The Kinematics of Anyon Multipole Configurations

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Acknowledgments

This talk first summarizes some earlier joint work with David H. Sharp, Los Alamos National Laboratory. New results are in collaboration with Hongyi Shen at Rutgers University, and form a part of his Ph.D. thesis in progress.

We are grateful to the local organizers of WGMP XLII for the opportunity to present these ideas.

A brief outlne of topics

1. Scope of the talk
2. A universal kinematical group for quantum mechanics
   - Fundamental assumptions
   - Mass (or charge) and momentum (resp., current) density observables
   - The semidirect product: momentum (resp., current) as the transport of mass (resp., charge
   - Structure constants of the Lie algebra
3. Unitary representations of the semidirect product group
4. Anyon dipole, quadrupole, and higher multipole configurations

Scope of the talk

The continuous unitary representations (CURs) of a certain infinite-dimensional Lie group describe the kinematics of all possible quantum systems with mass or charge in a specified physical space.

This group thus serves as a "**universal kinematical group**" for quantum mechanics.

Orignally it was constructed as a local current algebra from canonically quantized fields. We now obtain it straightforwardly from first principles.

This gives us a way to achieve quantum mechanics without any need to quantize a classical phase space. Last year at WGMP XLI, I presented this idea in detail. Today I first summarize briefly the reasoning behind this, before introducing some new results.

Scope of the talk (continued)

In 1980-81 our group at Los Alamos obtained one of the earliest predictions of possible intermediate particle exchange statistics in two space dimensions, from studying the unitary representations of this group. Such particles are now known as "anyons." Shortly thereafter, we published the first prediction of nonabelian anyons, similarly obtained.

With Hongyi Shen at Rutgers University, we show here how new interesting quantum phases can arise not only from the exchange statistics of anyons, but independently from their theoretically possible internal structure.

The mathematical origin of such phases is topological, as is the origin of the particle exchange statistics in our earlier development.

I will illustrate by discussing the possible kinematics of anyonic configurations in R^2 having dipole and quadrupole properties.

A universal kinematical group for QM: Fundamental assumptions

1. We describe **an arbitrary quantum system with mass or charge in a physical space**.

2. In quantum mechanics observables are described by **self-adjoint operators in a Hilbert space** with the usual measurement interpretations.

3. **Measurements** may be taken of the **mass (resp. charge) density in bounded regions** of space. The operators describing such observables **all commute**.

4. **Measurements** may be taken of the **momentum density (resp., current density) in bounded regions** of space.

Momentum density refers to **instantaneous transport of the mass density**. Likewise, current density refers to **instantaneous transport of the charge density**.

(continued)

Fundamental assumptions (continued)

5. The operators for **momentum density observables do not commute with** those for **mass density observables**.

   Likewise, the operators for **current density observables do not commute with** those for **charge density observables**.

Sharp and I construct from these assumptions an infinite-dimensional group G, whose inequivalent continuous unitary representations (CURs) in Hilbert space describe the quantum kinematics of all possible systems with mass (or charge) in M.

Mass (or charge) density observables: the commutative Lie group $\mathcal{D}(M)$

Mass (or charge) density observables in M are to be described by self-adjoint operators in a Hlilbert space H. As point-like particles are possible, such densities may be singular; i.e., **distributions** rather than functions.

E.g. for N point particles with masses $m_1, ...m_N$, located respectively at positions $x_1, ...x_N \in M$, the mass density is $\Sigma_{j=1}^{N} m_j \delta_{x_j}$.

The mass density may be any positive distribution on M. Charge densities need not be positive.

By our first three physical assumptions, we thus represent such density observables by a **self-adjoint operator-valued distribution** $\rho(x)$ on M, acting in H. The test function space can be the space $\mathcal{D}(M)$ of real, smooth, compactly supported functions on M.

Mass (or charge) density measurements (continued)

For a test function $f : M \to \mathbb{R}$, we write formally (in the usual notation for distributions):

$$\rho(f) = \int_M \rho(x) f(x) d^n x. \tag{2.1}$$

Then $\rho(f), f \in \mathcal{D}(M)$ is the self-adjoint operator describing the observable for mass or charge density averaged by the test function f over its region of support. Our assumption that such observables commute means that

$$[\rho(f_1), \rho(f_2)] = 0, \ \forall f_1, f_2 \in \mathcal{D}(M). \tag{2.2}$$

Now Stone's theorem associates with each self-adjoint operator $\rho(f)$ a strongly-continuous one-parameter unitary group:

$$\begin{aligned} U_s(f) &= \exp is\rho(f), s \in \mathrm{R}\,, \text{ where } s\rho(f) = \rho(sf), \text{ and} \\ \rho(f) &= \lim_{s \to 0} (1/is)[U_s(f) - I]. \end{aligned} \tag{2.3}$$

Because the $\rho(f)$'s all commute, we have $U(f_1)U(f_2) = U(f_1 + f_2), \ \forall f_1, f_2 \in \mathcal{D}(M)$.

Momentum (or current) density observables: the algebra of vector fields and the diffeomorphism group $\mathrm{Diff}_0(M)$

Our final two assumptions pertain to measurement of momentum density or current density. As the mass or charge density of a quantum system may be singular, the associated flux density may likewise be singular. Furthermore, it is a vector quantity.

Consequently, such measurements must be represented in $\mathcal{H}$ as an n-component operator-valued distribution $\mathrm{J}(x), \mathrm{J} = (J_1, ...J_n)$, where n is the dimensionality of M. The test functions for these observables are C^∞ **tangent vector fields** g on M. That is, for $x \in M$, $\mathrm{g}(x)$ is an element of $T_x(M)$, the tangent space to M at x.

The self-adjoint operator $J(\mathrm{g})$ describes the averaged momentum or current density, in the direction of $\mathrm{g}(x)$ at each point x and weighted there with the magnitude $|\mathrm{g}(x)|$:

$$J(\mathrm{g}) = \int_M \mathrm{J}(x) \cdot \mathrm{g}(x) d^n x \,.\, \text{where } \mathrm{J}(x) \cdot \mathrm{g}(x) = \Sigma_{k=1}^n \mathrm{J}_k(x) \mathrm{g}^k(x). \qquad (2.4)$$

Momentum (or current) density observables (continued)

For observations in bounded regions, each g has compact support. Let $\mathrm{vect}_0(M)$ denote the space of C^∞ compactly supported vector fields on M. Then the self-adjoint operators $J(\mathrm{g}), \mathrm{g} \in \mathrm{vect}_0(M)$ describe momentum (current) density observables.

Now each g generates a **smooth global flow** on M, the trajectory that a point carried by g would follow. Such a flow is a one-parameter group of C^∞ **diffeomorphisms** of M, denoted by $\phi_r^{\mathrm{g}} : M \to M$, where r is a real parameter. It is defined for all x and for all r due to the compact support of g. (A diffeomorphism is a smooth, invertible map ϕ from M to itself, whose inverse is also smooth.) We have,

$$\phi^{\mathrm{g}}_{r_1+r_2} = \phi^{\mathrm{g}}_{r_2} \circ \phi^{\mathrm{g}}_{r_1}, \text{ where } \partial\phi^{\mathrm{g}}_r(x)/\partial r = \mathrm{g}(\phi^{\mathrm{g}}_r(x)) \tag{2.5}$$

with initial condition $\phi^{\mathrm{g}}_{r=0}(x) \equiv x$. The product law is composition of diffeomorphisms. To first order in r, the point $x \in M$ moves (in local Euclidean coordinates) to $x + r\mathrm{g}(x)$.

Momentum (or current) density observables (continued)

Now $J(\mathrm{g})$ generates a continuous unitary represesention $V(\phi_r^{\mathrm{g}})$. Composing all such flows leads to a **CUR of a group of C^∞ diffeomorphisms of M**. For $\mathrm{g} \in \mathrm{vect}_0(M)$, ϕ_r^{g} is the identity map outside the support of g. Any product of such diffeomorphisms likewise has compact support, generating the group $\mathrm{Diff}_0(M)$.

Writing multiplication in $\mathrm{Diff}_0(M)$ as $\phi_1\phi_2 := \phi_2 \circ \phi_1$, where $\circ$ denotes composition, we have $V(\phi_1\phi_2) = V(\phi_1)V(\phi_2)$ without reversing the order of the diffeomorphisms.

An infinitesimal flow by g_1 followed by g_2, succeeded by an infinitesimal flow back by g_1 and then by g_2 yields an infinitesimal flow by the **Lie bracket** $[\mathrm{g}_1, \mathrm{g}_2]$. In the usual notation, $[\mathrm{g}_1, \mathrm{g}_2] = \mathrm{g}_1 \cdot \nabla \mathrm{g}_2 - \mathrm{g}_2 \cdot \nabla \mathrm{g}$. Thus J is a self-adjoint representation in $\mathcal{H}$ of the Lie algebra $\mathrm{vect}_0(M)$ equipped with the Lie bracket; i.e., $[J(\mathrm{g}_1), J(\mathrm{g}_2)] \propto iJ([\mathrm{g}_1, \mathrm{g}_2])$.

The semidirect product group $\mathcal{D}(M) \rtimes \mathrm{Diff}_0(M)$

Finally consider the commutation relations of the operators $\rho(f)$ with the operators $J(\mathrm{g})$. Here we use the physical assumption that momentum density (resp., current density) refers to the infinitesimal transport of mass density (resp., charge density).

This implies the proportionality $[\rho(f), J(\mathrm{g})] \propto i\rho(\mathrm{g} \cdot \nabla f)$. The right side is the density averaged with the derivative of f in the g-direction (i.e., the **Lie derivative** of f by g. By our final assumption, the constant of proportionality must be non-zero.

At the level of the unitary groups U and V, for $f \in \mathcal{D}(M)$ and $\phi \in \mathrm{Diff}_0(M)$, this gives us the semidirect product group action, $V(\phi)U(f) = U(f \circ \phi)V(\phi)$.

Physically, these equations assert that mass or charge density measured at the location to which it has been carried by momentum or current density takes the value measured at its original location before being transported.

The semidirect product group (continued)

To sum up, the kinematics of an arbitrary quantum system with mass or charge in the space M **is described universally** by a CUR $U(f)V(\phi)$ of the semidirect product group $\mathcal{D}(M) \rtimes \mathrm{Diff}_0(M)$. The group law is

$$(f_1, \phi_1)(f_2, \phi_2) = (f_1 + f_2 \circ \phi_1, \phi_2 \circ \phi_1). \tag{2.6}$$

In this development, we have found a very general way to arrive at quantum mechanics directly, **without the need for quantization** of a system in a classical phase space, and without assuming an underlying quantum field theory.

To my knowledge, the kinematics of every known "nonrelativistic" (i.e., Galilean) quantum-mechanical system corresponds to a CUR of this group. In a particular CUR, one may identity a configuration space on which it is modeled. The cotangent bundle of this configuration space serves as the classifical phase space for the system. The inverse problem of reconstructing the CUR from the phase space is "quantization."

Structure constants of the Lie algebra

In the relations $[\rho(f), J(\mathrm{g})] \propto \rho(\mathrm{g} \cdot \nabla f)$ and $[J(\mathrm{g}_1), J(\mathrm{g}_2)] \propto iJ([\mathrm{g}_1, \mathrm{g}_2])$, it is straightforward to determine carefully the units of the constants of proportionality.

For example, if the representation describes mass and momentum density observables, the constants of proportionality must both be in units ML^2/T, in order that the dimensionality of both expressions in each proportion be the same.

We therefore introduce the coefficient $\hbar$, whose magnitude is to be determined experimentally. Finally, we obtain the "Lie algebra of local currents" whose self-adjoint representations describe all possible quantum systems with mass in the space M. For $f_1, f_2, f \in \mathcal{D}(M)$ and $\mathrm{g}_1, \mathrm{g}_2, \mathrm{g} \in \mathrm{vect}_0(M)$,

$$\begin{aligned} [\rho(f_1), \rho(f_2)] &= 0, \\ [\rho(f), J(\mathrm{g})] &= i\hbar\rho(\mathrm{g} \cdot \nabla f), \\ [J(\mathrm{g}_1), J(\mathrm{g}_2)] &= -i\hbar J([\mathrm{g}_1, \mathrm{g}_2]). \end{aligned} \tag{2.7}$$

Unitary representations of $\mathcal{D}(M) \rtimes \mathrm{Diff}_0(M)$

Under very general conditions, a unitary representation $U(f)V(\phi)$ of the semidirect group in a Hilbert space $\mathcal{H}$ may be written, for $\Psi \in \mathcal{H}$,

$$[U(f)\Psi](\gamma) = e^{i<\gamma,f>}\Psi(\gamma),$$

$$[V(\phi)\Psi](\gamma) = \chi_\phi(\gamma)\Psi(\phi\gamma)\sqrt{\frac{d\mu_\phi}{d\mu}(\gamma)}\,. \tag{3.1}$$

In Eqs. (3.1), γ belongs to a **configuration space** Δ, which is a subset of the space of distributions modeled on $\mathcal{D}(M)$; that is, $\Delta \subset \mathcal{D}'(M)$. We denote by $< \gamma, f >$ the value that γ takes on $f \in \mathcal{D}'(M)$.

The group $\mathrm{Diff}_0(M)$ acts naturally on $\mathcal{D}'(M)$ as the dual to its action on $\mathcal{D}(M)$; that is, $< \phi\gamma, f > = < \gamma, f \circ \phi >$. The space Δ is invariant under this action.

Unitary representations (continued)

A measure μ exists on at least some such invariant submanifolds Δ, with the needed technical property of quasi-invariance under the group action. Letting μ_ϕ denote the measure transformed by ϕ, quasi-invariance ensures the existence (almost everywhere with respect to μ) of the Radon-Nikodym derivative $[d\mu_\phi/d\mu](\gamma)$ in Eqs. (3.1).

The Hilbert space $\mathcal{H}$ is realized as the space of square integrable functions on Δ taking values in a complex inner product space $\mathcal{W}$; i.e., $\mathcal{H} = L^2_\mu(\Delta, \mathcal{W})$. For scalar-valued wave functions, $\mathcal{W} = \mathbb{C}$. For vector-valued functions, $\mathcal{W}$ can be higher-dimensional.

For the representation to be irreducible, μ must be **ergodic**; that is, every invariant set under the action of the diffeomorphism group is either of full measure, or of measure zero. This can occur in two different ways – either Δ consists of a single orbit in $\mathcal{D}'(M)$ (as is the case for N-particle configuration spaces), or it is the uncountable union of orbits having zero measure (e.g., for infinite particle systems).

Unitary representations (continued)

Finally, χ is a **unitary 1-cocycle** acting in $\mathcal{W}$. That is, it satisfies the cocycle equation (almost everywhere in Δ),

$$\chi_{\phi_1\phi_2}(\gamma) = \chi_{\phi_1}(\gamma)\chi_{\phi_2}(\phi_1\gamma), \ \forall\phi_1, \phi_2 \in \mathrm{Diff}_0(M). \tag{3.2}$$

Cocycles matter greatly, because **inequivalent cocycles describe distinct topological effects** – e.g., the possible exchange statistics of particles associated with unitary representations of the fundamental group of Δ.

Thus one obtains Bose or Fermi statistics, parastatistics, the intermediate statistics of anyons, and nonabelian anyon statistics, as inequivalent irreducible continuous unitary representations of the same infinite-dimensional group.

Topology of the configuration space Δ

Briefly, for $\gamma \in \Delta$, there is a natural homomorphism from the subgroup of $\mathrm{Diff}_0(M)$ that leaves γ fixed (the stability subgroup), onto the fundamental group $\pi_1(\Delta)$. Then a unitary representation of the fundamental group immediately provides a CUR of the stability subgroup. One then obtains the cocycle *via* an inducing construction.

This in turn gives us a CUR of the full semidirect product group from Eq. (3.1), establishing the kinematics of a quantum system modeled on Δ.

For N indistinguishable point particles in $\mathbb{R}^3$, $\pi_1(\Delta)$ is the symmetric group S_N. Thus we obtain the exchange statistics of bosons, fermions, and paraparticles from unitary representations of S_N. In $\mathbb{R}^2$. the fundamental group is the braid group B_N, whose representations led us to the kinematics of anyons and nonabelian anyons.

This brings me to the last part of my talk (with Hongyi Shen), group-theoretic prediction of the kinematics of anyonic multipole configurations.

Anyonic dipoles

Consider configurations in the plane that include a dipole term. For $x \in M = \mathbb{R}^2$ and $\in T_x(\mathbb{R}^2)$, $T_x(M)$ being the tangent space to M at x, define the configuration $\gamma_{(x,\boldsymbol{\lambda})}$ by:

$$< \gamma_{(x,\boldsymbol{\lambda})}, f >:= qf(x) + \boldsymbol{\lambda} \cdot \nabla f(x). \tag{4.1}$$

Here q can be (for example) the charge of a particle at x, and $\boldsymbol{\lambda}$ its dipole moment.

Then the action of a diffeomorphism $\phi \in \mathrm{Diff}_0(M)$ on $\gamma_{(x,\boldsymbol{\lambda})}$ that enters Eqs. (3.1) is given by $\phi\gamma_{(x,\boldsymbol{\lambda})} = \gamma_{(x',\boldsymbol{\lambda}')}$, where

$$x' = \phi(x), \tag{4.2}$$

$$\boldsymbol{\lambda}' = J_\phi(x)\boldsymbol{\lambda}, \tag{4.3}$$

and where $J_\phi(x)$ is the matrix of derivatives of ϕ (i.e., the Jacobian matrix) at x. Because $\det J_\phi$ cannot vanish, if $\boldsymbol{\lambda} \neq 0$ then $J_\phi(x)\boldsymbol{\lambda} \neq 0$.

Physical consequences for dipole particles or excitations in two-space

In such an orbit of $\mathrm{Diff}_0(M)$, for each $x \in M$, the space of values taken by $\boldsymbol{\lambda}$ is without the origin. When $M = \mathbb{R}^2$, the fundamental group of Δ is nontrivial; we have $\pi_1(\Delta) = \mathbb{Z}$.

The stability subgroup of the configuration $\gamma_{(x,\boldsymbol{\lambda}}$ consists of diffeomorphisms for which $\phi(x) = x$ and $J_\phi(x)\boldsymbol{\lambda} = \boldsymbol{\lambda}$. When $M = \mathbb{R}^2$, each such diffeomorphism encodes **the number of counterclockwise windings of the dipole about itself**.

Consequently there are representations that introduce a **new intermediate phase** to the wave function, proportional to the number of local rotations by 2π performed by the diffeomorphism.

The spectrum of the local angular momentum operator generating the rotation shifts accordingly.

Physical consequences for anyonic dipoles (continued)

The magnitude of the phase coming from a local 2π rotation is, in principle, independent of the exchange phase associated with the counterclockwise exchange in position of indistinguishable pairs of anyonic dipoles in multi-particle systems.

One can construct a model wherein one pictures the dipole part of the system to be a tightly-bound (confined) composite formed by a pair of equal but oppositely-charged anyons, behaving like charged-particle/magnetic-flux-tube composites.

In such a picture, the "internal" phase associated with the number of 2π rotations is obtained from circulation of the charges about each others' flux tubes.

In this model, the respective attached fluxes (if equal in magnitude) should likewise be opposite in sign in order to contribute to a net nonzero rotational phase.

Quadrupoles (and higher multipoles)

Quadrupole and higher multipole configurations involve greater complexity, and require a more detailed discussion. We shall look at quadrupole configurations, where second derivative terms are included in the distributions defining configuration spaces. Higher derivatives of delta-distributions provide us with higher multipole configurations.

We write quadrupole configurations, summing over repeated indices (which range from 1 to the dimension of M) as follows. For $f \in \mathcal{D}(M)$,

$$< \gamma_{(x,\boldsymbol{\lambda},\boldsymbol{Q})}, f > := \; qf(x) + \lambda^i \partial_i f(x) + Q^{ij} \partial_i \partial_j f(x). \tag{4.4}$$

A particle configuration is described by its location $x \in M$, its charge q, its dipole moment vector $\boldsymbol{\lambda}$, and its quadrupole moment tensor $\boldsymbol{Q}$.

Quadrupoles configurations

Now the action of diffeomorphisms is given by $\phi\gamma_{(x,\boldsymbol{\lambda},\boldsymbol{Q})} = \gamma_{(x',\boldsymbol{\lambda}',\boldsymbol{Q}')}$, where

$$\begin{aligned} x' &= \phi(x), \\ (\lambda')^k &= [\partial_j \phi^k](x)\lambda^j + \frac{1}{2} Q^{pq}[\partial_p \partial_q \phi^k](x), \\ (Q')^{pq} &= [\partial_j \phi^p](x)[\partial_k \phi^q](x) Q^{jk}. \end{aligned} \tag{4.5}$$

The general formulas (4.5), and the classification of orbits, were discussed in this context in my joint publication with Ralph Menikoff back in 1985 on the quantum kinematics of tightly-bound composites.

With Hongyi Shen, we now focus on anyonic effects specific to two space dimensions.

Anyonic quadrupole configurationsv

In a CUR of the semidirect product group, wave functions having positive measure in **disjoint quadrupole orbits** cannot be connected by $U(f)V(\phi)$. Each orbit carries irreducible representations induced by unitary representations of its fundamental group.

Notice that if $\boldsymbol{Q} = 0$ in Eq.(4.5), it reduces to the dipole case. But for $\boldsymbol{Q} \neq 0$. the origin is no longer excluded from $\boldsymbol{\lambda}$-space due the quadrupole term. Thus the topology of quadrupole configuration space is wholly determined by the topology of quadrupole orbits under the action of diffeomorphisms.

The diffeent orbits descibe qualitatively different quantum particles – distinct particle types. The orbits are classified according the signs of the eigenvalues of $\boldsymbol{Q}$. We may have one nonzero eigenvalue (either + or -) or we may have two, with like or unlike signs.

Configurations in different orbits can be modeled by distinct composite anyonic structures.

A closer look at an anyonic quadrupole orbit

Let us explore an orbit for $\boldsymbol{Q}$ having just one nonzero eigenvalue. Such an orbit (with zero net charge) can be modeled by configurations of **three very tightly bound, very nearly colinear particles in the plane.** One particle must then have a charge different in sign from the other two, equal in magnitude to their sum.

Place the differently-charged particle between the other two, and let x be the coordinate of that particle. The sign of the eigenvalue is that of the outer pair of particles. The dipole moment λ and the quadrupole moment $\boldsymbol{Q}$ are defined about x. The orbit $(x, \boldsymbol{\lambda}, \boldsymbol{Q})$ has six degrees of freedom – two for x, two for $\boldsymbol{\lambda}$, and two for $\boldsymbol{Q}$.

Think of the separation between the "tightly bound" charges as first-order infinitesimal. The deviation from linearity is then second-order infinitesimal. This does not change the number of degress of freedom in the orbit.

An anyonic quadrupole orbit (continued)

The fundamental group of this orbit is again isomorphic to $\mathbb{Z}$. We can see easily from the model that a locally rigid rotation by π about x restores the quadrupole moment to its initial value, without affecting the dipole moment.

We again have nontrivial cocycles for the diffeomorphism group action, and the possibility of nontrivial phases introduced into wave functions in CURs of the semidirect product group.

One difference from the pure dipole case is that now local rotations about x by πn (rather than by $2\pi n$) belong to the subgroup of diffeomorphisms leaving a configuration fixed.

The three tightly-bound, colinear charges in our model are understood to be extremely large, but their sum is zero. If we would like to introduce a net charge q into the representation, we can do so in our model by attaching an additional charge to the structure at the point x.

Internal anyonic phases (continued)

By attaching a magnetic flux tube to each of the charged particles, one can adjust the magnitudes of the resulting phases under local rotations. If desired, one can set these to relate explicitly to the exchange phase of particles in multiplarticle systems, but there does not appear to be a necessary connection in either the dipole or quadrupole cases discussed.

Hongyi Shen is currently exploring further the classification and topology of higher multipole orbits, as part of his Ph.D. thesis.

Research in progress

It is interesting to speculate that anyonic multipole cofigurations may eventually play some theoretical role in our understanidng of phyiscal situations where anyons are relevant – e.g., surface phenomena in the presence of variable magnetic fields, or vortices in thin quantum fluids.

Mathematically, they are relevant (for example) to the study of coadjoint orbits and geometric quantization of exotic vortex configuraitions in classical superfluids.

Thank you for your attention!

References are in the slides that follow.

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