Rigged Hilbert Spaces: A contribution to the memory of Arno Bohm.

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A rigged Hilbert space is a triplet of spaces:

RHS

$$\Phi \subset \mathcal{H} \subset \Phi^\times \,, \tag{1}$$

such that:

1.- $\mathcal{H}$ is an infinite dimensional separable Hilbert space.

2.- Φ is a dense subspace endowed with its own locally convex topology, finer than the Hilbert space topology. Thus, the canonical injection $i : \Phi \longmapsto \mathcal{H}$ is continuous. Some other properties such as reflexivity for Φ are often considered.

3.- $\Phi^\times$ is the space of *continuous antilinear functionals* on $\Phi^\times$ (as mappings from Φ into $\mathbb{C}$). It is often called the antidual of Φ.

RHS have been used for...

1.- Giving a rigorous meaning to the Dirac formulation of Quantum Mechanics.

2.- Giving a proper mathematical meaning to Gamow vectors, i.e., vector states for the exponentially decay part of a quantum resonance.

3.- Extending quantum mechanics in order to accommodate the irreversible character of certain quantum processes such as the decay processes.

4.- Providing an appropriate context for spectral decompositions of Koopman and Frobenius-Perron operators for chaotic systems in terms of the Pollicot-Ruelle resonances, which are singularities of the power spectrum.

5.- Extending the formalism of Statistical Mechanics in order to include on it generalized states and singular structures, using either the Rigged Liouville Space or some other algebraic structure.

6.- Defining some elements that appear in the axiomatic formalism of Quantum Fields, such as Wightman Functional, Borchers Algebra, generalized states, etc.

7.- Describing White Noise or other stochastic processes.

8.- Dealing with singular solutions of some partial differential equations.

9.- Dealing with physics problems that require the use of distributions.

10.- Serve as a unifying framework for discrete and continuous basis, basis of special functions and symmetry Lie algebras represented by continuous operators.

Let S be the Schwartz space. A function $f(x) : \mathbb{R} \longmapsto \mathbb{C}$ is in S if

1.- $f(x) \in C^\infty(\mathbb{R})$

2.-

$$\lim_{|x|\to\infty} x^n \frac{d^m}{dx^m} f(x) = 0 \,, \qquad \forall\, n,m = 0,1,2,\ldots\,. \tag{2}$$

The Schwartz space is a vector space over $\mathbb{C}$.

Any $f(x) \in S$ is in $L^2(\mathbb{R})$ so that $S \subset L^2(\mathbb{R})$

S is dense in $L^2(\mathbb{R})$ with respect to the Hilbert space topology.

Topology

The topology on S is given by the following countable set of norms:

For any $f(x) \in S$, we have

$$p_{nm}(f) := \sup_{x \in \mathbb{R}} \left| x^n \frac{d^m}{dx^m} f(x) \right| , \qquad n, m = 0, 1, 2, \ldots , \tag{3}$$

or equivalently:

$$q_{nm}(f) = \sqrt{\int_{-\infty}^{\infty} \left| x^n \frac{d^m}{dx^m} f(x) \right|^2 dx} , \qquad n, m = 0, 1, 2, \ldots , \tag{4}$$

The set of normalized Hermite functions, $\{H_n\}$, is orthonormal and complete in $L^2(\mathbb{R})$, so that if $f(x) \in S$,

$$f(x) = \sum_{n=0}^{\infty} a_n \, H_n(x) \,. \tag{5}$$

Then, we may define the following set of norms

$$\pi_p(f) := \sqrt{\sum_{n=0}^{\infty} |a_n|^2 \, (n+1)^{2p}} \,, \qquad p = 0, 1, 2, \dots \tag{6}$$

which gives the same topology.
Then, S is a reflexive Frèchet space with the strong topology for $S^\times$ and

$$S \subset L^2(\mathbb{R}) \subset S^\times \tag{7}$$

is a RHS.

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Let us assume a scattering process produced by some potential, V, that we assume, for simplicity, to be short range and spherically symmetric. Thus, the scattering is produced by a Hamiltonian pair $\{H_0, H = H_0 + V\}$.
A quasistationary state is produced when a free evolving quantum state is captured by the interaction region for a large time and, then, released to evolve freely again. A resonance describes this later process, independently of the capture.
Let us assume that at a time $t = 0$ the state, $\psi(0) \equiv \psi$, has been captured and start the decay. The survival probability after a time t is defined as

$$P(t) := |\langle\psi|e^{-itH}\psi\rangle|^2 \,. \tag{8}$$

The survival probability has three regimes depending on the values of time:

1.- For very short values of time, deviations of the exponential decay law have been predicted and observed. This regime is usually called the Zeno regime.

2.- For intermediate values of time (observable regime), the decay is approximately exponential and the state energy distribution is approximately a Breit-Wigner one (Cauchy distribution function):

$$\omega(E) = \frac{\gamma}{\pi} \frac{1}{(E - E_R)^2 + \gamma^2/4}\,, \qquad E_{min} \leq E < \infty \tag{9}$$

3.- Due to the semiboundedness of H, deviations of the exponential law for very large values of t are predicted. The decay is slower than exponential (Khalfin regime).

The intermediate regime is exponential, which can be easily observed in the lab. One may think if this exponential decay behaviour could be described by a vector (function) state.

There are physical characterizations and mathematical definitions of what a resonance should be. Although not always equivalent, let us give an example of each one, useful for our purposes:

1.- Presence of a bump in the cross section centred at the energy E_R and with width γ.

2.- Pairs of poles of the analytic continuation of the scattering function $S(E)$ through the positive semiaxis, located at the points $z_R = E_R - i\gamma/2$ and $z_R^* = E_R + i\gamma/2$.

Gamow vectors

Vector states, called the Gamow vectors, ψ for the exponentially decaying quantum state have been defined as

$$H\psi = (E_R - i\gamma/2)\psi\,, \qquad (10)$$

since then,

$$U(t)\psi = e^{-itH}\psi = e^{-itE_R}\, e^{-t\,\gamma/2}\psi\,, \qquad (11)$$

and, then, ψ decays exponentially with time.

Warning: This is mathematically ill defined.

Consequence: The vector state ψ cannot be in the Hilbert space on which H acts.

Cure: RHS

We can construct a pair of RHS, admitting a representation using Hardy functions on a half plane:

Two RHS

$$\Phi_- \subset \mathcal{H} \subset \Phi_-^\times \,, \tag{12}$$

$$\Phi_+ \subset \mathcal{H} \subset \Phi_+^\times \,. \tag{13}$$

with

Invariance under H

$$H\Phi_- \subset \Phi_- \,, \qquad H\Phi_+ \subset \Phi_+ \,. \tag{14}$$

With the duality formula

Duality

$$\langle H\varphi_\pm|F_\pm\rangle = \langle\varphi_\pm|HF_\pm\rangle\,, \qquad \forall\varphi_\pm\in\Phi_\pm\,, \quad \forall\,F_\pm\in\Phi_\pm^\times\,, \tag{15}$$

one may extend H into the (anti)-duals, so that the formulas

Gamow

$$H\psi_D = (E_R - i\gamma/2)\psi_D\,, \qquad H\psi_G = (E_R + i\gamma/2)\psi_D\,, \tag{16}$$

are well defined for $\psi_D\in\Phi_+^\times$ and $\psi_G\in\Phi_-^\times$, respectively.

Time evolution

In addition, we have:

Time evolution

$$1. - \text{For any } t > 0\,,\, e^{itH}\Phi_+ \subset \Phi_+ \implies e^{-itH}\Phi_+^\times \subset \Phi_+^\times\,.$$

$$\text{If } t < 0\,,\, e^{itH}\Phi_+ \not\subset \Phi_+\,. \tag{17}$$

$$2. - \text{For any } t < 0\,,\, e^{itH}\Phi_- \subset \Phi_- \implies e^{-itH}\Phi_-^\times \subset \Phi_-^\times\,.$$

$$\text{If } t > 0\,,\, e^{itH}\Phi_- \not\subset \Phi_-\,. \tag{18}$$

Time evolution for Gamows

In addition,

Decaying and Growing Gamow states

$$e^{-itH}\psi_D = e^{-itE_R}\, e^{-t\gamma/2}\psi_D\,, \qquad t>0\,, \tag{19}$$

$$e^{-itH}\psi_G = e^{-itE_R}\, e^{t\gamma/2}\psi_G\,, \qquad t<0\,. \tag{20}$$

Decaying and growing Gamow vectors describe the same situation only that they are time reversal of each other.

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