle -

• **quantum geodesic equation** (Heisenberg)

— *e.g.* instantaneous frame of free-fall

$$\frac{d^2\hat{x}^\mu}{ds^2} + \frac{d\hat{x}^\nu}{ds}\Gamma^\mu_{\nu\sigma}(\hat{x})\frac{d\hat{x}^\sigma}{ds} = 0$$

→ maintaining (Weak) Equivalence Principle

— $$\{\hat{x}^a, \hat{p}_b\} = \delta^a_b, \quad \{\hat{x}^a, \hat{x}^b\} = 0 = \{\hat{p}_a, \hat{p}_b\}$$

$$\frac{\partial}{\partial \hat{x}^a} \equiv \{\cdot, \hat{p}_a\}, \qquad \frac{\partial}{\partial \hat{p}_a} \equiv -\{\cdot, \hat{x}^a\}$$

Goedesic from Free Particle Motion:-

- all positions coordinates Hermitian

- $\hat{H}_{\text{free}} = \frac{1}{2m}\,\hat{p}_{A}\,g^{Ab}(\hat{x})\,\hat{p}_{b}$, $\qquad \hat{p}^{b} = \hat{p}_{A}\,g^{Ab}(\hat{x})$

— four vectors : $\hat{V}'^{a} = \hat{V}^{i}\frac{\partial\hat{x}'^{a}}{\partial\hat{x}^{i}}$, $\quad \hat{W}'_{a} = \frac{\partial\hat{x}^{i}}{\partial\hat{x}'^{a}}\hat{W}_{i}$,

$$\hat{V}'^{A} \equiv \hat{V}'^{a\dagger} = \left(\frac{\partial\hat{x}'^{a}}{\partial\hat{x}^{i}}\right)^{\dagger}\hat{V}^{i\dagger} \equiv \left(\frac{\partial\hat{x}^{A}}{\partial\hat{x}'^{I}}\right)\hat{V}^{I} , \quad \hat{W}'_{A} = \hat{W}_{I}\left(\frac{\partial\hat{x}^{I}}{\partial\hat{x}'^{A}}\right).$$

— Schrödinger representation fails

- $\hat{x}^{a}$ and $\hat{p}^{a}$ as $\hat{g}$-Hermitian within the ref. frame

- Hamilton's Eqs. $\longrightarrow$ mass-indep. E.O.M.

. . .

- $\hat{p}^a = m\frac{d\hat{x}^a}{ds}$ (not $\hat{p}_b = \hat{g}_{bA}\hat{p}^B$) $g(\hat{x})$-Hermitian

- **Quantum Geodesic Equation :** $\frac{d^2\hat{x}^i}{ds^2} =$

$$\frac{d\hat{x}^h}{ds}\frac{\partial_J\hat{g}_{hK}}{2}\frac{d\hat{x}^K}{ds}\hat{g}^{Ji} - \frac{d\hat{x}^h}{ds}\frac{\partial_K\hat{g}_{hJ}}{2}\hat{g}^{Ji}\frac{d\hat{x}^K}{ds} - \frac{d\hat{x}^k}{ds}\hat{g}_{kM}\frac{d\hat{x}^h}{ds}\hat{g}^{Ml}\frac{\partial_h\hat{g}_{lJ}}{2}\hat{g}^{Ji}$$

— s a real number Hamiltonian evolution parameter

— proper time a quantum observable

- a quantum $\hat{s}$? $\qquad - d\hat{s}^2 = d\hat{x}^b\hat{g}_{bA}d\hat{x}^A$

— notion of q-number length in NCG (integral)

- Note : $d(\hat{H}\hat{K}) = d\hat{x}^a\frac{\partial\hat{H}\hat{K}}{\partial\hat{x}^a} = d\hat{x}^a\frac{\partial\hat{H}}{\partial\hat{x}^a}\hat{K} + \hat{H}d\hat{x}^a\frac{\partial\hat{K}}{\partial\hat{x}^a}$

— $d = (d\hat{x}^a\partial_a) \neq d\hat{x}^a\partial_a$ $\qquad$ (or $2d = [d\hat{x}^a\partial_a] + [\partial_A(\cdot)d\hat{x}^A]$)

(Naive) Quantum Einstein Equation :-

$$\hat{G}_{aB} + \Lambda \hat{g}_{aB} = \kappa \hat{T}_{aB}$$

- **Hermitian components of the geometric tensors**

$$\hat{G}_{ab} = \hat{R}_{ab} - \frac{1}{2}\hat{R}\,\hat{g}_{ab}$$

- **more intriguing : $\hat{\Lambda}$**

- **Schwarzschild solution, with plausibly $\hat{M}$**

Future (H & D $\rightarrow \ldots$) :-

- **q-number physics – Q gravity as GQR**

  more NC physics : $[\hat{x}^i, \hat{x}^j] \neq 0$?

- **q-number geometry – NC Geo. as symplectic**

  coordinate picture : 'Euclidean NC Geo.'

- **q-number theory – algebra beyond algebra**

- **q-number technology – q-information**

  Quine: To be is to be the (q-number) value of a (physical) variable.

- C^*-/*-algebra as observable algebra

— (Hilbert) space as NC symplectic geometry

— local NC canonical coordinates

: Gel'fand-Kirillov dimension

- Lie symmetry (*cf.* Heisenberg-Weyl)

— group C^*-algebra

/ universal enveloping algebra (*cf.* Weyl)

- extended to q-number bimodule

— Hilbert's Nullstellensatz

— Lie skew field : Gel'fand-Kirillov conjecture

THANK YOU !