Lie algebroid symmetry with bundle-valued n-plectic geometry

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Plan of the talk

1. Background of the talk
2. Homotopy Momentum Sections
   - E-valued n-plectic geometry
   - (E-valued) homotopy momentum sections
3. Symmetry for non-symplectic manifolds
   - HyperKähler case
   - Quaternionic Kähler case
4. Reduction
   - Vector valued 1-plectic cases
   - Compatible momentum sections
   - Reduction with compatible momentum sections

Background of the talk

Symmetries on a manifold are encoded via the action of a Lie group or a Lie algebra consistent with the underlyng geometric structure. Symmetries in Mechanics is formulated in the context of symplectic / Poisson geometry.

- (M, ω) : a symplectic manifold — a smooth manifold M equipped with a nondegenerate closed 2-form ω
- $\mathfrak{g}$: a Lie algebra that acts on M from the left by

$$\mathfrak{g} \longrightarrow \mathfrak{X}(M), \quad \xi \longmapsto \xi_M$$

so that $\mathcal{L}_{\xi_M}\omega = 0$ $(\forall \xi \in \mathfrak{g})$, equivalently

$$\omega([\xi, \eta]_M,\, \zeta_M) + \omega([\eta, \zeta]_M,\, \xi_M) + \omega([\zeta, \xi]_M,\, \eta_M) = 0$$

for $\xi, \eta, \zeta \in \mathfrak{g}$.

Hamiltonian symmetries for symplectic manifolds

The $\mathfrak{g}$-action is called a Hamiltonian action if there is a map $\mu : M \to \mathfrak{g}^*$, called an *equivariant momentum map*, such that

1. $\imath_{\xi_M}\omega = \mathrm{d}\mu^\xi$ $(\forall \xi \in \mathfrak{g})$, where $\mu^\xi(x) := \langle \mu(x), \xi \rangle$
2. μ is equivariant; $(\mathrm{d}\mu)_x(\xi_M) = -\mathrm{ad}^*_\xi \mu(x)$ $(x \in M)$

The first condition is equivalent to

$$\xi_M = X_{\mu^\xi} \quad (\forall \xi \in \mathfrak{g}).$$

■ The momentum maps of Kähler manifolds are defined to be those of symplectic manifolds.
■ The momentum maps of hyperKähler or quaternionic Kähler manifolds are defined in a way different from symplectic case.

Question

Can we unite those momentum maps in a single framework ?

Approach this matter by generalizing (equivariant) momentum maps in terms of a Lie algebroid.

Lie algebroid symmetries were discussed in

- A. Blaom
  Geometric structures as deformed infinitesimal symmetries. Trans. Amer. Math. Soc. 358(2006), no. 8.
- C. Blohmann, M.C.B. Fernandes and A. Weinstein
  Groupoid symmetry and constraints in general relativity. Commun. Contemp. Math. 15(1) (2013)
- M. Crampin and D. Saunders
  Cartan geometries and their symmetries – A Lie algebroid approach. Atlantis Stud. Var. Geom., 4 AtraIntis Press (2016)

In physics,

- A. S. Cattaneo and G. Felder
  Poisson sigma models and symplectic groupoids: in Quantization of Singular Symplectic Quotients, Progr. Math. 198 (2001)
- A. Alekseev and T. Strobl
  Current algebras and differential geometry. J. High. Energy Phys. (2005)

A *Hamiltonian Lie algebroid* over (pre-)symplectic manifold is a Lie algebroid together with a geometric structure that describes generalization of hamiltonian symmetries. The notion of that originally appears in

- C. Blohmann and A. Weinstein
  Hamiltonian Lie algebroids. Mem. Amer. Math. Soc. 295 (2024)

Definition (Hamiltonian Lie algebroids over symplectic manifolds)

Let $\mathcal{A}$ be a Lie algebroid over a (pre-)symplectic manifold (M, ω). An element $\mu \in \Gamma(\mathcal{A}^*)$ is called a ($\nabla^{\mathcal{A}}$-)momentum section if

$$\nabla^{\mathcal{A}^*}\mu = -\iota_\rho \omega$$

$\mathcal{A}$ together with μ is called a Hamiltonian Lie algebroid if both

$$\mho^{T^*M}\omega = 0$$

and

$$(\eth^{\mathcal{A}}\mu)(\alpha_1, \alpha_2) = \omega(\rho(\alpha_1),\, \rho(\alpha_2))$$

are satisfied.

A Hamiltonian Lie algebroid has been extended to the one over

- Multisymplectic manifolds
  1. Y.H. & N. Ikeda
     Homotopy momentum sections for (pre-)n-plectic manifolds.
     J. Geom. Phys. 182 (2022)
  2. Y.H. & N. Ikeda
     Geometry of bundle-valued multisymplectic structures with Lie algebroids. J. Geom. Phys. 203 (2024).

- Poisson manifolds
  3. C. Blohmann, S. Ronchi and A. Weinstein
     Hamiltonian Lie algebroids over Poisson manifolds.
     J. Symplectic Geom. 22 (2024).

Definition (Lie algebroid)

Let M be a smooth manifold. A smooth vector bundle $\mathcal{A} \to M$ is called a *Lie algebroid* if it is equipped with

- a bundle morphism $\rho : \mathcal{A} \to TM$
- a Lie bracket $[\cdot, \cdot]$ on $\Gamma(\mathcal{A})$

such that

$$[\alpha, f\beta] = f[\alpha, \beta] + \left(\mathcal{L}_{\rho(\alpha)} f\right)\beta$$

for any $\alpha, \beta \in \Gamma(\mathcal{A})$ and $f \in C^\infty(M)$.

ρ is called the anchor map of $\mathcal{A}$. It preserves Lie algebra structures.

$$\rho\big([\alpha, \beta]\big) = [\rho(\alpha), \rho(\beta)]$$

Example (Lie algebras)

Any Lie algebra is a Lie algebroid over a point $\{*\}$.

Example (Tangent algebroids)

The tangent bundle TM of M is a Lie algebroid whose anchor is the identity map of TM, and whose bracket is the commutatator bracket on $\mathfrak{X}(M)$.

Example

Let ω be a closed 2-form on M. $TM \oplus \mathbb{R}$ is a Lie algebroid with bracket on $\mathfrak{X}(M) \oplus C^\infty(M)$ by

$$[X \oplus f, Y \oplus g] := [X, Y] \oplus \big(X[g] - Y[f] + \omega(X, Y)\big)$$

and with the natural projection to TM as the anchor.

Example (Action algebroids)

If M is endowed with a Lie algebra *left* action $\Theta : \mathfrak{g} \to \mathfrak{X}(M)$,

$$[\alpha, \beta](z) = [\alpha(z), \beta(z)] - \Theta(\alpha(z))_z\beta + \Theta(\beta(z))_z\alpha$$

defines a Lie algebra on $\Gamma(\mathfrak{g} \times M) = C^\infty(M, \mathfrak{g})$. Then, $\mathfrak{g} \times M$ is a Lie algebroid with anchor $\rho(\alpha(x), x) := -\Theta(\alpha(x))_x$.

Example (Cotangent bundles on Poisson manifolds)

If (M, Π) is a Poisson manifold, Π defines a Lie bracket on $\Omega^1(M)$ as

$$\{\xi, \eta\} := \mathcal{L}_{\Pi^\sharp\xi}\eta - \mathcal{L}_{\Pi^\sharp\eta}\xi + \mathrm{d}\big(\langle \xi \wedge \eta, \Pi\rangle\big),$$

where $\xi, \eta \in \Omega^1(M)$. For $f, g \in C^\infty(M)$,

- $\mathrm{d}\{f, g\}_\Pi = -\{\mathrm{d}f, \mathrm{d}g\}$.
- $[\Pi^\sharp(\mathrm{d}f), \Pi^\sharp(\mathrm{d}g)] = \Pi^\sharp(\{\mathrm{d}f, \mathrm{d}g\})$

$\mathcal{A}$ associates with a differential operator $\eth^{\mathcal{A}}$, named *$\mathcal{A}$-differential operator* of degree $+1$, defined by

$$\begin{aligned}(\eth^{\mathcal{A}}\theta)&(\alpha_1,\cdots,\alpha_{k+1})\\ &:=\sum_{i=1}^{k+1}(-1)^{i+1}\mathcal{L}_{\rho(\alpha_i)}\big(\theta(\alpha_1,\cdots,\check{\alpha}_i,\cdots,\alpha_{k+1})\big)\\ &+\sum_{i<j}(-1)^{i+j}\theta([\alpha_i,\alpha_j],\alpha_1,\cdots,\check{\alpha}_i,\cdots,\check{\alpha}_j,\cdots,\alpha_{k+1}),\end{aligned}$$

where $\theta\in\Gamma(\wedge^k\mathcal{A}^*)$ and $\alpha_1,\cdots,\alpha_{k+1}\in\Gamma(\mathcal{A})$.

$$\cdots\xrightarrow{\eth^{\mathcal{A}}}\Gamma(\wedge^{k-1}\mathcal{A}^*)\xrightarrow{\eth^{\mathcal{A}}}\Gamma(\wedge^{k}\mathcal{A}^*)\xrightarrow{\eth^{\mathcal{A}}}\Gamma(\wedge^{k+1}\mathcal{A}^*)\xrightarrow{\eth^{\mathcal{A}}}\cdots$$

Example (de Rham differentials)

If $\mathcal{A}=TM$, $\eth^{TM}$ is just $\mathrm{d}:\Omega^\bullet(M)\to\Omega^{\bullet+1}(M)$.

E-valued n-plectic geometry

- M : a smooth manifold.
- E : a vector bundle over M with a vector bundle connection ∇^E.
- d^E_∇ : the covariant exterior derivative of ∇^E.

$$\cdots \xrightarrow{\mathrm{d}^E_\nabla} \Omega^{\bullet-1}(M,E) \xrightarrow{\mathrm{d}^E_\nabla} \Omega^{\bullet}(M,E) \xrightarrow{\mathrm{d}^E_\nabla} \Omega^{\bullet+1}(M,E) \xrightarrow{\mathrm{d}^E_\nabla} \cdots$$

Definition (E-n-plectic structure)

Let $n \geqq 1$. An $(n+1)$-form $\omega \in \Omega^{n+1}(M,E)$ on M is called an *E-valued n-plectic form* (or structure) if both $\mathrm{d}^E_\nabla \omega = 0$ and non-degenerate in the sense that, at each point $x \in M$, the following map is injective:

$$\omega^\flat_x : T_xM \longrightarrow \bigwedge^n T^*_xM \otimes_{\mathbb{R}} E_x, \quad X \longmapsto \imath_X \omega_x.$$

We call M equipped with ω an E-valued (pre-)n-plectic manifold and denote it by (M, ω, E, ∇^E).

Example 1

(Pre)symplectic manifolds, (pre)-n-plectic manifolds, Polysymplectic manifolds and V-symplectic manifolds.

Example 2 (Lie groups)

A compact connected Lie group G is a $\underline{\mathfrak{g}}_G$-valued pre-1-plectic manifold, where $\omega = \mathrm{d}\lambda_R$ and $\nabla^E = \mathbf{d}$.

Example 3 (Curvature 2-forms)

Let E be a vector bundle over M with a connection ∇^E. A curvature $R^E_\nabla \in \Omega^2(M, \mathrm{End}E)$ of ∇^E is an $\mathrm{End}E$-valued pre-1-plectic form on M.

- $\mathcal{M} := (M, \omega, E, \nabla^E)$: an E-valued (pre-)n-plectic manifold.

Definition (Covariant Lie derivatives)

For $\varphi \in \Omega^k(M, E)$ and $X_1, \cdots, X_k \in \mathfrak{X}(M)$. define

$$(\mathcal{L}^{\nabla}_X \varphi)(X_1, \cdots, X_k) := \nabla^E_X(\varphi(X_1, \cdots, X_k)) - \sum_{i=1}^{k} \varphi(X_1, \cdots, [X, X_i], \cdots, X_k).$$

Proposition (Extended Cartan's formulas)

1. $\mathcal{L}^{\nabla}_X = \imath_X \circ \mathrm{d}^E_\nabla + \mathrm{d}^E_\nabla \circ \imath_X$
2. $\imath_{[X,Y]} = \mathcal{L}^{\nabla}_X \imath_Y - \imath_Y \circ \mathcal{L}^{\nabla}_X$
3. $(\mathcal{L}^{\nabla}_X \circ \mathrm{d}^E_\nabla)\varphi = (\imath_X R^E_\nabla) \wedge \varphi + (\mathrm{d}^E_\nabla \circ \mathcal{L}^{\nabla}_X)\varphi$

Definition (pseudo-Hamiltonian)

$\varphi \in \Omega^{n-1}(M, E)$ is said to be a *pseudo-Hamiltonian* if there exists a vector field $X_\varphi \in \mathfrak{X}(M)$ such that $\mathrm{d}^E_\nabla \varphi = \imath_{X_\varphi}\omega$. X_φ is the *pseudo-Hamiltonian vector field* corresponding to φ.

$\mathrm{pHam}^{n-1}(\mathcal{M})$: the set of all pseudo-Hamiltonian $(n-1)$-forms.

For $\varphi, \psi \in \mathrm{pHam}^{n-1}(\mathcal{M})$,

$$\{\varphi,\, \psi\} := \imath_{X_\psi}\imath_{X_\varphi}\omega = \imath_{X_\varphi \wedge X_\psi}\omega \in \Omega^{n-1}(M, E).$$

In general, $\{\varphi, \psi\}$ fails to be a pseudo-Hamiltonian since

$$\imath_{[X_\psi, X_\phi]}\omega = \mathrm{d}^E_\nabla\{\varphi,\, \psi\} - \imath_{X_\varphi}(\psi \wedge R^E_\nabla) + \imath_{X_\psi}(\varphi \wedge R^E_\nabla).$$

(E-valued) homotopy momentum sections

Given

- $\mathcal{M}$: an E-valued (pre-)n-plectic manifold (M, ω, E, ∇^E).
- $\mathcal{A}$: a Lie algebroid over M with a connection $\nabla^{\mathcal{A}}$,

$$\nabla^{\mathcal{A}} : \Gamma(\mathcal{A}) \longrightarrow \Omega^1(M, \mathcal{A}) := \Gamma(T^*M \otimes \mathcal{A}).$$

Set

$$\Omega^q_{\mathcal{A}}(M) := \Gamma(\wedge^q \mathcal{A}^*), \qquad \Omega^q_{\mathcal{A}}(M, E) := \Gamma(\wedge^q \mathcal{A}^* \otimes E)$$

and

$$\begin{aligned}\Omega^{p,q}(M, \mathcal{A})^E &:= \Gamma(\wedge^p T^*M \otimes \mathrm{Hom}(\wedge^q \mathcal{A}, E)) \\ &= \Omega^p(M) \otimes_{C^\infty(M)} \Omega^q_{\mathcal{A}}(M) \otimes_{C^\infty(M)} \Gamma(E)\end{aligned}$$

for each $p \geqq 0, q \geqq 1$.

Extend $\nabla^{\mathcal{A}}$ to a connection on $\wedge^q \mathcal{A}^* \otimes E$ ($q \geqq 1$) as

$$\begin{aligned}&\left(\nabla_X^{\mathcal{A}^*\otimes E}(\theta \otimes s)\right)(\alpha_1, \cdots, \alpha_q)\\ :=&\nabla_X^E\left(\theta(\alpha_1, \cdots, \alpha_q)s\right) - \sum_{i=1}^{q} \theta(\alpha_1, \cdots, \nabla_X^{\mathcal{A}}\alpha_i, \cdots, \alpha_q)s,\end{aligned}$$

where $\theta \otimes s \in \Omega^q_{\mathcal{A}}(M, E)$, $\alpha_1, \cdots, \alpha_q \in \Gamma(\mathcal{A})$ and $X \in \mathfrak{X}(M)$.

Denote by $\mathrm{d}_\nabla^{\mathcal{A}^*\otimes E}$ a covariant exterior derivative of $\nabla^{\mathcal{A}^*\otimes E}$

$$\mathrm{d}_\nabla^{\mathcal{A}^*\otimes E} : \Omega^{p,q}(M, \mathcal{A})^E \longrightarrow \Omega^{p+1,q}(M, \mathcal{A})^E.$$

Remark that

- $\mathrm{d}_\nabla^{\mathcal{A}^*\otimes E}(\eta \otimes \phi) = \mathrm{d}\eta \otimes \phi + (-1)^p \eta \wedge \nabla^{\mathcal{A}^*\otimes E}\phi.$
- $\mathrm{d}_\nabla^{\mathcal{A}^*\otimes E} = \nabla^{\mathcal{A}^*\otimes E}$ when $p = 0$.

Define an $\mathbb{R}$-linear map $\mho^{TM} : \mathfrak{X}(M) \to \Omega^1_{\mathcal{A}}(M, TM)$ as

$$\mho^{TM}_{\alpha} X := \rho(\nabla^{\mathcal{A}}_X \alpha) + [\rho(\alpha), X]$$

for $\alpha \in \Gamma(\mathcal{A})$, $X \in \mathfrak{X}(M)$. Then, $\mho^{TM}$ is shown to be a Lie algebroid connection on TM.

Lie algebroid connection

An $\mathcal{A}$-connection $\mho^B$ on a vector bundle B is defined to be an $\mathbb{R}$-linear mapping

$$\mho^B : \Gamma(B) \longrightarrow \Gamma(\mathcal{A}^* \otimes B), \quad s \longmapsto \mho^B s$$

which satisfies

$$\mho^B_{\alpha}(fs) = f\, \mho^B_{\alpha} s + \imath_{\rho(\alpha)}(\mathrm{d}f)\, s$$

for any $\alpha \in \Gamma(\mathcal{A}), s \in \Gamma(B)$ and $f \in C^{\infty}(M)$.

Extend $\mho^{TM}$ to a Lie algebroid connection on $\wedge^p T^*M \otimes E$ by

$$\big(\mho_\alpha^{T^*M\otimes E}(\eta \otimes s)\big)(X_1, \cdots, X_p)$$
$$:= \nabla^E_{\rho(\alpha)}\big(\eta(X_1, \cdots, X_p)s\big) - \sum_{i=1}^p \eta(X_1, \cdots, \mho_\alpha^{TM} X_i, \cdots, X_p)s,$$

where $\eta \otimes s \in \Omega^p(M) \otimes \varGamma(E)$, $\alpha \in \varGamma(\mathcal{A})$ and $X_1, \cdots, X_p \in \mathfrak{X}(M)$.

Denote by $\eth_\mho^{T^*M\otimes E}$ an $\mathcal{A}$-covariant exterior derivative of $\mho^{T^*M\otimes E}$

$$\eth_\mho^{T^*M\otimes E} : \Omega^{p,q}(M, \mathcal{A})^E \longrightarrow \Omega^{p,q+1}(M, \mathcal{A})^E,$$

Remark that

- $\eth_\mho^{T^*M\otimes E}(\theta \otimes \phi) = \eth^{\mathcal{A}}\theta \otimes \phi + (-1)^q \theta \wedge \mho^{T^*M\otimes E}\phi.$
- $\mho_\alpha^{T^*M\otimes E} = \nabla^E_{\rho(\alpha)}$ when $p = 0$.

- $$\big(\mathrm{d}_\nabla^{\mathcal{A}^*\otimes E}(\eta\otimes\phi)\big)(X_1,\cdots,X_{p+1})$$
  $$:=\sum_{i=1}^{p+1}(-1)^{i-1}\nabla_{X_i}^{\mathcal{A}^*\otimes E}\big(\eta(X_1,\cdots,\check{X}_i,\cdots,X_{p+1})\,\phi\big)$$
  $$+\sum_{i<j}(-1)^{i+j}\eta([X_i,\,X_j],X_1,\cdots,\check{X}_i,\cdots,\check{X}_j,\cdots,X_{p+1})\,\phi,$$
  where $\eta\in\Omega^p(M)$, $\phi\in\Omega^q_{\mathcal{A}}(M,E)$ and $X_1,\cdots,X_{p+1}\in\mathfrak{X}(M)$.
- $$\big(\eth_{\mho}^{T^*M\otimes E}(\theta\otimes\varphi)\big)(\alpha_1,\cdots,\alpha_{q+1})$$
  $$:=\sum_{i=1}^{q+1}(-1)^{i-1}\mho_{\alpha_i}^{T^*M\otimes E}(\theta(\alpha_1,\cdots,\check{\alpha}_i,\cdots,\alpha_{q+1})\,\varphi)$$
  $$+\sum_{i<j}(-1)^{i+j}\theta([\alpha_i,\,\alpha_j],\alpha_1,\cdots,\check{\alpha}_i,\cdots,\check{\alpha}_j,\cdots,\alpha_{q+1})\,\varphi,$$
  where $\theta\in\Omega^q_{\mathcal{A}}(M),\varphi\in\Omega^p(M,E)$ and $\alpha_1,\cdots,\alpha_{q+1}\in\Gamma(\mathcal{A})$.

Let $k, m \in \mathbb{N}$ with $k \leqq m$. Define an operator

$$\imath_\rho^k : \Omega^m(M, E) \to \Omega^{m-k}(M, E) \otimes_{C^\infty(M)} \Gamma(\wedge^k \mathcal{A}^*)$$

by

$$(\imath_\rho^k(\eta \otimes s))(\alpha_1, \cdots, \alpha_k) := \imath_{\rho(\alpha_1)\wedge\cdots\wedge\rho(\alpha_k)}\eta \otimes s \in \Omega^{m-k}(M, E)$$

Definition (E-valued homotopy momentum section)

An *E-valued homotopy momentum section* with respect to $(\mathcal{A}, \nabla^{\mathcal{A}})$ is a formal sum $\mu = \sum_{k=0}^{n-1} \mu_k$ with $\mu_k \in \Omega^{k,n-k}(M, \mathcal{A})^E$ satisfying

$$\left(\mathrm{d}_\nabla^{\mathcal{A}^*\otimes E} + \eth_\mho^{T^*M\otimes E}\right)\mu = \sum_{k=0}^{n} (-1)^{n+1-k} \imath_\rho^{n+1-k}\omega.$$

Alternative description of homotopy momentum sections

An E-valued homotopy momentum section is a formal sum $\mu = \sum_{k=0}^{n-1} \mu_k$ satisfyng

- $\eth_{\mho}^{T^*M\otimes E}\mu_0 = (-1)^{n+1}\iota_\rho^{n+1}\omega \in \Omega_{\mathcal{A}}^{n+1}(M,E)$
- for $k = 1, 2, \cdots, n-1$
$$\mathrm{d}_\nabla^{\mathcal{A}^*\otimes E}\mu_{k-1} + \eth_{\mho}^{T^*M\otimes E}\mu_k = (-1)^{n+1-k}\iota_\rho^{n+1-k}\omega$$
- $\mathrm{d}_\nabla^{\mathcal{A}^*\otimes E}\mu_{n-1} = -\iota_\rho^1\omega \in \Omega^{n,1}(M,\mathcal{A})^E$,

where

- $\omega \in \Omega^{n+1}(M,E)$: E-valued (pre-)n-plectic form.
- for each $k = 0, 1, \cdots, n-1$
$$\mu_k \in \Omega^{k,n-k}(M,\mathcal{A})^E \cong \Omega^k(M) \otimes_{C^\infty(M)} \Omega_{\mathcal{A}}^{n-k}(M,E)$$

Definition

Let $\mathcal{S}$ be a subset of $\Gamma(\mathcal{A})$. We say that μ is a homotopy momentum section on $\mathcal{S}$ if the above conditions holds for any element in $\mathcal{S}$.

$$
\begin{tikzcd}
\Omega_E^{0,n+1} & & & & & \\
\Omega_E^{0,n} \arrow[u, "\eth_{\mho}^{E}"] \arrow[r, "\nabla^{\mathcal{A}^*\otimes E}"] & \Omega_E^{1,n} & & & & \\
\vdots \arrow[u, "\eth_{\mho}^{E}"] \arrow[r, "\nabla^{\mathcal{A}^*\otimes E}"] & \vdots \arrow[u, "\eth_{\mho}^{T^*M\otimes E}"'] \arrow[r, "\mathrm{d}_{\nabla}^{\mathcal{A}^*\otimes E}"] & \Omega_E^{n+1-k,k} & & & \\
\Omega_E^{0,2} \arrow[u, "\eth_{\mho}^{E}"] \arrow[r, "\nabla^{\mathcal{A}^*\otimes E}"] & \Omega_E^{1,2} \arrow[u, "\eth_{\mho}^{T^*M\otimes E}"'] \arrow[r, "\mathrm{d}_{\nabla}^{\mathcal{A}^*\otimes E}"] & \dots \arrow[u, "\eth_{\mho}^{T^*M\otimes E}"'] \arrow[r, "\mathrm{d}_{\nabla}^{\mathcal{A}^*\otimes E}"] & \Omega_E^{n-1,2} & & \\
\Omega_E^{0,1} \arrow[u, "\eth_{\mho}^{E}"] \arrow[r, "\nabla^{\mathcal{A}^*\otimes E}"] & \Omega_E^{1,1} \arrow[u, "\eth_{\mho}^{T^*M\otimes E}"'] \arrow[r, "\mathrm{d}_{\nabla}^{\mathcal{A}^*\otimes E}"] & \dots \arrow[u, "\eth_{\mho}^{T^*M\otimes E}"'] \arrow[r, "\mathrm{d}_{\nabla}^{\mathcal{A}^*\otimes E}"] & \Omega_E^{n-1,1} \arrow[u, "\eth_{\mho}^{T^*M\otimes E}"'] \arrow[r, "\mathrm{d}_{\nabla}^{\mathcal{A}^*\otimes E}"] & \Omega_E^{n,1} & \\
\Gamma(E) \arrow[u, "\nabla_{\rho}^{E}"] \arrow[r, "\nabla^{E}"] & \Omega_E^{1,0} \arrow[u, "\mho^{T^*M\otimes E}"'] \arrow[r, "\mathrm{d}_{\nabla}^{E}"] & \dots \arrow[u, "\mho^{T^*M\otimes E}"'] \arrow[r, "\mathrm{d}_{\nabla}^{E}"] & \Omega_E^{n-1,0} \arrow[u, "\mho^{T^*M\otimes E}"'] \arrow[r, "\mathrm{d}_{\nabla}^{E}"] & \Omega_E^{n,0} \arrow[u, "\mho^{T^*M\otimes E}"'] &
\end{tikzcd}
$$

1-plectic case

$$\begin{array}{ccc} \Omega^{0,2}(M,\mathcal{A})^E & & \\ \uparrow{\scriptstyle \eth_{\mho}^{T^*M\otimes E}} & & \\ \Omega^{0,1}(M,\mathcal{A})^E & \xrightarrow{\nabla^{\mathcal{A}^*\otimes E}} & \Omega^{1,1}(M,\mathcal{A})^E \end{array}$$

- $\nabla^{\mathcal{A}^*\otimes E}\mu_0 = -\imath_\rho^1\omega \in \Omega^{1,1}(M,\mathcal{A})^E$
- $\eth_{\mho}^{T^*M\otimes E}\mu_0 = \imath_\rho^2\omega \in \Omega^2_{\mathcal{A}}(M,E)$

The 2nd formula can be expressed as

$$(\eth_{\mho}^{T^*M\otimes E}\mu_0)(\alpha_1,\alpha_2) = \omega\big(\rho(\alpha_1),\rho(\alpha_2)\big)$$

for $\alpha_1,\alpha_2 \in \Gamma(\mathcal{A})$.

A usual momentum map for symplectic manifold is a momentum section on the constant sections of the case where $\mathcal{A} = \mathfrak{g}\times M$, $\nabla^{\mathcal{A}} = \mathrm{d}$, $E = M\times\mathbb{R}$ and $n=1$.

2-plectic case

$$\begin{tikzcd}
\Omega^{0,3}(M,\mathcal{A})^E & & \\
\Omega^{0,2}(M,\mathcal{A})^E \arrow[u, "\eth_{\mho}^{T^*M\otimes E}"] \arrow[r, "\nabla^{\mathcal{A}^*\otimes E}"] & \Omega^{1,2}(M,\mathcal{A})^E & \\
\Omega^{0,1}(M,\mathcal{A})^E \arrow[u, "\eth_{\mho}^{T^*M\otimes E}"] \arrow[r, "\nabla^{\mathcal{A}^*\otimes E}"] & \Omega^{1,1}(M,\mathcal{A})^E \arrow[u, "\eth_{\mho}^{T^*M\otimes E}"'] \arrow[r, "\mathrm{d}_{\nabla}^{\mathcal{A}^*\otimes E}"] & \Omega^{2,1}(M,\mathcal{A})^E
\end{tikzcd}$$

- $\eth_{\mho}^{T^*M\otimes E}\mu_0 = -\imath_\rho^3\omega \in \Omega^3_{\mathcal{A}}(M,E)$
- $\nabla^{\mathcal{A}^*\otimes E}\mu_0 + \eth_{\mho}^{T^*M\otimes E}\mu_1 = \imath_\rho^2\omega \in \Omega^{1,2}(M,\mathcal{A})^E$
- $\mathrm{d}_{\nabla}^{\mathcal{A}^*\otimes E}\mu_1 = -\imath_\rho^1\omega \in \Omega^{2,1}(M,\mathcal{A})^E$

Example 4

Let (M,ω) be a (pre)-n-plectic manifold endowed with a $\mathfrak{g}$-action. Consider $\mathcal{A}=\mathfrak{g}\times M$ and $E=M\times\mathfrak{g}$.
If each $\mu_k\in\Omega^{k,n-k}(M,\mathfrak{g}\times M)$ $(k=0,1,\cdots,n-1)$ satisfies

$$\mho_{\alpha}^{T^*M}(\mu_k(\alpha_1,\cdots,\alpha_{n-k}))$$
$$=\sum_{i=1}^{n-k}(-1)^{i+1}\mu_k([\alpha,\alpha_i],\alpha_1,\cdots,\breve{\alpha}_i,\cdots,\alpha_{n-k}),$$

then $\mu=\sum_{k=0}^{n-1}\mu_k$ is a homotopy momentum map introduced in

- M. Callies, Y. Frégier, C.L. Rogers and M. Zambon *Homotopy Momentum Maps.* Adv. Math. 303 (2016)

Example 5

Consider a symplectic structure on $\mathbb{R}^{2n}$ by

$$\omega = \sum_{i=1}^{n} (\exp x_i + \exp(-y_i))\, \mathrm{d}x_i \wedge \mathrm{d}y_i.$$

Define a connection on $T\mathbb{R}^{2n}$ by

$$\begin{aligned}
&A_{i,i} = -2\mathrm{d}y_i, \quad A_{i+n,i+n} = 2\mathrm{d}x_i, \quad (i = 1, 2, \cdots, n)\\
&A_{i+n,i} = -\mathrm{d}y_i, \quad A_{i,i+n} = \mathrm{d}x_i \quad (i = 1, 2, \cdots, n)\\
&A_{ij} = 0 \quad (\text{otherwise})
\end{aligned}$$

Then, the differential 1-form

$$\mu = \sum_{i=1}^{n} (\exp(-y_i)\mathrm{d}x_i + \exp x_i \mathrm{d}y_i)$$

is a bracket-compatible momentum section.

HyperKähler case

A *hyperKähler manifold* is a $4n$-dimensional Riemannian manifold (M, g) endowed with three almost complex structures I_1, I_2, I_3 which satisfy

$$I_a \circ I_b = -\delta_{ab}\mathrm{id} + \varepsilon_{abc} I_c \quad (a, b, c = 1, 2, 3).$$

Associate symplectic forms

$$\omega_a(X, Y) := g(I_a X, Y)$$

Any hyperKähler manifold is considered as an $\mathbb{R}^3$-valued 1-plectic manifold, $\omega^{(3)} := \sum_i^3 \omega_i \otimes \boldsymbol{e}_i$, $E = M \times \mathbb{R}^3$ and $\nabla^E = \mathrm{d}$.

- G : a compact Lie subgroup of the isometry group $\curvearrowright M$, properly and freely.
- $\mathcal{L}_X \omega_i = 0$ (X: a Killing vector field, $\mathcal{L}_X g = 0$)

Let $\mu_i : M \to \mathfrak{g}^*$ $(i = 1, 2, 3)$ be an equivariant momentum maps for the Hamiltonian G-action. Remark that each μ_i satisfies

1. $\imath_{\xi_M}\omega_i = \mathrm{d}\mu_i^\xi$ $(\forall \xi \in \mathfrak{g})$.
2. $(\mathrm{d}\mu_i)_x(\xi_M) = -\mathrm{ad}_\xi^*\mu(x)$ $(x \in M)$.

Definition

A momentum map of hyperKähler manifold is defined to be the map $\mu : M \to \mathfrak{g}^* \otimes \mathbb{R}^3$ by $\mu = (\mu_1, \mu_2, \mu_3)$.

Theorem (H & Ikeda, '24)

A momentum map μ of a hyper-Kähler manifold is an $\mathbb{R}^3$-valued homotopy momentum section on the space of $\mathfrak{g}$-valued constant functions with respect to $(\mathcal{A} = \mathfrak{g} \ltimes M,\ \nabla^{\mathcal{A}} = \mathbf{d})$.

$\because$) Remark that $\nabla^{\mathcal{A}} = \nabla^{E} = \mathbf{d}$ and $\mho_{\xi}^{T^*M \otimes E} = -\nabla^{E}_{\xi_M}$ for $\xi \in \mathfrak{g}$. Then,

$$\begin{aligned}\big(\nabla^{\mathcal{A}\otimes E}_X(\mu_i \otimes \boldsymbol{e}_i)\big)(\xi) &= \nabla^{E}_X(\mu_i^{\xi}\boldsymbol{e}_i) - \mu_i(\nabla^{\mathcal{A}}_X \boldsymbol{e}_i) \\ &= \mathrm{d}\mu_i^{\xi}(X)\boldsymbol{e}_i\end{aligned}$$

for any constant section ξ of $\mathfrak{g} \ltimes M$ and $X \in \mathfrak{X}(M)$. On the other hand, from the definition of a momentum map,

$$\begin{aligned}\big(\imath_\rho^1 \omega_i(\xi)\big)(X) &= \omega_i(\rho(\xi), X) = -\omega_i(\xi_M, X) \\ &= -\mathrm{d}\mu_i^{\xi}(X).\end{aligned}$$

Consequently,

$$\nabla^{\mathcal{A}\otimes E}\mu = \mathbf{d}\mu = -\imath_\rho^1 \omega^{(3)}.$$

By the definition of a momentum map,

$$\eth_{\mho}(\mu_i \otimes \boldsymbol{e}_i) = -\nabla^E_{\xi_M}(\mu_i^{\eta}\boldsymbol{e}_i) + \nabla^E_{\eta_M}(\mu_1(\xi)\boldsymbol{e}_i) - \mu_i([\xi,\, \eta])\boldsymbol{e}_i$$
$$= 2\omega_i(\xi_M, \eta_M)\boldsymbol{e}_i - \mu_i([\xi, \eta])\boldsymbol{e}_i.$$

On the other hand,

$$\left\langle (\mathrm{d}\mu_i)_x(\xi_M|_x),\, \eta_M|_x \right\rangle = (\mathrm{d}\mu^{\eta})_x(\xi_M|_x) = -(\omega_i)_x(\xi_M|_x,\, \eta_M|_x).$$

$$\langle -\mathrm{ad}^*_{\xi}\mu_i(x),\, \eta\rangle = -\langle \mu_i(x),\, [\xi,\, \eta]\rangle = -\mu_i^{[\xi,\eta]}(x).$$

So, the condition $(\mathrm{d}\mu_i)_x(\xi_M) = -\mathrm{ad}^*_{\xi}\mu(x)$ is equvalent to

$$\mu_i^{[\xi,\eta]} = \omega_i(\xi_M,\, \eta_M) \quad (i = 1, 2, 3).$$

Therefore,

$$(\eth_{\mho}\mu)(\xi, \eta) = \omega(\xi_M,\, \eta_M) = \imath_{\rho}^{(2)}\omega. \qquad \square$$

Quaternionic Kähler case

Let

- (M, g) : a $4n$-dimensional Riemannian manifold ($n > 1$).

A *quaternionic structure* is a collection of locally $(1,1)$-tensors $\{J_1^{(\alpha)}, J_2^{(\alpha)}, J_3^{(\alpha)}\}_\alpha$ on each $U_\alpha \subset M$ satisfying

$$J_a^{(\alpha)} \circ J_b^{(\alpha)} = -\delta_{ab}\mathrm{id} + \varepsilon_{abc} J_c^{(\alpha)} \quad (a, b, c = 1, 2, 3).$$

Transition functions $J_a^{(\alpha)} = \tau_{\alpha\beta} J_b^{(\beta)}$ are elements in $SO(3)$. One has a three-dimensional subbundle $\mathcal{Q} \subset \mathrm{End}\, TM$.

Definition

M together with $\mathcal{Q}$ is said to be *almost quaternionic*.

Suppose that g satisfies

$$g(JX, JY) = g(X, Y) \text{ for any } J \in \Gamma(\mathcal{Q}).$$

$\mathcal{Q}$ can be regarded as a subbundle in $\wedge^2 T^*M$ by an isometric embedding $\mathcal{Q} \hookrightarrow \wedge^2 T^*M$.

- ω_i : a nondegenerate 2-form associated to J_i

$$\omega_a(X, Y) := g(J_a X,\ Y) \quad (a = 1, 2, 3)$$

- ∇^g : the Levi-Civita connection with respect to a metric g.

Then, the 4-form

$$\Theta^\wedge := \textstyle\sum_{i=1}^3 \omega_i \wedge \omega_i$$

is defined globally on M.

Definition

$(M, g, \mathcal{Q})$ is called a quaternionic Kähler manifold if $\nabla^g \Theta^\wedge = 0$.

∇^g induces the de Rham complex sequence

$$\Gamma(\mathcal{Q}) \xrightarrow{\nabla^g} \Omega^1(M.\mathcal{Q}) \xrightarrow{\mathrm{d}^g_\nabla} \Omega^2(M.\mathcal{Q}) \xrightarrow{\mathrm{d}^g_\nabla} \cdots$$

- $\mathfrak{K}$: the Lie algebra of Killing vector fields $\mathcal{L}_V\Theta^\wedge = 0$.
- $\Theta_V := \sum_i^3 (\imath_V \omega_i) \otimes \omega_i \quad (V \in \mathfrak{K})$

Theorem (K. Galicki & H. B. Lawson '87, '88)

Suppose that the scalar curvature of M is not zero. Then, for any $V \in \mathfrak{K}$, there exists a unique section $f_V \in \Gamma(\mathcal{Q})$ which satisfies $\nabla^g f_V = \Theta_V$.

Definition (K. Galicki & H. B. Lawson '87, '88)

A momentum map f for M is defined to be a section of $\mathfrak{K}^* \otimes \mathcal{Q}$ satisfying $\nabla^g f_V = \Theta_V$ for any $V \in \mathfrak{K}$.

Questions

How can we describe the momentum map of G.-L.'s in the context of homotopy momentum sections ?

Define

- $\Theta := \sum_{i=1}^{3} \omega_i \otimes \omega_i, \quad \Theta(X,Y) = \sum_{i=1}^{3} \omega_i(X,Y)\omega_i.$

Proposition

A quaternionic Kähler manifold M is a $\mathcal{Q}$-valued 1-plectic manifold.

Sketch of the proof

Θ is non-degenerate from the non-degeneracy of each ω_i. The closedness follows from

$$\mathrm{d}^{g}_{\nabla}\Theta = \sum_{i=1}^{3} \left(\mathrm{d}\omega_i + \sum_{j=1}^{3} \alpha_{ij} \wedge \omega_j \right) \otimes \omega_i$$

and the structural equations.

Consider $E = \mathcal{Q}$, $\mathcal{A} = \mathfrak{K}$.

Theorem (H & Ikeda, '24)

The G.-L. momentum map f is a $\mathcal{Q}$-valued homotopy momentum section if

$$f_{[V_1,V_2]} = \{f_{V_1}, f_{V_2}\} \ ; \ V_1, V_2 \in \mathfrak{K}.$$

Conversely, if $\mu \in \Gamma(\mathfrak{K}^* \otimes \mathcal{Q})$ is a $\mathcal{Q}$-valued homotopy momentum section, then μ is the G.-L. momentum map.

Sketch of the proof

- $\nabla^g f_V = \Theta_V \iff \nabla^g f = \imath^1_{\mathrm{id}}\Theta$
- $(\eth_{\mho}^{T^*M\otimes\mathcal{Q}} f)(V_1, V_2) = \nabla^g_{V_1} f_{V_2} - \nabla^g_{V_2} f_{V_1} - f_{[V_1,V_2]}$
$$= -2\sum_i \omega_i(V_1, V_2)\omega_i - f_{[V_1,V_2]}.$$

Reduction of V-1-plectic manifolds

- V : real vector space of finite dimension.
- (M, V) : a V-valued symplectic manifold.
- $\mathcal{A}$: a Lie algebroid over (M, V) endowed with $\nabla^{\mathcal{A}}$.
- $\mu \in \Gamma(\mathcal{A}^* \otimes V)$: a homotopy momentum section.

Lie algebroid action

A *Lie algebroid action* of $\mathcal{A}$ on a smooth manifold N consists of a smooth map $J : N \to M$, and a Lie algebra homomorphism $\varrho : \Gamma(\mathcal{A}) \to \mathfrak{X}(N)$ which satisfy the following conditions:

- $\forall \alpha \in \Gamma(\mathcal{A}), \forall f \in C^\infty(M) \;\; \varrho(f\alpha) = (J^* f)\varrho(\alpha)$.
- $\forall \alpha \in \Gamma(\mathcal{A}), q \in N \;\; \rho(\alpha)_{J(q)} = (\mathrm{d}J)_q\big(\varrho(\alpha)\big)_q$

$\mathcal{A}$ acts on (M, V) by the anchor map ρ, i.e., $J = \mathrm{id} : M \to M$ and $\varrho = \rho : \Gamma(\mathcal{A}) \to \mathfrak{X}(M)$.

■ Assume that the zero locus of μ

- $M_\mu := \mu^{-1}(\mathbf{0}) = \{x \in M \,|\, \mu_x = \mathbf{0} \in \mathcal{A}^*|_x \otimes V\} \subset M$

is an embedded submanifold of M such that

$$T_x M_\mu + \mathrm{Im}\,\rho_x = T_x M.$$

Then, one has

$$\mathcal{A}_\mu = \textstyle\coprod_{x \in M_\mu} \{a \in \mathcal{A}_x \,|\, \rho(a) \in T_x M_\mu\}$$

■ $\mathcal{A}_\mu$ associates with the characteristic distribution of ρ

$$M_\mu \ni x \longmapsto \mathcal{D}_\mu(x) := \mathrm{span}\{\rho(\alpha)_x \,|\, \alpha \in \Gamma(\mathcal{A}_\mu)\} \subset T_x M_\mu.$$

■ The set of the local flows of vector fields in $\mathcal{D}_\mu$ by

$$\mathcal{E}_\mu := \{\phi_t^X \,|\, \text{the local flow of } X = \rho(\alpha) \text{ for some } \alpha \in \Gamma(\mathcal{A}_\mu)\}$$

■ The pseudogroup generated by $\mathcal{E}_\mu$

$$\mathcal{P}_\mu := \{\mathrm{id}_{M_\mu}\} \bigcup \left\{ \phi_{t_1}^{X_1} \circ \cdots \circ \phi_{t_k}^{X_k} \;\middle|\; \phi_{t_j}^{X_j} \in \mathcal{E}_\mu \text{ or } (\phi_{t_j}^{X_j})^{-1} \in \mathcal{E}_\mu \right\}$$

■ $\mathcal{P}_\mu$-orbit through $x \in M_\mu$ is the set

$$\mathcal{P}_\mu \cdot x := \left\{ \phi_{\boldsymbol{t}}(x) \,|\, \phi_{\boldsymbol{t}} \in \mathcal{P}_\mu, \ \mathrm{Dom}\, \phi_{\boldsymbol{t}} \ni x, \right\},$$

where $\phi_{\boldsymbol{t}} = \phi_{t_1}^{X_1} \circ \cdots \circ \phi_{t_k}^{X_k}$ for $\boldsymbol{t} = (t_1, \cdots, t_k) \in \mathbb{R}^k$.
We say that $x, x' \in M_\mu$ are $\mathcal{P}_\mu$-equivalent

$$x \sim x' \quad \overset{\text{def}}{\Longleftrightarrow} \quad \exists \phi_{\boldsymbol{t}} \in \mathcal{P}_\mu \ x' = \phi_{\boldsymbol{t}}(x).$$

From the Stefan-Sussmann theorem,

- $\mathcal{D}_\mu$ is integrable, and each $\mathcal{D}_\mu(x)$ is the tangent space to the leaf of the singular foliation associated to $\mathcal{D}$.
- $\mathcal{D}_\mu$ is invariant with respect to $\mathcal{P}_\mu$:
$$(\mathrm{d}\phi_{\boldsymbol{t}})_x(\mathcal{D}_\mu(x)) = \mathcal{D}_\mu\big(\phi_{\boldsymbol{t}}(x)\big), \quad x \in \mathrm{Dom}\, \phi_{\boldsymbol{t}}.$$

The tangent space $T_x(\mathcal{P}_\mu \cdot x)$ is expressed as

$$\mathrm{span}\Big\{ (\mathrm{d}\phi_{\boldsymbol{t}})_y(\rho_y(\alpha_y)) \,|\, \phi_{\boldsymbol{t}} \in \mathcal{P}_\mu,\ \alpha \in \Gamma(\mathcal{A}_\mu),\ x = \phi_{\boldsymbol{t}}(y) \Big\}$$

Let $W \subset T_xM$. Define an *ω-orthogonal space* W^ω of W as

$$W^\omega = \{\boldsymbol{u} \in T_xM \,|\, \forall \boldsymbol{v} \in T_xM\ \omega_x(\boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{0}_V\}.$$

If $\dim V = 1$, $\dim T_xM = \dim W + \dim W^\omega$ and $W = (W^\omega)^\omega$.

Lemma

Let μ be a V-valued homotopy momentum section.

$$T_xM_\mu \subset \ker(\nabla^{\mathcal{A}^*\otimes V}\mu) \cap \big(\mathcal{D}_\mu(x)\big)^\omega$$

hold at each $x \in M_\mu$.

Denote the orbit space by $\mathcal{P}_\mu$ by

$$\mathcal{M}_\mu := M_\mu / \mathcal{P}_\mu.$$

- a natural projection $\pi_\mu : M_\mu \to \mathcal{M}_\mu$
- a natural inclusion $\iota : M_\mu \hookrightarrow M$

Theorem (H & Ikeda, '24)

Suppose that

$$\forall \phi_{\boldsymbol{t}} \in \mathcal{P}_\mu, \forall x \in \mathrm{Dom}\, \phi_{\boldsymbol{t}}; \quad (\phi_{\boldsymbol{t}}^* \omega)_x = \omega_x$$

Then, $\mathcal{M}_\mu$ is a V-presymplectic manifold whose V-presymplectic structure ω_μ is characterized uniquely as

$$\pi_\mu^* \omega_\mu = \iota^* \omega.$$

Sketch of the proof

Construction of ω_μ

For any $\boldsymbol{u}, \boldsymbol{v} \in T_x M_\mu$,

$$(\omega_\mu)_{[x]}(\bar{\boldsymbol{u}},\ \bar{\boldsymbol{v}}) := \omega_x(\boldsymbol{u},\ \boldsymbol{v}),$$

where $\bar{\boldsymbol{u}} = (\mathrm{d}\pi_\mu)_z(\boldsymbol{u})$ and $\bar{\boldsymbol{v}} = (\mathrm{d}\pi_\mu)_z(\boldsymbol{v})$

Well-definedness of ω_μ

If $\boldsymbol{u}'$ and $\boldsymbol{v}'$ are tangent vectors of M_μ to $z' = \phi_{\boldsymbol{t}}(z)$ such that

$$(\mathrm{d}\pi_\mu)_x(\boldsymbol{u}) = (\mathrm{d}\pi_\mu)_x(\boldsymbol{u}') \quad \text{and} \quad (\mathrm{d}\pi_\mu)_x(\boldsymbol{v}) = (\mathrm{d}\pi_\mu)_x(\boldsymbol{v}'),$$

then

$$\omega_{x'}(\boldsymbol{u}',\ \boldsymbol{v}') = \omega_x(\boldsymbol{u},\ \boldsymbol{v}) + \text{"the term including } \{\mu^\alpha,\ \mu^\beta\}(x')\text{"}$$

Compatible momentum sections

Given data:

- $\mathcal{M}$: an E-valued 1-plectic manifold (M, ω, E, ∇^E).
- $\mathcal{A}$: a Lie algebroid with a connection $\nabla^{\mathcal{A}}$.
- $\mu \in \Omega^{0,1}(M, \mathcal{A})^E$: E-homotopy momentum section.

Definition

We say that μ is compatible with $\mathcal{A}$ if μ satisfies

$$\forall \alpha \in \Gamma(\mathcal{A}); \quad (\imath_\alpha \circ \nabla^{\mathcal{A}^* \otimes E})\mu = (\nabla^E \circ \imath_\alpha)\mu.$$

Here, $\imath_\alpha$ stands for the interior product by $\alpha \in \Gamma(\mathcal{A})$ defined by

$$(\imath_\alpha \theta)(X_1, \cdots, X_k) := \theta^\alpha(X_1, \cdots, X_k) := \langle \theta(X_1, \cdots, X_k), \alpha \rangle$$

where $X_1, \cdots, X_k \in \mathfrak{X}(M)$ and $\theta \in \Omega^k(M, \mathcal{A}^*)$ with $k \geqq 1$.

If μ is compatible with $\mathcal{A}$, we have

$$\nabla^E \mu^\alpha = -\iota_{\rho(\alpha)}\omega \quad \text{and} \quad (\eth_{\mho}^{T^*M\otimes E}\mu)(\alpha, \beta) = \omega(\rho(\alpha), \rho(\beta)).$$

for any $\alpha, \beta \in \Gamma(\mathcal{A})$.

Proposition

If a homotopy momentum section μ is compatible with $\mathcal{A}$, it holds that

$$\mu^{[\alpha,\beta]} = \{\mu^\alpha, \mu^\beta\}$$

for any $\alpha, \beta \in \Gamma(\mathcal{A})$, and moreover, the set

$$\Gamma_\mu(E) := \{\mu^\alpha \,|\, \alpha \in \Gamma(\mathcal{A})\} \subset \Gamma(E)$$

is a Lie algebra with respect to the bracket $\{\cdot, \cdot\}$.

Reduction with compatible momentum sections

- (M, V) : a V-valued symplectic manifold.
- A : a Lie algebroid over (M, V) endowed with ∇^A.
- $\mu \in \Gamma(A^* \otimes V)$: a homotopy momentum section **compatible with** A.

■ Assume that the zero locus of μ

- $M_\mu := \mu^{-1}(\mathbf{0}) = \{x \in M \,|\, \mu(x) = \mathbf{0} \in A^*|_x \otimes V\} \subset M$

is an embedded submanifold of M.

■ A associates with the characteristic distribution of ρ

$$M \ni x \longmapsto \mathcal{D}_\rho(x) := \mathrm{span}\{\rho(\alpha)_x \,|\, \alpha \in \Gamma(A)\} \subset T_x M.$$

■ The set of the local flows of vector fields in $\mathcal{D}_\rho$ by

$$\mathcal{E}_\rho := \{\varphi_t^X \,|\, \text{the local flow of } X = \rho(\alpha) \text{ for some } \alpha \in \Gamma(\mathcal{A})\}$$

$$\mathcal{P}_\rho := \{\mathrm{id}_M\} \bigcup \left\{ \varphi^{X_1}_{t_1} \circ \cdots \circ \varphi^{X_k}_{t_k} \;\middle|\; k \in \mathbb{N},\ \varphi^{X_j}_{t_j} \in \mathcal{E}_\rho \text{ or } (\varphi^{X_j}_{t_j})^{-1} \in \mathcal{E}_\rho \right\}$$

Furthermore,

- $\mathcal{P}^0_\rho := \left\{ \varphi_{\boldsymbol{t}} \in \mathcal{P}_\rho \,\middle|\, \mu \circ \varphi_{\boldsymbol{t}} = \hat{0}|_{\mathrm{Dom}\varphi_{\boldsymbol{t}}} \right\} \subset \mathcal{P}_\rho$
- $\mathcal{P}^0_\rho \cdot x := \left\{ \varphi_{\boldsymbol{t}(x)} \,\middle|\, \varphi_{\boldsymbol{t}} \in \mathcal{P}^0_\rho, \mathrm{Dom}\ \varphi_{\boldsymbol{t}} \ni x, \right\};\ x \in M$

For each $z \in M_\mu$, $\mathcal{P}^0_\rho \cdot z = \mathcal{L}(z) \cap M_\mu$, where $\mathcal{L}(z)$ means the leaf by $\mathcal{D}_\rho$ containing z.

Theorem (H & Ikeda, '24)

Suppose that

$$\forall \varphi_{\boldsymbol{t}} \in \mathcal{P}^0_\rho,\ \forall z \in \mathrm{Dom}\,\varphi_{\boldsymbol{t}}; \quad (\varphi^*_{\boldsymbol{t}}\omega)_z = \omega_z$$

Then, $\mathcal{M}_\mu = M_\mu/\mathcal{P}^0_\rho$ is a V-presymplectic manifold whose V-presymplectic structure ω_ρ is characterized uniquely as

$$\pi^*_\rho \omega_\rho = \iota^* \omega.$$

Thank you very much for your attention.