Nijenhuis operators on Banach homogeneous spaces

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On the Kähler geometry of infinite dimensional coadjoint orbits of unitary groups
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Plan of the talk

1. Finite dimensional setting and motivation
2. Main results
3. Examples
4. Hierarchy of integrable systems on Banach Lie–Poisson spaces related to the restricted Grassmannian

- Nijenhuis (1951): study of the behaviour of distributions spanned by eigenvectors
- a vector bundle map $\mathcal{N} : TM \to TM$ (covering identity) — or (1,1) tensor field on M
- *Nijenhuis torsion* of $\mathcal{N}$ is defined as

$$\Omega_{\mathcal{N}}(X, Y) = \mathcal{N}[\mathcal{N}X, Y] + \mathcal{N}[X, \mathcal{N}Y] - [\mathcal{N}X, \mathcal{N}Y] - \mathcal{N}^2[X, Y]$$

- $\Omega_{\mathcal{N}}$ is actually a tensor

- a vector bundle map $\mathcal{J} : TM \to TM$ covering the identity such that $\mathcal{J}^2 = -1$
- problem of integrability of $\mathcal{J}$: Eckmann (1951), Frölicher (1955), Newlander–Nirenberg (1957):

$$\text{integrability} \iff \Omega_{\mathcal{J}} = 0$$

- $\mathcal{N}$ is a Nijenhuis operator if $\Omega_{\mathcal{N}} = 0$

- allows to define a new bracket on vector fields

$$[X, Y]_{\mathcal{N}} = [\mathcal{N}X, Y] + [X, \mathcal{N}Y] - \mathcal{N}[X, Y],$$

  lead to deformation of Poisson brackets, recursion operators...
  (Kosmann-Schwarzbach, Magri 1980, Bolsinov, Konyaev, Matveev 2022–, ...)

To the best of our knowledge, so far Nijenhuis operators were only studied in finite dimensional context or formally.

In the infinite-dimensional setting, the Newlander–Nirenberg theorem is no longer true. An example of an infinite-dimensional smooth almost complex Banach manifold with a vanishing Nijenhuis tensor, that is not integrable, was given by Patyi (2000).

Almost complex structures on Banach homogeneous spaces were studied by Beltiţă (2005).

Our motivation was application of the obtained results in the study of almost Kähler structures on the coadjoint orbits of the unitary groups.

Definition (Homogeneous spaces)

Let K be an immersed Banach–Lie subgroup of G with Banach–Lie subalgebra $\mathfrak{k} \subset \mathfrak{g}$. We say that G/K is a *homogeneous space* of G if the quotient space for the right action of K on G has a Hausdorff Banach manifold structure such that the quotient map $\pi : G \to G/K$, $g \mapsto gK$ is a submersion.

Immersion (resp. submersion) — a map such that the differential at every point is an injection with closed range (resp. a surjection).

Notation

- base point $p_0 = \pi(K)$
- group G acts on G/K:

$$\alpha : G \times G/K \to G/K \qquad \alpha(g,p) = \pi(gh) = ghK$$

  for $p = \pi(h) = hK \in G/K$
- $\alpha_g(p) = \alpha(g,p)$
- If $f : \mathfrak{M}_1 \to \mathfrak{M}_2$ is a smooth map, its differential will be denoted by

$$f_* : T\mathfrak{M}_1 \to T\mathfrak{M}_2,$$

  or if there is a need to specify the point $m \in \mathfrak{M}_1$, it will be denoted

$$f_{*m} : T_m\mathfrak{M}_1 \to T_{f(m)}\mathfrak{M}_2.$$

Definition (Homogeneous Banach vector bundle maps)

A smooth Banach vector bundle map $\mathcal{N} : T(G/K) \to T(G/K)$ covering the identity is a smooth map preserving fibers, i.e. $\pi \circ \mathcal{N} = \pi$ such that the restriction to each fiber $\mathcal{N}_{|\pi^{-1}(p)}$ is linear and bounded. A bundle map $\mathcal{N}$ is called *homogeneous* if it is invariant with respect to the left action of G by automorphisms α_g:

$$(\alpha_g)_* \mathcal{N}_p = \mathcal{N}_{\alpha_g(p)} (\alpha_g)_* \qquad \textit{for any } p \in G/K \textit{ and any } g \in G.$$

By homogeneity, any homogeneous bundle map $\mathcal{N} : T(G/K) \to T(G/K)$ comes from some $\mathcal{N}_{p_0} \in \mathcal{B}(T_{p_0}(G/K))$ at the base point p_0.

From the point of view of the Lie algebra $\mathfrak{g}$ of the group G: we are interested in a set of bounded operators on $\mathfrak{g}$, which descend to a homogeneous bundle map $\mathcal{N} : T(G/K) \to T(G/K)$.

Definition (Admissible operators)

The set of admissible operators $\mathcal{A}(G,K)$ on $\mathfrak{g}$ is defined as the set of $N \in \mathcal{B}(\mathfrak{g})$ satisfying the following conditions

$$N\mathfrak{k} \subset \mathfrak{k}, \tag{1}$$

$$\mathrm{Ran}(\mathrm{Ad}_k N - N \,\mathrm{Ad}_k) \subset \mathfrak{k}, \tag{2}$$

Condition (1) ensures that N correctly defines an operator $\mathcal{N}_{p_0} \in \mathcal{B}(T_{p_0}(G/K))$ at the base point p_0.

The condition (2) ensures that $\mathcal{N}_{p_0}$ can be propagated to $T(G/K)$.

Proposition

An admissible operator $N \in \mathcal{A}(G,K)$ *induces a homogeneous Banach vector bundle map* $\mathcal{N} : T(G/K) \to T(G/K)$ *given at each* $p = \pi(g) \in G/K$ *by*

$$\mathcal{N}_p = (\alpha_g)_* \mathcal{N}_{p_0} (\alpha_g)_*^{-1},$$

where $\mathcal{N}_{p_0} : T_{p_0}G/K \to T_{p_0}G/K$ *is defined as*

$$\mathcal{N}_{p_0} \pi_{*1} v := \pi_{*1} N v$$

for $v \in \mathfrak{g}$.

Theorem

Consider $\mathcal{N}$ given by an admissible operator $N \in \mathcal{A}(G,K)$. Let X, Y be vector fields on G/K. Let $p = \pi(g)$ and choose $v, w \in \mathfrak{g}$ such that $X(p) = (\alpha_g)_*\pi_{*1}v$ and $Y(p) = (\alpha_g)_*\pi_{*1}w$. Then the Nijenhuis torsion of $\mathcal{N}$ can be expressed as

$$\Omega_{\mathcal{N}}(X,Y)(p) =$$

$$= (\alpha_g)_*\pi_{*1}\big(N[v,Nw]_{\mathfrak{g}} + N[Nv,w]_{\mathfrak{g}} - [Nv,Nw]_{\mathfrak{g}} - N^2[v,w]_{\mathfrak{g}}\big),$$

where $[\cdot,\cdot]_{\mathfrak{g}}$ denotes the bracket in the Lie algebra $\mathfrak{g}$. In particular $\mathcal{N}$ is a Nijenhuis operator if and only if

$$N[v,Nw]_{\mathfrak{g}} + N[Nv,w]_{\mathfrak{g}} - [Nv,Nw]_{\mathfrak{g}} - N^2[v,w]_{\mathfrak{g}} \in \mathfrak{k}$$

for all $v, w \in \mathfrak{g}$.

- This theorem is known in finite dimensional case, but the existing proofs employ certain properties of vector fields that do not always hold in Banach context. In particular the proof by Frölicher involves the existence of local cross sections of the quotient map $\pi : G \to \mathfrak{M} = G/K$.
- The proof for almost complex structures in the Banach setting (it doesn't deal with Nijenhuis operators) assumed that $\mathfrak{k}$ is complemented in $\mathfrak{g}$. We also note that there is a problem with the proof as the used properties do not hold, even in the finite dimensional case.
- Our approach avoids and clarifies the problem, but at the same time extends the result to the case when $\mathfrak{k}$ is not complemented in $\mathfrak{g}$.

Definition

A C^*-*algebra* is a complex Banach algebra $\mathfrak{A}$ (i.e. complex Banach space equipped with a bilinear associative multiplication) with an anti-linear involution denoted by $*$ such that

$$\|ab\| \leqslant \|a\|\|b\|, \qquad (ab)^* = b^*a^*, \qquad \|aa^*\| = \|a\|^2$$

for all $a, b \in \mathfrak{A}$. We say that the C^*-algebra is unital if it contains a multiplicative unit, which we denote by $\mathbb{1}$.

Let $\mathfrak{k}$ be a closed two-sided non-trivial *-ideal in $\mathfrak{A}$. Then the quotient space $\mathfrak{A}/\mathfrak{k}$ possesses a structure of a C^*-algebra.

- $\mathfrak{A}^\times$ — group of invertible elements
- K — subgroup of $\mathfrak{A}^\times$ defined as

$$K = \mathfrak{A}^\times \cap (\mathbb{1} + \mathfrak{k})$$

Proposition

The quotient space G/K is a homogeneous space of G. Moreover it has a structure of a real analytic Banach–Lie group with Lie algebra $\mathfrak{A}/\mathfrak{k}$.

Proposition

Let $\iota : \mathfrak{A}^\times/(\mathbb{1}+\mathfrak{k})^\times \to (\mathfrak{A}/\mathfrak{k})^\times$ *be defined as* $\iota(\pi(a)) = Q(a) = a + \mathfrak{k}$*. Then* ι *is a Banach–Lie group isomorphism onto an open Banach–Lie subgroup of* $(\mathfrak{A}/\mathfrak{k})^\times$*, and the identity components of* $\mathfrak{A}^\times/(\mathbb{1}+\mathfrak{k})^\times$ *and* $(\mathfrak{A}/\mathfrak{k})^\times$ *are isomorphic as Banach–Lie groups.*

If $\mathfrak{A}^\times$ and $(\mathfrak{A}/\mathfrak{k})^\times$ are connected, then $\mathfrak{A}^\times/(\mathbb{1}+\mathfrak{k})^\times$ is isomorphic to $(\mathfrak{A}/\mathfrak{k})^\times$. This happens, for instance, if $\mathfrak{A}$ and $\mathfrak{A}/\mathfrak{k}$ are von Neumann algebras.

- rank one operators:

$$\ell(\mathfrak{k}) = 0, \qquad \ell(\mathbb{1}) = 1$$

  (extend to $\mathfrak{A}$ via Hahn–Banach theorem)

$$N(a) = \ell(a) \cdot n$$

  for a fixed $n \notin \mathfrak{k}$.

- left and right multiplication:

$$N(a) = AaB$$

  for some fixed $A, B \in \mathfrak{A}$.

- adjoint action:

$$N(a) = \mathrm{ad}_d \, a = [d, a]$$

  for some $d \in \mathfrak{A}$

- rank one operators:

$$\ell(w)\ell([v,n]) + \ell(v)\ell([n,w]) - \ell([v,w])\ell(n) = 0,$$

There are two possibilities. If $\ell(n) \neq 0$, then $\mathcal{N}$ is a Nijenhuis operator iff

$$\ell([v,w]) = 0 \qquad \text{for all } v, w \in \mathfrak{A}.$$

If $\ell(n) = 0$, then $\mathcal{N}$ is a Nijenhuis operator iff

$$\ell([v,n]) = 0 \qquad \text{for all } v \in \mathfrak{A}.$$

It is never an almost complex structure.

- left and right multiplication:
  If $A = \mathbb{1}$ or $B = \mathbb{1}$ then $\mathcal{N}$ is a Nijenhuis operator.

  If we choose A and B such that A^2, B^2 are in the center of $\mathfrak{A}$ and $A^2B^2 = -\mathbb{1}$, we get an almost complex structure on G/K.

- adjoint action: $\mathcal{N}$ is a Nijenhuis operator iff

$$\big[[d,a],[d,b]\big] \in \mathfrak{k}$$

1. the C^*-algebra of bounded operators $\mathcal{B}(\mathcal{H})$ on a separable Hilbert space $\mathcal{H}$
2. the set of continuous functions $C(X)$ on a compact set X
3. Toeplitz algebra
4. Crossed product algebra $C(X) \rtimes_\alpha \mathbb{Z}$ related to a dynamical system (X, α)

Brief summary of the results

Existence of Nijenhuis operators: in each case the existence of Nijenhuis operators is marked by a $+$ sign, the non-existence of Nijenhuis operators is marked by a $-$ sign, and the fact that all Nijenhuis operators are identically zero is marked by a 0. A question marked is used when the existence of Nijenhuis operators is an open question for some particular situation

	Rank one	LR mult.	adjoint
bounded operators	-/+	?/+	?
continuous functions	+	+	0
Toeplitz algebra	+	+	0
crossed product (only fixed points)	+	+	0
crossed product (generic situation)	-	?/+	-/?

Possibility of almost complex structures: in each case the fact that an almost-complex structure can be constructed using the considered class of Nijenhuis operators is marked by a $+$ sign, when no almost-complex structure can be constructed we use a $-$ sign, and when the question is still open we use a ? sign.

	Rank one	LR mult.	adjoint
bounded operators	-	+	-
continuous functions	-	+	-
Toeplitz algebra	-	+	-
crossed product (only fixed points)	-	+	-
crossed product (generic situation)	-	+	?