Confluent algorithm and spectral design in quantum mechanics

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Introduction

- The idea is to factorize the Hamiltonian

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + V(x)$$

in terms of two first-order differential operators

$$B^{\pm} = \frac{1}{\sqrt{2}}\left(\mp\frac{d}{dx} + \beta(x)\right)$$

in order to solve the eigevalue equation

$$H\psi(x) = E\psi(x)$$

with appropriate boundary conditions

Introducción

- Origin of the method (physicist): in 1935 Dirac factorized the harmonic oscillator
- In 1940 Schrödinger addressed the hydrogen atom through the factorization method
- Between 1941 and 1951 Infeld focussed in the method, ending up with his seminal paper with a classification of the potentials that can be solved through such a technique
- In 1984 Mielnik introduced the generalized factorization to generate new families of solvable potentials departing from the harmonic oscillator

Introducción

- In 1984 DF applied the generalized factorization to the hydrogen atom, Andrianov set up the connection between factorization and Darboux transformation, and Nieto realized the connection between factorization method and SUSY QM
- In 1985 Sukumar formulated the generalized factorization for a general solvable potential and an arbitrary factorization energy, as well as the corresponding iterations
- Departing from 1984 our group at Cinvestav has done important contributions to the subject

First-order intertwining

Suppose we have 2 Hamiltonians H, $\widetilde{H}$ and a first-order differential operator B^+ fulfilling

$$\widetilde{H} B^+ = B^+ H$$

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + V(x)$$

$$\widetilde{H} = -\frac{1}{2}\frac{d^2}{dx^2} + \widetilde{V}(x)$$

$$B^+ = \frac{1}{\sqrt{2}}\left(-\frac{d}{dx} + \beta(x)\right)$$

First-order intertwining

Thus, the following equations must be fulfilled

$$\beta' + \beta^2 = 2(V - \epsilon) \quad \Rightarrow \quad V = \frac{\beta'}{2} + \frac{\beta^2}{2} + \epsilon$$

$$-\beta' + \beta^2 = 2(\widetilde{V} - \epsilon) \quad \Rightarrow \quad \widetilde{V} = -\frac{\beta'}{2} + \frac{\beta^2}{2} + \epsilon$$

$$\Rightarrow \widetilde{V} = V - \beta'$$

Making the change $\beta = u'/u$ it is obtained

$$-\frac{u''}{2} + Vu = \epsilon u$$

The following factorizations as well arise

$$H = B^- B^+ + \epsilon$$

$$\widetilde{H} = B^+ B^- + \epsilon$$

First-order intertwining

The seed solution u must be nodeless for avoiding singularities in $\beta \Rightarrow \epsilon \leq E_0$

If H is a given solvable Hamiltonian with eigenfunctions ψ_n and eigenvalues E_n then $\widetilde{H}$ is a new solvable Hamiltonian with eigenfunctions $\widetilde{\psi}_n$ and eigenvalues $\widetilde{E}_n$ such that

$$\widetilde{\psi}_n = \frac{B^+\psi_n}{\sqrt{E_n - \epsilon}}$$

$$\widetilde{E}_n = E_n \quad \text{if} \quad \epsilon \neq E_0$$

$$\widetilde{E}_{n-1} = E_n \quad \text{if} \quad \epsilon = E_0$$

The following options for spectral manipulation arise

First-order intertwining

- Deleting the ground state: this spectral manipulation appears if $u = \psi_0,\ \epsilon = E_0$ for which $\widetilde{E}_{n-1} = E_n,\ n = 1, 2, \ldots$
- Creating a new ground state: this manipulation appears if u is a nodeless seed solution such that $\epsilon < E_0$. Thus, Sp($\widetilde{H}$) = Sp(H)$\cup\{\epsilon\}$
- However, we cannot create a level at $\epsilon > E_0$ by using the first-order intertwining since singularities appear in β

Second-order intertwining

Suppose now that

$$\widetilde{H} B^{+} = B^{+} H$$
$$B^{+} = \frac{1}{2}\left(\frac{d^2}{dx^2} - \eta(x)\frac{d}{dx} + \gamma(x)\right)$$

Thus

$$\gamma = \frac{\eta'}{2} + \frac{\eta^2}{2} - 2V + \epsilon_1 + \epsilon_2$$
$$V = \frac{\eta''}{4\eta} - \frac{\eta'^2}{8\eta^2} + \frac{\eta'}{2} + \frac{\eta^2}{8} + \frac{(\epsilon_1-\epsilon_2)^2}{2\eta^2} + \frac{(\epsilon_1+\epsilon_2)}{2}$$
$$\widetilde{V} = \frac{\eta''}{4\eta} - \frac{\eta'^2}{8\eta^2} - \frac{\eta'}{2} + \frac{\eta^2}{8} + \frac{(\epsilon_1-\epsilon_2)^2}{2\eta^2} + \frac{(\epsilon_1+\epsilon_2)}{2}$$
$$\Rightarrow \quad \widetilde{V} = V - \eta'$$

Second-order intertwining

If V is given we need to solve the second-order non-linear differential equation for η. Using the Ansatz

$$\eta' = -\eta^2 + 2\beta\eta + 2\xi$$

it turns out that $\xi = \pm(\epsilon_1 - \epsilon_2)$; thus

$$\eta' = -\eta^2 + 2\beta_1\eta + 2(\epsilon_1 - \epsilon_2)$$
$$\eta' = -\eta^2 + 2\beta_2\eta - 2(\epsilon_1 - \epsilon_2)$$

By subtracting these equations we will get

$$\eta = -\tfrac{2(\epsilon_1 - \epsilon_2)}{\beta_1 - \beta_2}$$

where

Second-order intertwining

$$\beta_i' + \beta_i^2 = 2(V - \epsilon_i), \quad i = 1, 2$$

In terms of the corresponding seed solutions u_i of the Schrödinger equation we will get

$$\eta = \frac{W'(u_1, u_2)}{W(u_1, u_2)}$$

There are two possible cases: $\epsilon_1 \neq \epsilon_2$ and $\epsilon_1 \to \epsilon_2$. Our interest in this talk is the last case, called confluent algorithm

Confluent second-order intertwining

For the confluent algorithm it is fulfilled

$$\eta' = -\eta^2 + 2\beta\eta$$

Solving this equation for η we will get

$$\eta = \frac{e^{2\int \beta(x)dx}}{w_0 + \int e^{2\int \beta(x)dx} dx}$$

In terms of the Schrödinger seed solution $u(x)$

$$\eta = \frac{u^2(x)}{w_0 + \int u^2(x)dx} = \frac{w'(x)}{w(x)}$$

Confluent second-order intertwining

where

$$w(x) = w_0 + \int_{x_0}^{x} u^2(y)dy$$

In order to avoid singularities in η we must choose $u(x)$ such that

$$\lim_{x\to\infty} u(x) = 0, \quad I_+ = \int_{x_0}^{\infty} u^2(y)dy < \infty \qquad \text{or}$$

$$\lim_{x\to-\infty} u(x) = 0, \quad I_- = \int_{-\infty}^{x_0} u^2(y)dy < \infty$$

Confluent second-order intertwining

In the first case the domain of w_0 where $w(x)$ is nodeless is $w_0 \leq -I_+$; in the second case it will be $w_0 \geq I_-$; if u is one bound state of H both conditions are fulfilled, thus in this case the w_0-domain will be the union of the two previous domains

Depending of the choice of u several spectral manipulation possibilities arise

- Isospectral: if $\epsilon = E_m$, $u(x) = \psi_m(x)$, and w_0 is inside the regularity domain $\Rightarrow$ Sp$(\widetilde{H}) =$ Sp(H)
- Deleting one level: if $\epsilon = E_m$, $u(x) = \psi_m(x)$, and w_0 is one of the borders of the regularity domain $\Rightarrow$ Sp$(\widetilde{H}) =$ Sp$(H)/E_m$

Confluent second-order intertwining

- Creating one level: if $\epsilon < E_0$, $u(x)$ is nodeless and vanishes either to the right or to the left, and w_0 is inside the corresponding regularity domain $\Rightarrow$ Sp$(\widetilde{H})$ = Sp$(H) \cup \epsilon$
- Isospectral: if $\epsilon < E_0$, $u(x)$ is nodeless and vanishes either to the right or to the left, and w_0 is one of the borders of the regularity domain $\Rightarrow$ Sp$(\widetilde{H})$ = Sp(H)

Confluent second-order intertwining

- Let us stress that $w(x)$ is proportional to $W(u, v)$ with u such that $(H - \epsilon)u = 0$ while v fulfills:

$$(H - \epsilon)v = u$$

- By deriving $(H - \epsilon)u = 0$ with respect to ϵ we can see that $\partial_\epsilon u$ is a particular solution $\Rightarrow$ $W(u, v) = W_0 + W(u, \partial_\epsilon u)$
- Thus, we can use either the integral formula or the differential one for this algorithm for calculating the new potential $\widetilde{V}$

Example: harmonic oscillator

The general solution to the SE for $V(x) = \frac{x^2}{2}$ with $a = (1-2\epsilon)/4$:

$$u(x) = e^{-\frac{x^2}{2}}[{}_1F_1(a, \tfrac{1}{2}; x^2) + Cx\,{}_1F_1(a+\tfrac{1}{2}, \tfrac{3}{2}; x^2)]$$

The eigenfunctions ψ_n associated to the eigenvalues $E_n = n + \frac{1}{2}$ are:

$$\psi_n(x) = \sqrt{\frac{1}{2^n n! \sqrt{\pi}}}\, e^{-\frac{x^2}{2}} H_n(x)$$

If we delete the ground state we must use the FOI and choose $\epsilon = E_0 = \frac{1}{2}$ and $u(x) = \psi_0(x) \propto e^{-\frac{x^2}{2}}$. Thus:

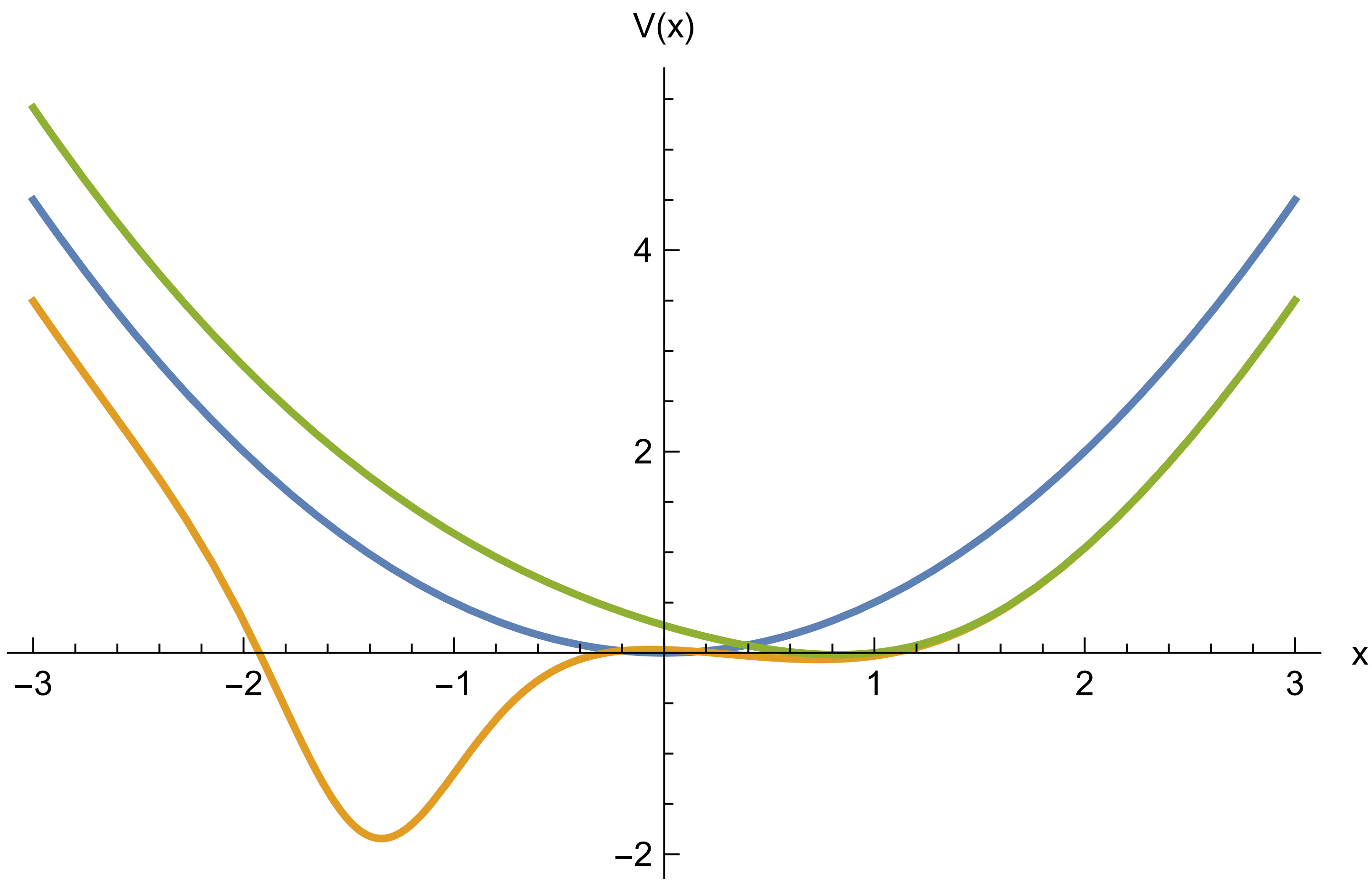
Example: harmonic oscillator

$$\widetilde{V}(x) = x^2 + 1$$

On the other hand, for $\epsilon = -\frac{1}{2}$ we recover the potentials derived by Mielnik in 1984:

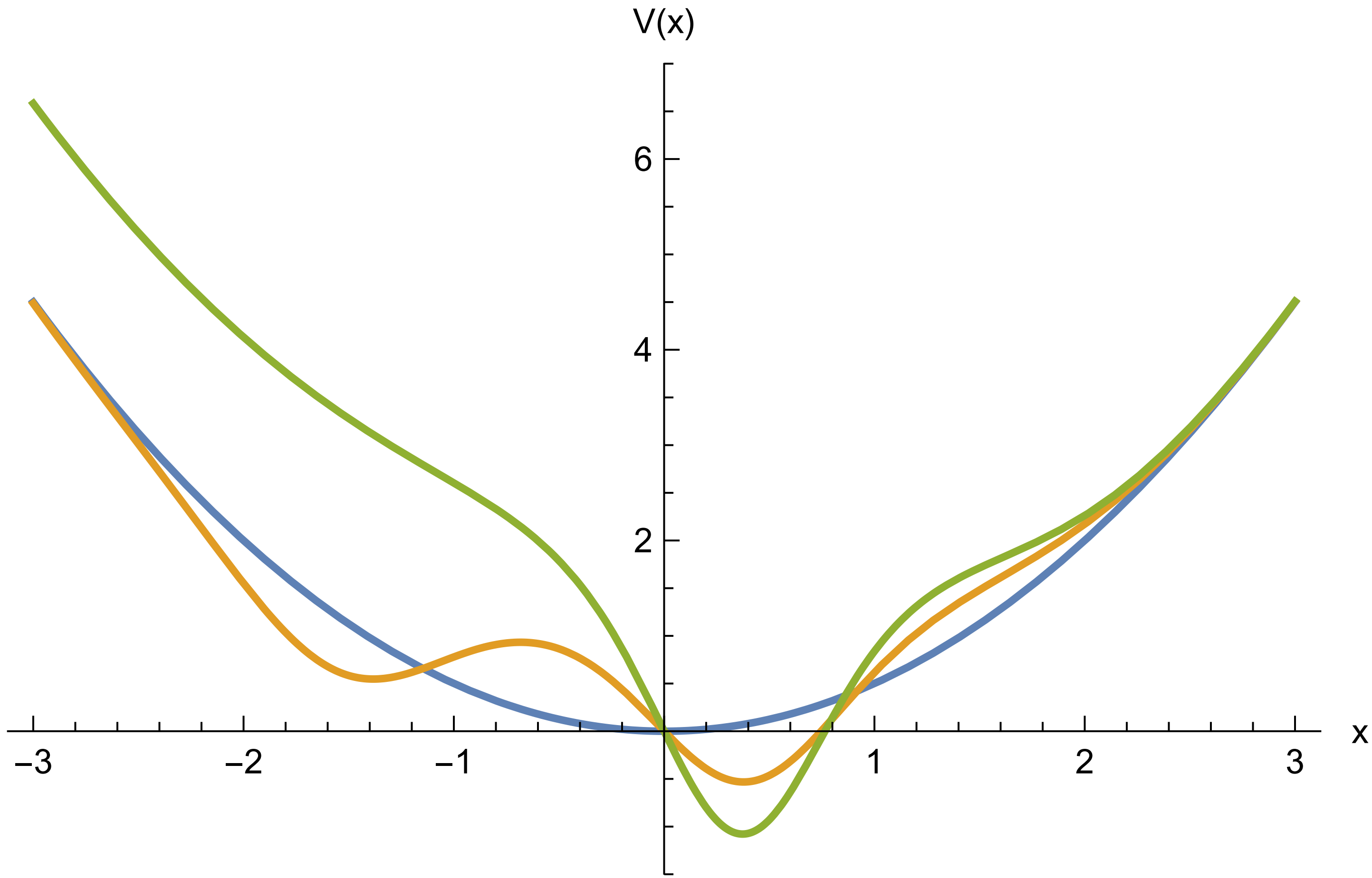
$$\widetilde{V}(x) = \frac{x^2}{2} - \left(\frac{2\nu e^{-\frac{x^2}{2}}}{\sqrt{\pi}[1 + \nu \mathrm{Erf}(x)]} \right)' - 1$$

Example: harmonic oscillator



First-order SUSY with $\epsilon = -\frac{1}{2}$

Example: harmonic oscillator



Confluent second-order SUSY with $\epsilon = \frac{3}{2}$

Conclusions

- We have shown that the confluent algorithm allows to create one new level at an arbitrary energy position on the real axis as compared with the initial spectrum
- It allows as well to delete one arbitrary energy level of the initial spectrum
- The confluent algorithm represents one of the simplest tools for manipulating spectra of one-dimensional quantum mechanical systems

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