A Purely Geometrical Aharonov–Bohm Effect

Affine Covariant Integral Quantization of the Punctured Plane

Jean–Pierre Gazeau[1,2] Tomoi Koide[3] Romain Murenzi[4] Aidan Zlotak[4]

[1]University of Białystok, Poland
[2]Université Paris Cité, CNRS, APC, France
[3]Instituto de Física, UFRJ, Brazil
[4]Department of Physics, Worcester Polytechnic Institute, Ma, USA

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Talk roadmap

1. Motivation
2. Model setup
3. Phase space as SIM(2)
4. Affine Covariant Integral Quantization
5. Results & Interpretation
6. Outlook

Why revisit the Aharonov–Bohm (AB) effect?

- Paradigmatic example of *topology* in quantum mechanics.
- Experimental reality settled since Tonomura *et al.* (1982).
- Ongoing debates: role of gauge potentials vs. fields, information–theoretic angles.
- Our twist: derive AB–like gauge structure *purely* from topology via Affine Covariant Integral Quantization (ACIQ).

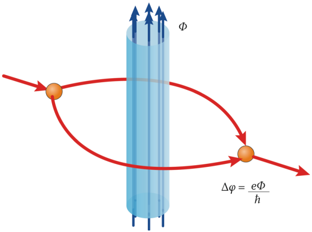
AB: Overview

The **Aharonov–Bohm (AB) effect** shows that electromagnetic *potentials* can influence quantum states for particles even in regions where the corresponding *fields* vanish.

- Classical: Only **E**, **B** affect motion.
- Quantum: Potentials **A**, ϕ have physical consequences.

Experimental Setup

- Electron beam split into two paths.
- A solenoid lies between paths; $\mathbf{B} \neq \mathbf{0}$ inside, but zero outside.
- Outside: $\mathbf{A} \neq \mathbf{0}$, $\mathbf{B} = \mathbf{0}$.



$$\Delta\varphi = \frac{e}{\hbar} \oint \mathbf{A} \cdot \mathrm{d}\boldsymbol{\ell} = \frac{e\Phi}{\hbar}$$

- Proposed: Aharonov&Bohm, *Phys. Rev.* **115** (1959)
- Confirmed experimentally: Tonomura et al. *PRL* **48** (1982)

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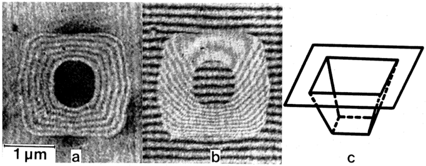


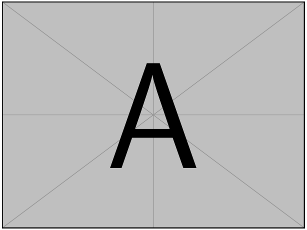
FIG. 4. Interference micrographs of the toroidal magnet shown in Fig. 1. (a) Contour map of electron phase. (b) Interferogram of electron phase. (c) Schematic form of the wave front.

Physical Implications

- Potentials have observable effects (not just fields).
- Quantum theory is **nonlocal**.
- Gauge potentials are physically relevant.

Infinite solenoid: classical picture

- Infinitely long, infinitesimal–radius coil along $\mathbf{e}_3$.
- Particle of charge $\mathfrak{q}$ confined to transverse (x_1, x_2) plane with basis $\mathbf{e}_1$, $\mathbf{e}_2$.
- $\sim$ Outside region is punctured: origin removed.



Vector potential (Coulomb gauge)

$$\mathbf{A}^{\mathrm{sol}} = \frac{\Phi_0}{2\pi r^2}(-x_2,\, x_1)\,, \qquad \Phi_0 = \text{const.}$$

Hamiltonian on the punctured plane

$$H_0(\mathbf{x},\mathbf{p}) = \frac{1}{2m}\left(\mathbf{p} - \mathfrak{q}\,\mathbf{A}^{\mathrm{sol}}\right)^2, \qquad \mathbf{x} \in \mathbb{R}^2_*.$$

- Standard canonical quantization *fails*: translation symmetry broken.
- Need quantization that respects dilations + rotations: enter **SIM(2)**.

The similitude group SIM(2)

Elements $(a, \theta, \mathbf{b})$ act as

$$(a, \theta, \mathbf{b})\,\mathbf{x} = a\,R(\theta)\mathbf{x} + \mathbf{b}\,, \quad R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\,.$$

- New variables $q = \dfrac{1}{a}\,,\ \mathbf{q} = (q, \theta)\,,\ \mathbf{b} \equiv \mathbf{p}$.
- Phase space $\Gamma = \left\{(\mathbf{q}, \mathbf{p}) \in \mathbb{R}^2_* \times \mathbb{R}^2\right\} \simeq \mathrm{SIM}(2)$.
- SIM(2) group structure: $(\mathbf{1}, \mathbf{0})$ (unit),

$$(\mathbf{q}, \mathbf{p})(\mathbf{q}', \mathbf{p}') = \left(\mathbf{q}\mathbf{q}', \frac{\mathbf{p}'}{\mathbf{q}^*} + \mathbf{p}\right)\,, \quad (\mathbf{q}, \mathbf{p})^{-1} = \left(\mathbf{q}^{-1}, -\mathbf{q}^{\,*}\mathbf{p}\right)\,,$$

  where $\mathbf{q}, \mathbf{p}$ are viewed as complex numbers.
- Non–unimodular: left Haar $\mathrm{d}^2\mathbf{q}\,\mathrm{d}^2\mathbf{p}$.

ACIQ in one slide

- UIR of SIM(2): $(U(\mathbf{q},\mathbf{p})\psi)(\mathbf{x}) = \dfrac{e^{i\mathbf{p}\cdot\mathbf{x}}}{q}\psi\left(\dfrac{\mathbf{x}}{\mathbf{q}}\right)$ for $\psi \in \mathcal{H} = L^2(\mathbb{R}^2_*, \mathrm{d}^2\mathbf{x})$.
- Choose weight $\varpi(\mathbf{q},\mathbf{p})$ such that

$$\mathsf{M}^{\varpi} = \int_{\Gamma} \frac{\mathrm{d}^2\mathbf{q}\,\mathrm{d}^2\mathbf{p}}{4\pi^2}\,\varpi(\mathbf{q},\mathbf{p})QU(\mathbf{q},\mathbf{p})Q\,, \quad Q\psi(\mathbf{x}) = x\psi(\mathbf{x})\,.$$

is bounded, self-adjoint, <u>unit trace</u> (i.e., $\varpi(\mathbf{1},\mathbf{0}) = 1$)

- Resolution of identity:

$$\int_{\Gamma} \frac{\mathrm{d}^2\mathbf{q}\,\mathrm{d}^2\mathbf{p}}{c_{\mathsf{M}^{\varpi}}}\mathsf{M}^{\varpi}(\mathbf{q},\mathbf{p}) = \mathbb{1}_{\mathcal{H}}\,, \quad \mathsf{M}^{\varpi}(\mathbf{q},\mathbf{p}) = U(\mathbf{q},\mathbf{p})\mathsf{M}^{\varpi}U^{\dagger}(\mathbf{q},\mathbf{p})\,.$$

- Covariant map

$$f \mapsto \mathsf{Op}^{\varpi}_f = \int_{\Gamma} \frac{\mathrm{d}^2\mathbf{q}\,\mathrm{d}^2\mathbf{p}}{c_{\mathsf{M}^{\varpi}}}\, f(\mathbf{q},\mathbf{p})\,\mathsf{M}^{\varpi}(\mathbf{q},\mathbf{p})\,, \quad \mathrm{Op}^{\varpi}_{f=1} = \mathbb{1}_{\mathcal{H}}\,.$$

Key benefit: geometry/topology enter via choice of weight; no *ad hoc* gauge fields.

Covariance of the map

Quantization map is covariant with respect to the unitary 2-D affine action U:

$$U(\mathbf{q}_0,\mathbf{p}_0)\,\mathrm{Op}^{\varpi}_{f}\;U^{\dagger}(\mathbf{q}_0,\mathbf{p}_0)=\mathrm{Op}^{\varpi}_{\mathfrak{U}(\mathbf{q}_0,\mathbf{p}_0)f}\,,$$

with

$$\left(\mathfrak{U}(\mathbf{q}_0,\mathbf{p}_0)f\right)(\mathbf{q},\mathbf{p})=f\left((\mathbf{q}_0,\mathbf{p}_0)^{-1}(\mathbf{q},\mathbf{p})\right)=f\left(\frac{\mathbf{q}}{\mathbf{q}_0},\mathbf{q}_0^{*}(\mathbf{p}-\mathbf{p}_0)\right)\,,$$

The technics

With

- $\Omega(\mathbf{q}) := \int_{\mathbb{R}^2_*} \frac{\mathrm{d}^2\mathbf{x}}{x^2}\, \widehat{\varpi}_p\,(\mathbf{q}, \pm\mathbf{x})$ obeying $0 < \Omega(1) < \infty$,
- and where $\widehat{f_p}$ is the partial 2-D Fourier transform of $f(\mathbf{q}, \mathbf{p})$ with respect to the variable $\mathbf{p}$

the action of Op^{ϖ}_f on ϕ in $C_0^\infty(\mathbb{R}^2_*)$ is given in the form of the linear integral operator

$$(\mathrm{Op}^{\varpi}_f\,\phi)(\mathbf{x}) = \int_{\mathbb{R}^2_*} \mathcal{A}^{\varpi}_f(\mathbf{x}, \mathbf{x}')\,\phi(\mathbf{x}')\,\mathrm{d}^2\mathbf{x}'\,,$$

where the kernel $\mathcal{A}^{\varpi}_f$ reads as

$$\mathcal{A}^{\varpi}_f(\mathbf{x}, \mathbf{x}') = \frac{1}{c_{\mathrm{M}^{\varpi}}}\,\frac{x^2}{x'^2} \int_{\mathbb{R}^2_*} \frac{\mathrm{d}^2\mathbf{q}}{q^2}\, \widehat{\varpi}_p\left(\frac{\mathbf{x}}{\mathbf{x}'}, -\mathbf{q}\right)\, \hat{f}_p\left(\frac{\mathbf{x}}{\mathbf{q}}, \mathbf{x}' - \mathbf{x}\right)\,,$$

and the constant $c_{\mathrm{M}^{\varpi}}$ is given by

$$c_{\mathrm{M}^{\varpi}} = 2\pi\Omega(1)\,.$$

Emergent quantum operators

With $\Omega(\mathbf{q}) := \int_{\mathbb{R}^2_*} \frac{\mathrm{d}^2\mathbf{x}}{x^2}\, \widehat{\varpi}_p(\mathbf{q}, \pm\mathbf{x})$ obeying $0 < \Omega(1) < \infty$, and where $\widehat{\varpi}_p$ is the partial 2-D Fourier transform of ϖ with respect to the variable $\mathbf{p}$, one obtains

$$\mathrm{Op}^{\varpi}_{\mathbf{p}} = \mathbf{P} + \frac{\mathrm{i}}{\mathbf{Q}^*}\left(2 + \frac{\nabla\Omega(1)}{\Omega(1)}\right), \quad \mathbf{P} = -\mathrm{i}\boldsymbol{\nabla}, \quad \mathbf{Q}\psi(\mathbf{x}) = \mathbf{x}\psi(\mathbf{x})$$

$$\mathrm{Op}^{\varpi}_{\mathbf{p}^2} = (\mathbf{P} - \mathfrak{q}\,\mathbf{A}^{\varpi}(\mathbf{Q}))^2 + \frac{K^{\varpi}}{Q^2}.$$

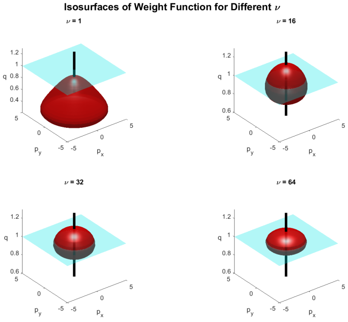
- **Affine vector potential** $\mathbf{A}^{\varpi}(\mathbf{Q}) = -\frac{\mathrm{i}}{\mathfrak{q}\mathbf{Q}^*}\left(2 + \frac{\nabla\Omega(1)}{\Omega(1)}\right)$.
- **Centrifugal scalar potential** $V^{\varpi}(\mathbf{Q}) = \frac{K^{\varpi}}{Q^2}$.

AB gauge field from topology

- with the example of well localised about SIM(2) unit $(\mathbf{1},\mathbf{0})$: $\varpi(\mathbf{q},\mathbf{p}) \propto e^{-\nu(q+1/q)} e^{\mathrm{i}\mu \arg \mathbf{q}} \left(1 - q\frac{p^2}{2\sigma^2}\right) e^{-q\frac{p^2}{2\sigma^2}} \Rightarrow$ flux *quantised* when $\mu \in \mathbb{Z}$.
- one obtains

$$\mathbf{A}^{\varpi}(\mathbf{Q}) = \frac{\Phi_0^{\varpi}}{2\pi Q^2}(-Q_2,\, Q_1), \qquad \Phi_0^{\varpi} = \frac{2\pi\hbar\mu}{\mathfrak{q}}.$$

- No external solenoid required: AB phase *emerges* from punctured topology.



Role of the scalar potential

- Repulsive $V^{\varpi}(\mathbf{Q})$ regularises the origin; ensures essential self–adjointness.
- Physically acts like a "centrifugal core", preventing wave-function leakage into the singularity.

Extensions & open questions

- 3-D analogue: affine quantization on $\mathbb{R}^3_* \times \mathbb{R}^3$ (possible link to magnetic monopoles!).
- Experimental signatures of the scalar barrier in mesoscopic rings.
- Other topologically non-trivial manifolds (e.g. genus g surfaces).

Take-away messages

1. Affine symmetry of the punctured plane naturally yields AB-type vector potential.
2. No need to postulate external gauge field: topology (and $\hbar$!) do the job.
3. Scalar potential is not a bug but a feature; secures well-posed quantum dynamics.

Key references I

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Thank you!

PHYSICAL REVIEW

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Significance of Electromagnetic Potentials in the Quantum Theory

Y. AHARONOV AND D. BOHM
H. H. Wills Physics Laboratory, University of Bristol, Bristol, England
(Received May 28, 1959; revised manuscript received June 16, 1959)

In this paper, we discuss some interesting properties of the electromagnetic potentials in the quantum domain. We shall show that, contrary to the conclusions of classical mechanics, there exist effects of potentials on charged particles, even in the region where all the fields (and therefore the forces on the particles) vanish. We shall then discuss possible experiments to test these conclusions; and, finally, we shall suggest further possible developments in the interpretation of the potentials.

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Observation of Aharonov-Bohm Effect by Electron Holography

Akira Tonomura, Tsuyoshi Matsuda, Ryo Suzuki, Akira Fukuhara, Nobuyuki Osakabe, Hiroshi Umezaki, Junji Endo, Kohsei Shinagawa, Yutaka Sugita, and Hideo Fujiwara
Central Research Laboratory, Hitachi Ltd., Kokubunji, Tokyo 185, Japan
(Received 16 February 1982)

In this experiment, an electron- and optical-holographic technique is employed with small toroidal ferromagnets each forming a magnetic-flux closure. The holographic interferometry proves that a phase difference between two electron beams having passed through the field-free regions agrees well with the fundamental relation known as the Aharonov-Bohm effect. It is also confirmed from the same hologram that flux leakage from the toroids does not affect the conclusion.