Integrable systems from Poisson reductions of generalized Hamiltonian torus actions

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We deal with classical integrable systems adopting the following definition.

Let $(\mathcal{M}, P_{\mathcal{M}})$ be a finite dimensional, connected, C^∞ Poisson manifold, and $\mathfrak{H}$ an Abelian Poisson subalgebra of $C^\infty(\mathcal{M})$ subject to the conditions:

1. As a commutative algebra of functions $\mathfrak{H}$ has functional dimension $\mathrm{ddim}(\mathfrak{H}) = \ell$.
2. The Hamiltonian vector fields of the elements of $\mathfrak{H}$ are complete and span an ℓ dimensional subspace of the tangent space over a dense open subset of $\mathcal{M}$.
3. The commutant $\mathfrak{F}$ of $\mathfrak{H}$ in $C^\infty(\mathcal{M})$, which contains the joint constants of motion of the Hamiltonians $\mathcal{H} \in \mathfrak{H}$, has functional dimension $\mathrm{ddim}(\mathfrak{F}) = \mathrm{dim}(\mathcal{M}) - \ell$.

We refer to the quadruple $(\mathcal{M}, P_{\mathcal{M}}, \mathfrak{H}, \mathfrak{F})$, or simply $\mathfrak{H}$, as a *(degenerate) integrable system of rank* ℓ. The standard notion of Liouville integrability results if $\mathcal{M}$ is a symplectic manifold and $\ell = \mathrm{dim}(\mathcal{M})/2$. Liouville integrability on Poisson manifolds is the case for which $\ell = k$, where k is half the dimension of the maximal symplectic leaves. When $\ell < k$, both on Poisson and symplectic manifolds, then one obtains the case of degenerate integrability, alternatively called superintegrability. A single Hamiltonian is called (super)integrable if it is a member of $\mathfrak{H}$ obeying the definition.

The basic idea

Suppose that on $(\mathcal{M}, P_{\mathcal{M}})$ we have a proper and effective Hamiltonian action of a 'generalized torus'

$$L = \mathrm{U}(1)^{\ell_1} \times \mathbb{R}^{\ell_2}, \qquad \ell_1 + \ell_2 = \ell$$

with equivariant momentum map $\Phi : \mathcal{M} \to \mathbb{R}^\ell$. Then, the 'collective Hamiltonians'

$$\mathfrak{H}_\Phi := \{h \circ \Phi \mid h \in C^\infty(\mathbb{R}^\ell)\}$$

form an integrable system, with $\mathfrak{F} = C^\infty(\mathcal{M})^L$ (the L-invariant functions). This holds because $\mathrm{ddim}(C^\infty(\mathcal{M})^L) = \dim(\mathcal{M}) - \ell$. The system is superintegrable if ℓ is small relative to $\dim(\mathcal{M})$.

We wish to obtain interesting integrable systems by applying Hamiltonian reduction to integrable master systems with large symmetries on higher dimensional phase spaces. The master systems often come together with 'generalized action variables' that generate a 'torus' action on a dense open submanifold. Under certain conditions, the torus action descends to yield generalized action variables for the reduced systems (meaning that the Hamiltonians are members of $\mathfrak{H}_\Phi^{\mathrm{red}}$ after some restriction), and this implies their integrability. We developed sufficient conditions that ensure that this mechanism works, and applied it to several examples.

A simple example with $\ell = 1$ is when one starts with an unreduced Hamiltonian circle action generated by a 'periodic' Hamiltonian. Starting with an isotropic harmonic oscillator, Hamiltonian reduction explains the maximal superintegrability of rational Calogero models in confining external potential and their various spin extensions. This basically well-known result is elaborated with new applications in [L.F., arXiv:2409.19349].

Our notion of generalized action variables

Consider an integrable system of rank ℓ on a connected Poisson manifold $(\mathcal{M}, P_{\mathcal{M}})$ given by the Abelian Poisson algebra $\mathfrak{H}$. Suppose that we have ℓ smooth functions $H_1, \ldots, H_\ell$ on a connected dense open submanifold $\check{\mathcal{M}} \subset \mathcal{M}$ subject to the properties:
(i) The map $(H_1, \ldots, H_\ell) : \check{\mathcal{M}} \to \mathbb{R}^\ell$ is an equivariant momentum map for a proper and effective action of an ℓ-dimensional 'generalized torus' $\mathrm{U}(1)^{\ell_1} \times \mathbb{R}^{\ell_2}$ on $\check{\mathcal{M}}$.
(ii) The restriction of the elements of $\mathfrak{H}$ on $\check{\mathcal{M}}$ can be expressed in terms of $H_1, \ldots, H_\ell$ and the span of the exterior derivatives of the elements of $\mathfrak{H}$ coincides with the span of the exterior derivatives $dH_1, \ldots, dH_\ell$ at every point of $\check{\mathcal{M}}$.
Then, we say that the functions $H_1, \ldots, H_\ell$ are generalized action variables on $\check{\mathcal{M}}$ for the integrable system $\mathfrak{H}$.

Semi-locally, in a neighbourhood of any principal orbit of $\mathrm{U}(1)^{\ell_1} \times \mathbb{R}^{\ell_2}$, the generalized action variables are part of generalized action-angle coordinates. If $\mathcal{M}$ is symplectic with $\ell_2 = 0$, then this follows from a generalization of the classical Liouville–Arnold theorem due to `Nekhoroshev, 1972`]. In the Poisson case, with $\ell_2 = 0$, it follows from a similar result of `[Laurent-Gengoux, Miranda and Vanhaecke, arXiv:0805.1679]`.

Remark: One may allow disconnected $\check{\mathcal{M}}$ in the definition, and require the conditions separately on the connected components, with varying (ℓ_1, ℓ_2).

Reminder: Suppose that a Lie group $\mathbb{G}$ acts on a C^∞ manifold Y and denote the smooth action map $\mathbb{G} \times Y \to Y$ by juxtaposition $(g, y) \mapsto gy$. This is a *proper* action if for any sequence (g_n, y_n) $(n \in \mathbb{N})$ for which both y_n and $g_n y_n$ converge, there exists a convergent subsequence of the sequence g_n.

Our result about generalized action-angle coordinates

Theorem 1. Assume that $(\mathcal{M}, P_{\mathcal{M}}, \mathfrak{H})$ is an integrable system on a connected smooth Poisson manifold of dimension d that admits generalized action variables $H_1, \ldots, H_\ell$ on a connected dense open submanifold $\breve{\mathcal{M}}$. Let y_0 be a point of $\breve{\mathcal{M}}$ with trivial isotropy group for the generalized torus action, and put $p_i := H_i - H_i(y_0)$. Then, there exist a $\mathrm{U}(1)^{\ell_1} \times \mathbb{R}^{\ell_2}$-invariant open neighbourhood $\mathcal{U} \subset \mathcal{M}$ around y_0 and functions $\theta_1, \ldots, \theta_\ell, z_1, \ldots, z_{d-2\ell} : \mathcal{U} \to \mathbb{R}$ that possess the following properties:

(i) The functions $(e^{\mathrm{i}\theta_1}, \ldots, e^{\mathrm{i}\theta_{\ell_1}}, \theta_{\ell_1+1}, \ldots, \theta_\ell, p_1, \ldots, p_\ell, z_1, \ldots, z_{d-2\ell})$ define a diffeomorphism $\mathcal{U} \longrightarrow (\mathrm{U}(1)^{\ell_1} \times \mathbb{R}^{\ell_2}) \times C_\epsilon^{d-\ell}$ for some $\epsilon > 0$, with $C_\epsilon^{d-\ell}$ denoting a hypercube of dimension $d-\ell$, and y_0 corresponds to $(e, 0)$.

(ii) The Poisson structure can be written in terms of these coordinates as

$$P_{\mathcal{M}}\big|_{\mathcal{U}} = \sum_{i=1}^{\ell} \frac{\partial}{\partial \theta_i} \wedge \frac{\partial}{\partial p_i} + \sum_{\substack{a,b=1 \\ a<b}}^{d-2\ell} f_{ab}(z) \frac{\partial}{\partial z_a} \wedge \frac{\partial}{\partial z_b},$$

for some smooth functions f_{ab} depending only on $z_1, \ldots, z_{d-2\ell}$.

Moreover, $\mathcal{U}$ can be chosen in such a manner that the 'action coordinates' p_i and 'transversal coordinates' z_a can be expressed in terms of restrictions of elements of the Abelian Poisson algebra $\mathfrak{H}$ and its constants of motion $\mathfrak{F}$, respectively.

Plan of the rest of the talk

- **A general scenario and its consequences**
- A bird's-eye view of examples
- Lie algebraic preparation
- (Reductions of cotangent bundles and Heisenberg doubles)
- (Recall of quasi-Poisson basics)
- **Examples based on moduli spaces of flat connections**
- The torus with one hole
- The sphere with four holes
- (Further examples on character varieties – as time permits)
- Concluding remarks

Coming to the crux, now we formulate a scenario (a set of assumptions) leading to integrable systems by Poisson reduction. We have many realizations of this scenario.

Scenario. We have a manifold M equipped with a bivector P_M and a smooth action of a connected compact Lie group K fitting into one of the following three types:

- (M, P_M) is a Poisson manifold with a K-invariant Poisson bivector field.
- (M, P_M) is a Poisson manifold, K is a Poisson–Lie group and the action map $K \times M \to M$ is Poisson.
- (M, P_M, K) is a quasi-Poisson manifold.

The following four conditions hold:

(1) M is equipped with an Abelian Poisson subalgebra $\mathfrak{H} \subset C^\infty(M)^K$ and the vector fields $V_{\mathcal{H}}$, $\mathcal{H} \in \mathfrak{H}$ are all are complete.

(2) On a dense, open, connected K-invariant submanifold $Y \subset M$ we have ℓ smooth K-invariant functions $H_1, \ldots, H_\ell \in C^\infty(Y)^K$ that are independent at every point of Y, satisfy $\{H_i, H_j\} = 0$ and the vector fields V_{H_i} (defined with the aid of the bivector P_M) generate a proper, effective action of a Lie group $L = \mathrm{U}(1)^{\ell_1} \times \mathbb{R}^{\ell_2}$ $(\ell_1 + \ell_2 = \ell)$ on Y.

(3) The restriction of the elements of $\mathfrak{H}$ on Y can be expressed in terms of $H_1, \ldots, H_\ell$ and the span of the exterior derivatives of the elements of $\mathfrak{H}$ coincides with the span of the exterior derivatives $dH_1, \ldots, dH_\ell$ at every point of Y.

(4) Denote by $Y_0 \subset Y$ the principal isotropy type submanifold for the combined action of the direct product group $\mathbb{G} = K \times L$ on Y. Then, for any $y \in Y_0$ the isotropy group $\mathbb{G}_y$ is equal to $K_y \times \{e\}$.

The first consequences of the Scenario

Consider the algebra of first integrals $\mathfrak{F} := \{\mathcal{F} \in C^\infty(M) \mid \{\mathcal{F}, \mathcal{H}\} = 0,\ \forall \mathcal{H} \in \mathfrak{H}\}$, which is closed under the bracket defined by P_M.

Proposition 1. The Scenario implies the equality $\mathrm{ddim}(\mathfrak{H}) + \mathrm{ddim}(\mathfrak{F}) = \dim(M)$. Therefore, $(M, P_M, \mathfrak{H}, \mathfrak{F})$ defines an integrable system of rank(ℓ) *before reduction* (with the quoted definition applied to quasi-Poisson manifolds as well).

The functions $H_1, \ldots, H_\ell$ on $Y \subset M$ represent generalized action variables for the integrable system $(M, P_M, \mathfrak{H}, \mathfrak{F})$ (with definition extended to the quasi-Poisson case).

Proposition 2. Suppose that the Scenario holds and denote by Y_0^1 and Y_0 the principal isotropy type submanifolds of Y with respect to the K and $K \times L$-actions, respectively. Then, the natural identifications

$$C^\infty(Y_0^1/K) \simeq C^\infty(Y_0^1)^K \quad \text{and} \quad C^\infty(Y_0/K) \simeq C^\infty(Y_0)^K$$

induce Poisson structures on the smooth manifolds Y_0^1/K and Y_0/K. The L-action descends to a proper and effective Hamiltonian action of the group L on the connected Poisson manifold Y_0^1/K, which is generated by the momentum map

$$(\mathcal{H}_1, \ldots, \mathcal{H}_\ell) : Y_0^1/K \to \mathbb{R}^\ell$$

defined by the relation $\mathcal{H}_i \circ \pi_0^1 = H_i$ on Y_0^1, with the projection $\pi_0^1 : Y_0^1 \to Y_0^1/K$. This action restricts to a proper and free Hamiltonian action on the connected, dense open submanifold $Y_0/K \subset Y_0^1/K$.

Theorem 2 (main theorem). Suppose that the Scenario holds and let $M_* \subset M$ be the principal isotropy type submanifold with respect to the K-action. Then, the Abelian Poisson algebra $\mathfrak{H}$ descends to an integrable system of rank ℓ on the Poisson manifold M_*/K. The restrictions of the system to the Poisson manifolds Y_0/K and Y_0^1/K possess *action variables* given by $(\mathcal{H}_1, \ldots, \mathcal{H}_\ell)$, and the corresponding Hamiltonian L-action ($L = \mathrm{U}(1)^{\ell_1} \times \mathbb{R}^{\ell_2}$) is free on Y_0/K. As a result, $\mathfrak{H}$ induces integrable systems of rank ℓ with action variables defined by the restrictions of $(\mathcal{H}_1, \ldots, \mathcal{H}_\ell)$

- on every sympectic leaf of Y_0/K;
- and on every such symplectic leaf of Y_0^1/K that intersects Y_0/K.

In most examples, the systems described by Theorem 2 are *superintegrable* since ℓ is small in comparison to the dimensions of the manifolds involved in our statements.

Recall that $Y_0/K \subset Y_0^1/K \subset M_*/K$ are dense open submanifolds. Also, $M_*/K \subset M/K$ is a dense open subset, but M/K is not a smooth manifold in general. Generically, all these subsets are (expected to be) proper subsets, and it is largely unexplored what happens on the complements of the proper subsets.

When (M, P_M) is a Poisson manifold, then the compactness of K was used only to ensure that the action of $K \times L$ is proper if the L-action is proper.

From the next slide on, K is a connected and simply connected compact Lie group with simple Lie algebra $\mathfrak{k}$.

What kind of master systems do we reduce?

Let K be a (connected and simply connected) compact Lie group with simple Lie algebra $\mathfrak{k}$. Denote $\mathfrak{k}^{\mathbb{C}}$ and $K^{\mathbb{C}}$ the complexifications, and define $\mathfrak{P} := \exp(i\mathfrak{k}) \subset K^{\mathbb{C}}$.
Example: $K = \mathrm{SU}(n)$, $K^{\mathbb{C}} = \mathrm{SL}(n, \mathbb{C})$, $\mathfrak{P} = \{X \in \mathrm{SL}(n, \mathbb{C}) \mid X^\dagger = X,\ X \text{ positive}\}$.
I. One has 3 'classical doubles' of K:

Cotangent bundle $\quad T^*K \simeq K \times \mathfrak{k}^* \simeq K \times \mathfrak{k}$

Heisenberg double $\quad K^{\mathbb{C}} \simeq K \times K^* \simeq K \times \mathfrak{P}$

Internally fused quasi-Poisson double $\quad \mathbb{D}(K) = K \times K$

The pull-backs of the relevant rings of invariants $C^\infty(K)^K$, $C^\infty(\mathfrak{k})^K$, $C^\infty(\mathfrak{P})^K$ give rise to two 'master integrable systems' on each double. Reductions were analysed before (see e.g. [LF, arXiv:2208.03728]), but open problems remained.

II. Let $\Sigma_{m,n}$ be a surface of genus m with n holes. It is known from [Arthamonov and Reshetikhin, arXiv:1909.08682] that the moduli space of flat principal K-connections, identified as the character variety $\mathrm{Hom}(\pi_1(\Sigma_{m,n}), K)/K$, supports a plethora of superintegrable systems. The commuting Hamiltonians of an integrable system arise from the K-invariant functions of the holonomies along a collection of (homotopy classes of) pairwise non-intersecting simple closed curves on $\Sigma_{m,n}$.
We can describe many of these systems, and get action variables for them, by applying reduction to master systems living on the quasi-Poisson manifolds

$$M_{m,n} := \mathbb{D}(K) \circledast \cdots \circledast \mathbb{D}(K) \circledast K \circledast \cdots \circledast K,$$

obtained from m copies of $\mathbb{D}(K)$ and n copies of K by successive fusion.

Lie algebraic preparation

Let $\mathbb{T} < K$ be a maximal torus, with Lie algebra $\mathfrak{t} < \mathfrak{k}$. Let us realize $\mathfrak{k}$ as

$$\mathfrak{k} = \mathrm{span}_{\mathbb{R}}\{\mathrm{i}h_{\alpha_j},\ (e_\alpha - e_{-\alpha}),\ \mathrm{i}(e_\alpha + e_{-\alpha}) \mid \alpha_j \in \Delta,\ \alpha \in \mathfrak{R}^+\},$$

using a Weyl–Chevalley basis of $\mathfrak{k}^{\mathbb{C}}$:

$$e_\alpha,\ e_{-\alpha},\ h_{\alpha_j} \quad \text{with} \quad \alpha \in \mathfrak{R}^+,\ j = 1, \dots, \ell,$$

where $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ is a base of the root system $\mathfrak{R}$ of $\mathfrak{k}^{\mathbb{C}}$ with respect to $\mathfrak{t}^{\mathbb{C}}$. Define the open Weyl chamber $\mathcal{C} \subset \mathrm{i}\mathfrak{t}$ and the open Weyl alcove $\mathcal{A} \subset \mathcal{C} \subset \mathrm{i}\mathfrak{t}$ as follows:

$$\mathcal{C} := \{X \in \mathrm{i}\mathfrak{t} \mid 0 < \alpha_j(X),\ \forall j = 1, \dots, \ell\},$$

$$\mathcal{A} := \{X \in \mathrm{i}\mathfrak{t} \mid 0 < \alpha_j(X),\ \forall j = 1, \dots, \ell, \text{ and } \vartheta(X) < 2\pi\},$$

where ϑ is the highest root with respect to the base Δ. Then, introduce the *smooth* mappings $\underline{\varphi} : \mathfrak{k}^{\mathrm{reg}} \to \mathcal{C}$ and $\underline{\chi} : K^{\mathrm{reg}} \to \mathcal{A}$ by the following recipes:

$$\underline{\varphi}(J) = \xi \quad \text{if} \quad \mathrm{i}\xi = \mathrm{Ad}_{\Gamma_1(J)}(J) \equiv \Gamma_1(J) J \Gamma_1(J)^{-1} \quad \text{for some} \quad \Gamma_1(J) \in K,$$

$$\underline{\chi}(g) = \xi \quad \text{if} \quad e^{\mathrm{i}\xi} = \Gamma_2(g) g \Gamma_2(g)^{-1} \quad \text{for some} \quad \Gamma_2(g) \in K.$$

The formulae

$$\varphi_j := \langle h_{\alpha_j}, \underline{\varphi} \rangle \quad \text{and} \quad \chi_j := \langle h_{\alpha_j}, \underline{\chi} \rangle, \qquad \forall j = 1, \dots, \ell,$$

define real-analytic, K-invariant real functions φ_j on $\mathfrak{k}^{\mathrm{reg}}$ and χ_j on K^{reg}, respectively. They can be extended to globally continuous functions, but not to smooth functions.

Reductions of cotangent bundles (the simplest examples)

We consider the cotangent bundle with its canonical Poisson (symplectic) structure

$$M := T^*K = K \times \mathfrak{k} = \{(g, J) \mid g \in K,\, J \in \mathfrak{k}\}.$$

The group K acts on M by the mapping

$$K \times M \ni (\eta, (g, J)) \mapsto (\eta g \eta^{-1}, \eta J \eta^{-1}) \in M.$$

This is a Hamiltonian action generated by the momentum map $\Phi(g, J) = J - g^{-1} J g$.

Let $p_{\mathfrak{k}} : M \to \mathfrak{k}$ and $p_K : M \to K$ denote the natural projections, and consider the K-invariant functions

$$\varphi \circ p_{\mathfrak{k}},\ \forall \varphi \in C^\infty(\mathfrak{k})^K \quad \text{and} \quad \chi \circ p_K,\ \forall \chi \in C^\infty(K)^K.$$

For an arbitrary initial value $(g_0, J_0) \in M$, the integral curve of the evolution equation generated by the Hamiltonian $\varphi \circ p_{\mathfrak{k}}$ is

$$(g(\tau), J(\tau)) = (\exp(\tau d\varphi(J_0))\, g_0, J_0), \quad \forall \tau \in \mathbb{R},$$

and for the Hamiltonian $\chi \circ p_K$ it is

$$(g(\tau), J(\tau)) = (g_0, J_0 - \tau \nabla \chi(g_0)), \quad \forall \tau \in \mathbb{R}.$$

Thus, M carries the following two Abelian Poisson subalgebras of $C^\infty(M)^K$ having complete flows,

$$\mathfrak{H} := p_{\mathfrak{k}}^*(C^\infty(\mathfrak{k})^K) \quad \text{and} \quad \tilde{\mathfrak{H}} := p_K^*(C^\infty(K)^K),$$

and these represent two degenerate integrable systems on M.

Action variables for the master systems

Consider the K-invariant dense open, connected submanifolds of M,

$$Y := K \times \mathfrak{k}^{\text{reg}} \quad \text{and} \quad \tilde{Y} := K^{\text{reg}} \times \mathfrak{k}.$$

Using $H_j := \varphi_j \circ p_{\mathfrak{k}}$ and $\tilde{H}_j := \chi_j \circ p_K$, Define the K-invariant mappings

$$(H_1, \ldots, H_\ell) : Y \to \mathbb{R}^\ell \quad \text{and} \quad (\tilde{H}_1, \ldots, \tilde{H}_\ell) : \tilde{Y} \to \mathbb{R}^\ell.$$

Introduce also the *diffeomorphisms* $\mathcal{T} : \mathbb{R}^\ell \to \mathfrak{t}$ and $T : \mathbb{R}^\ell/(2\pi\mathbb{Z})^\ell \to \mathbb{T}$:

$$\mathcal{T}(\underline{\tau}) := -\mathrm{i} \sum_{j=1}^{\ell} \tau_j h_{\alpha_j} \quad \text{and} \quad T(\underline{\tau}) := \exp\Big(-\mathrm{i} \sum_{j=1}^{\ell} \tau_j h_{\alpha_j}\Big), \quad \forall \underline{\tau} = (\tau_1, \ldots, \tau_\ell) \in \mathbb{R}^\ell.$$

Lemma 1. The map $(H_1, \ldots, H_\ell)$ is the momentum map for the free Hamiltonian action of the torus $\mathbb{T}$ on Y that works according to the formula

$$(T(\underline{\tau}), (g, J)) \mapsto (\Gamma_1(J)^{-1} T(\underline{\tau}) \Gamma_1(J) g, J), \quad \forall \underline{\tau} \in \mathbb{R}^\ell,\ (g, J) \in Y.$$

The map $(\tilde{H}_1, \ldots, \tilde{H}_\ell)$ serves as the momentum map generating the free and proper Hamiltonian action of $\mathbb{R}^\ell$ on $\tilde{Y}$ that operates as

$$(\underline{\tau}, (g, J)) \mapsto (g, J - \Gamma_2(g)^{-1} \mathcal{T}(\underline{\tau}) \Gamma_2(g)), \quad \forall \underline{\tau} \in \mathbb{R}^\ell,\ (g, J) \in \tilde{Y}.$$

These $\mathbb{T}$- and $\mathbb{R}^\ell$-actions commute with the K-actions restricted on Y and on $\tilde{Y}$.

Over Y and $\tilde{Y}$, respectively, the elements of $\mathfrak{H}$ and $\tilde{\mathfrak{H}}$ can be expressed as functions of the above momentum maps, which represent *generalized action variables*.

Results concerning the $M = T^*K$ example

Using the K-action, we apply Hamiltonian reduction to the Abelian Poisson algebras of the globally smooth Hamiltonians, $\mathfrak{H}$ and $\tilde{\mathfrak{H}}$, as well as to their action variables.

Let $M_* \subset M$ and $Y_0^1 \subset Y$ be the principal orbit type submanifolds for the K-action, and $Y_0 \subset Y$ the principal isotropy type submanifold for the action of $K \times \mathbb{T}$. The principal isotropy groups are $\mathcal{Z}(K)$ and $\mathcal{Z}(K) \times \{e\}$. We have $Y_0 \subset Y_0^1 \subset Y \subset M$ and $Y_0^1 \subset M_*$.

Proposition 3. Theorem 2 is applicable to the Abelian Poisson algebra $\mathfrak{H}$ on $M = T^*K$ with the various data of the Scenario described above. Thus, $\mathfrak{H}$ engenders integrable systems of rank ℓ (on M_*/K as well as on symplectic leaves of Y_0^1/K and of Y_0/K) whose action variables arise from $H_1, \dots, H_\ell$. Moreover, the analogous statements hold if we replace $(\mathfrak{H}, Y, H_1, \dots, H_\ell, K \times \mathbb{T})$ by $(\tilde{\mathfrak{H}}, \tilde{Y}, \tilde{H}_1, \dots, \tilde{H}_\ell, K \times \mathbb{R}^\ell)$.

Except for $\ell = 1$ and a few very small symplectic leaves, all these reduced systems are superintegrable.

The physical interpretation of the reduced systems as trigonometric spin Sutherland systems and rational spin Ruijsenaars type systems is well-known. Integrability on **generic symplectic leaves** of the reduced phase space M/K has been shown previously by Reshetikhin [`arXiv:math/0202245`] studying the algebras of constants of motion, without using action variables.

Examples based on the Heisenberg double $M = K^{\mathbb{C}}$

Let us decompose the real Lie algebra $\mathfrak{k}^{\mathbb{C}}$ as the vector space direct sum $\mathfrak{k}^{\mathbb{C}} = \mathfrak{k} + \mathfrak{b}$ with the 'Borel' subalgebra

$$\mathfrak{b} = \text{span}_{\mathbb{R}}\{h_{\alpha_j},\, e_\alpha,\, \mathrm{i}e_\alpha \mid \alpha_j \in \Delta,\, \alpha \in \mathfrak{R}^+\}.$$

A well-known non-degenerate Poisson bracket on $C^\infty(M)$ is given by

$$\{F, H\} := \langle DF, \varrho DH\rangle_{\mathbb{I}} + \langle D'F, \varrho D'H\rangle_{\mathbb{I}} \quad \text{with} \quad \varrho := \frac{1}{2}(\pi_{\mathfrak{k}} - \pi_{\mathfrak{b}}),$$

where $\pi_{\mathfrak{k}}$ and $\pi_{\mathfrak{b}}$ are the projections from $\mathfrak{k}^{\mathbb{C}}$ onto the subalgebras $\mathfrak{k}$ and $\mathfrak{b}$. Here, $\langle -,-\rangle_{\mathbb{I}} := \Im\langle -,-\rangle$ is the imaginary part of the complex Killing form and the $\mathfrak{k}^{\mathbb{C}}$-valued derivatives DF and $D'F$ of any real function $F \in C^\infty(M)$ are defined by

$$\langle Z_1, DF(X)\rangle_{\mathbb{I}} + \langle Z_2, D'F(X)\rangle_{\mathbb{I}} := \left.\frac{d}{dt}\right|_{t=0} F(e^{tZ_1} X e^{tZ_2}), \quad \forall X \in M,\, Z_1, Z_2 \in \mathfrak{k}^{\mathbb{C}}.$$

Every element $X \in K^{\mathbb{C}}$ admits the unique Iwasawa decompositions

$$X = g_L b_R^{-1} = b_L g_R^{-1} \quad \text{with} \quad g_L, g_R \in K,\, b_L, b_R \in B,$$

which define the smooth maps $\Xi_L, \Xi_R : M \to K$ and $\Lambda_L, \Lambda_R : M \to B$ by

$$\Xi_L(X) := g_L, \quad \Xi_R(X) := g_R, \quad \Lambda_L(X) := b_L, \quad \Lambda_R(X) := b_R.$$

Then, $\Lambda := \Lambda_L \Lambda_R$ is a group valued Poisson–Lie momentum map, and it generates an action of K on M. The pertinent action map reads

$$K \times M \ni (\eta, X) \mapsto \eta X \Xi_R(\eta \Lambda_L(X)) \in M, \qquad \forall (\eta, X) \in K \times M,$$

and is a Poisson map w.r.t. the standard Poisson–Lie group structure on K. This is the 'quasi-adjoint action' of K that first appeared in [`Klimčík, arXiv:math-ph/0602048`]. Note that in this example, and actually in all of our examples, the principal isotropy group for the K-action on M is the center $\mathcal{Z}(K) < K$.

To continue, we need to introduce the closed submanifold $\mathfrak{P} := \exp(\mathrm{i}\mathfrak{k}) \subset K^{\mathbb{C}}$, and the diffeomorphism $\mathcal{P} : B \to \mathfrak{P}$ for which $\mathcal{P}(b) := bb^\dagger$.

The group K acts on $\mathfrak{P}$ by conjugations. Let $\mathfrak{P}^{\mathrm{reg}}$ denote the dense open submanifold of principal orbit type, which is diffeomorphic to $\mathfrak{k}^{\mathrm{reg}}$ by the exponential map. On $\mathfrak{P}^{\mathrm{reg}}$ we define K-invariant smooth functions ϕ_j $(j = 1, \ldots, \ell)$ by

$$\phi_j(P) := \frac{1}{2}\varphi_j(\mathrm{i}\log(P)) \quad \text{using } \varphi_j \in C^\infty(\mathfrak{k}^{\mathrm{reg}})^K \text{ introduced before.}$$

Now, we obtain two Abelian subalgebras of $C^\infty(M)^K$ as follows:

$$\mathfrak{H} = (\mathcal{P} \circ \Lambda_R)^*(C^\infty(\mathfrak{P})^K) \quad \text{and} \quad \tilde{\mathfrak{H}} = \Xi_R^*(C^\infty(K)^K).$$

Then, we introduce two K-invariant submanifolds of M,

$$Y := (\mathcal{P} \circ \Lambda_R)^{-1}(\mathfrak{P}^{\mathrm{reg}}) \quad \text{and} \quad \tilde{Y} := \Xi_R^{-1}(K^{\mathrm{reg}}),$$

and define K-invariant maps $(H_1, \ldots, H_\ell) : Y \to \mathbb{R}^\ell$ and $(\tilde{H}_1, \ldots, \tilde{H}_\ell) : \tilde{Y} \to \mathbb{R}^\ell$ by

$$H_j := \phi_j \circ \mathcal{P} \circ \Lambda_R \text{ and } \tilde{H}_j := \chi_j \circ \Xi_R \text{ with } \chi_j \in C^\infty(K^{\mathrm{reg}})^K \text{ used before.}$$

Proposition 4. The map $(H_1, \ldots, H_\ell)$ is the momentum map for a free Hamiltonian action of the maximal torus $\mathbb{T} < K$ on Y and the map $(\tilde{H}_1, \ldots, \tilde{H}_\ell)$ is the momentum map for a free and proper Hamiltonian action of the Abelian group $\mathbb{R}^\ell$ on $\tilde{Y}$. The explicit formulas of these actions can be used to verify that all assumptions of the Scenario are satisfied by $(K^{\mathbb{C}}, Y, \mathfrak{H}, H_1, \ldots, H_\ell)$ as well as by $(K^{\mathbb{C}}, \tilde{\mathfrak{H}}, \tilde{Y}, \tilde{H}_1, \ldots, \tilde{H}_\ell)$. Consequently, the general Theorem 2 is applicable to both of these cases, in full analogy to the cotangent bundle T^*K. (This completes previous results of [L.F. arXiv:1809.01529, arXiv:2208.03728, arXiv:2402.02990].)

Quasi-Poisson basics [Alekseev et al, arXiv:math/0006168]

A quasi-Poisson K-manifold with momentum map Φ is given by (M, A, P_M, Φ), where A is the action map $A: K \times M \to M$, P_M is a K-invariant bivector field and $\Phi: M \to K$ is an equivariant map such that:
The Schouten bracket $[P_M, P_M]$ reads

$$[P_M, P_M] = -\frac{1}{12}\langle [E_a, [E_b, E_c]\rangle E^a_M \wedge E^b_M \wedge E^c_M.$$

Here, $\{E_a\}$ and E^b are dual bases of $\mathfrak{k}$, $\langle E_a, E^b\rangle = \delta^b_a$, and E^a_M is the vector field on M induced by $\exp(tE^a) \in K$.

The momentum map and the quasi-Poisson bracket are related by

$$\{f, F \circ \Phi\} = \frac{1}{2} E^a_M[f](E^L_a + E^R_a)[F] \circ \Phi, \qquad \forall f \in C^\infty(M),\ F \in C^\infty(G),$$

where $E^L_a[F]$ and $E^a_M[f]$ denote the respective derivatives of the functions F and f. Note that the 'quasi-Poisson bracket' and the 'quasi-Hamiltonian vector' field V_h are defined by the formula $\{f, h\} := P_M(df, dh) =: V_h[f]$ for all $f, h \in C^\infty(M)$.

Then, $C^\infty(M)^K$ is a Poisson algebra. The momentum map is constant along the integral curves of V_h for any invariant function $h \in C^\infty(M)^K$, and these vector fields can be projected onto the quotient Poisson space M/K.

The invariant functions of the form $\chi \circ \Phi$ with $\chi \in C^\infty(K)^K$ are in the center of the Poisson algebra $C^\infty(M)^K$, and $\Phi^{-1}(\mathcal{O})/K$ is a (stratified) Poisson subspace of M/K for any conjugacy class $\mathcal{O} \subset K$ for which $\Phi^{-1}(\mathcal{O})$ is not empty.

If $(M_i, A_i, P_{M_i}, \Phi_i)$ for $i = 1, 2$ are quasi-Poisson K-manifolds, then the direct product $M = M_1 \times M_2$ becomes such a manifold by using the diagonal K-action, generated by the vector fields

$$E^a_M(x_1, x_2) = E^a_{M_1}(x_1) + E^a_{M_2}(x_2) \quad \text{using} \quad T_{(x_1,x_2)}M = T_{x_1}M_1 + T_{x_2}M_2,\ \forall (x_1, x_2) \in M,$$

and the 'fused bivector'

$$P_M = P_{M_1} + P_{M_2} + P_{\text{fus}} \quad \text{with} \quad P_{\text{fus}} := -\frac{1}{2} E^a_{M_1} \wedge E^{M_2}_a.$$

The momentum map Φ on M is the product $\Phi(x_1, x_2) = \Phi_1(x_1)\Phi_2(x_2)$. One denotes $M_1 \times M_2$ with the fused quasi-Poisson structure as $M_1 \circledast M_2$.

The simplest example is (K, P_K) with the conjugation action of K, $P_K = \frac{1}{2} E^R_a \wedge E^a_L$, and identity map as the momentum map.

Another important example is the so-called internally fused double of K:

$$\mathbb{D}(K) = K \times K = \{(A, B)\}$$

with the diagonal K-action, momentum map $\Phi_{\mathbb{D}(K)}(A, B) = ABA^{-1}B^{-1} =: [A, B]$ and

$$P_{\mathbb{D}(K)} = \frac{1}{2}\big(E^{(1),R}_a \wedge E^a_{(1),L} - E^{(2),R}_a \wedge E^a_{(2),L} + E^{(1),L}_a \wedge (E^a_{(2),L} + E^a_{(2),R}) + E^{(1),R}_a \wedge (E^a_{(2),L} - E^a_{(2),R})\big),$$

where $E^{(1),L}_a$ and $E^a_{(2),R}$ are left- and right-invariant vector fields on the two factors.

A few consequences of the definitions

(1) The commutant C_H of $H \in C^\infty(M)^K$ is closed under the quasi-Poisson bracket.

(2) For $\chi \in C^\infty(K)^K$, the integral curve of the Hamiltonian $\chi \circ \Phi$ with initial value $X_0 \in M$ is given by

$$A(\exp(t\nabla\chi(\Phi(X_0))), X_0),$$

where $A : K \times M \to M$ is the action map. The function $\chi \circ \Phi$ is in the center of the Poisson algebra $C^\infty(M)^K$; its flow becomes trivial after projection on M/K.

(3) Consider the fusion product $M = M_1 \circledast M_2$ and an invariant function of the form $H(X_1, X_2) = h(X_1)$ for $h \in C^\infty(M_1)^K$. Then, the second component X_2 is constant along the integral curves of H, and the first component follows the integral curves of h in the constituent quasi-Poisson manifold M_1. A similar result holds for functions depending on X_2 as well as for multiple fusion products.

(4) Observe from the last property that the functions of the form $\chi \circ \Phi_1$ and $\chi \circ \Phi_2$, $\chi \in C^\infty(K)^K$ generate non-trivial dynamics on $M_1 \circledast M_2$, for which the explicit formula follows from (2) and (3).

(5) The fusion of quasi-Poisson spaces enjoys the associativity property. It is commutative up to a certain non-trivial diffeomorphism between $M_1 \times M_2$ and $M_2 \times M_1$.

Let $\Sigma_{m,n}$ be a surface of genus m with n holes (boundary components). It is well-known that the character variety, $(\mathrm{Hom}(\pi_1(\Sigma_{m,n}), K))/K$, is a stratified Poisson space. There exist many interesting description of its Poisson structure. We prefer the one developed in [Alekseev, Malkin and Meinrenken, arXiv:dg-ga/9707021] and [Alekseev, Kosmann-Schwarzbach and Meinrenken, arXiv:math/0006168], which states that

$$\mathrm{Hom}(\pi_1(\Sigma_{m,n}), K)/K \equiv M_{m,n}//K := \Phi_{m,n}^{-1}(e)/K,$$

where $M_{m,n} := \mathbb{D}(K) \circledast \cdots \circledast \mathbb{D}(K) \circledast K \circledast \cdots \circledast K$ obtained by the 'fusion procedure'.

As a manifold, $M_{m,n} = K^{\times(2m+n)} = \{X \mid X = (A_1, B_1, \dots, A_m, B_m, C_1, \dots, C_n)\}$ and $\Phi_{m,n}(X) = [A_1, B_1] \cdots [A_m, B_m] C_1 \cdots C_n$ (where $[A, B] := ABA^{-1}B^{-1}$). We assume that $n \geq 1$, and eliminate C_n by the condition $\Phi_{m,n}(X) = e$. This leads to the identification

$$C^\infty(\mathrm{Hom}(\pi_1(\Sigma_{m,n}), K))^K = C^\infty(M_{m,n-1})^K =: C^\infty(M_{m,n-1}/K).$$

We obtain (super)integrable systems from quasi-Hamiltonian systems on $M_{m,n-1}$.

The first building block is (K, P_K) with the conjugation action of K and $P_K = \frac{1}{2} E_a^R \wedge E_L^a$. The second is the double $\mathbb{D}(K) = K \times K = \{(A, B)\}$ with the diagonal K-action and $P_{\mathbb{D}(K)} = \frac{1}{2}\big(E_a^{(1),R} \wedge E^a_{(1),L} - E_a^{(2),R} \wedge E^a_{(2),L} + E_a^{(1),L} \wedge (E^a_{(2),L} + E^a_{(2),R}) + E_a^{(1),R} \wedge (E^a_{(2),L} - E^a_{(2),R})\big)$, where $E_a^{(1),L}$ and $E^a_{(2),R}$ are left- and right-invariant vector fields on the factors.

The case of the torus with one hole

We start with the internally fused double $M := \mathbb{D}(K)$ of the connected and simply connected compact Lie group K. As a manifold $M = K \times K = \{(A, B)\}$. We introduce the Abelian Poisson subalgebras $\mathfrak{H}$ and $\tilde{\mathfrak{H}}$ of the Poisson algebra $C^\infty(M)^K$ by

$$\mathfrak{H} := p_1^*(C^\infty(K)^K) \quad \text{and} \quad \tilde{\mathfrak{H}} := p_2^*(C^\infty(K)^K),$$

where p_1 and p_2 denote the projections from M onto the first and second K factors.

For any $\chi \in C^\infty(K)^K$, the integral curves of the quasi-Hamiltonian vector fields $V_{p_1^*\chi}$ and $V_{p_2^*\chi}$ are, respectively,

$$(A(\tau), B(\tau)) = (A_0, B_0 \exp(-\tau \nabla \chi(A_0))) \text{ and } (A(\tau), B(\tau)) = (A_0 \exp(\tau \nabla \chi(B_0)), B_0).$$

In this case, we define

$$Y := p_1^{-1}(K^{\text{reg}}) \quad \text{and} \quad \tilde{Y} := p_2^{-1}(K^{\text{reg}}),$$

and the K-invariant maps

$$(H_1, \ldots, H_\ell) : Y \to \mathbb{R}^\ell \quad \text{and} \quad (\tilde{H}_1, \ldots, \tilde{H}_\ell) : \tilde{Y} \to \mathbb{R}^\ell$$

with $H_j := \chi_j \circ p_1$ and $\tilde{H}_j := \chi_j \circ p_2$ using $\chi_j \in C^\infty(K^{\text{reg}})^K$ introduced before.

The joint flows of $(H_1, \ldots, H_\ell)$ generate a free action of the maximal torus $\mathbb{T} < K$ on Y, and the flows of $(\tilde{H}_1, \ldots, \tilde{H}_\ell)$ work similarly on $\tilde{Y}$.

The principal isotropy group for the K-action on Y as well as for the K-action on $\tilde{Y}$ is $\mathcal{Z}(K)$, and the principal isotropy group for the actions of $\mathbb{G} = K \times \mathbb{T}$ is $\mathcal{Z}(K) \times \{e\}$.

All assumptions of our general Scenario are satisfied and thus we obtain

Proposition 5. The Abelian Poisson algebra $\mathfrak{H}$ descends to an integrable system of rank ℓ on the Poisson manifold $\mathbb{D}(K)_*/K$ (for the principal orbit type submanifold $\mathbb{D}(K)_* \subset \mathbb{D}(K)$).
The restrictions of the system to the Poisson manifolds Y_0/K and Y_0^1/K (for $Y \subset \mathbb{D}(K)$) and its principal isotropy submanifolds Y_0^1 and Y_0 w.r.t the K- and $\mathbb{G} = K \times \mathbb{T}$-actions) possess *action variables* given by $(H_1, \ldots, H_\ell)$, and the corresponding $\mathbb{T}$-action is free on Y_0/K. As a result, $\mathfrak{H}$ induces integrable systems of rank ℓ with action variables given by the restriction of $(H_1, \ldots, H_\ell)$ on every sympectic leaf of Y_0/K and every such symplectic leaf of Y_0^1/K that intersects Y_0/K.
Furthermore, the above statements hold if we replace $(Y, \mathfrak{H})$ by $(\tilde{Y}, \tilde{\mathfrak{H}})$.

It would have been enough to consider only one of $\mathfrak{H}$, Y and $\tilde{\mathfrak{H}}$, $\tilde{Y}$ since they are exchanged by the 'duality automorphism' $\mathfrak{S} : (A, B) \mapsto (B^{-1}, B^{-1}AB)$ of the double.

Without action variables, the integrability on $\mathbb{D}(K)_*$ and on its maximal symplectic leaves was shown previously in [L.F. arXiv:2208.03728, arXiv:2309.16245]. This is degenerate integrabiliy except for $K = \mathrm{SU}(2)$.

The minimal symplectic leaves for $K = \mathrm{SU}(n)$ carry the compact(tified) trigonometric Ruijseanaars–Schneider system, which is 'only' Liouville integrable [L.F. and Klimčík arXiv:1101.1759, L.F. and Kluck, arXiv:1312.0400].

The sphere with four holes

Now our starting point is $M_{0,3} = K \circledast K \circledast K = \{(C_1, C_2, C_3)\}$. We let $\mathfrak{H} \subset C^\infty(M_{0,3})^K$ be the set of functions of the form

$$H(C_1, C_2, C_3) = \chi(C_1 C_2), \qquad \forall \chi \in C^\infty(K)^K.$$

The integral curve of V_H with initial value (c_1, c_2, c_3) reads

$$(C_1(\tau), C_2(\tau), C_3(\tau)) = (e^{\tau \nabla \chi(c_1 c_2)} c_1 e^{-\tau \nabla \chi(c_1 c_2)}, e^{\tau \nabla \chi(c_1 c_2)} c_2 e^{-\tau \nabla \chi(c_1 c_2)}, c_3).$$

We introduce the dense open, connected submanifold $Y \subset M_{0,3}$ by

$$Y := \{(C_1, C_2, C_3) \in M_{0,3} \mid C_1 C_2 \in K^{\text{reg}}\}.$$

The form of the integral curves implies that instead of $\mathbb{T} < K$ now the adjoint torus $\mathbb{T}^{\text{ad}} = \mathbb{T}/\mathcal{Z}(K)$ acts. In fact, an action of $\mathbb{T}^{\text{ad}}$ on Y is generated by the joint flows of the Hamiltonians

$$(H_1^{\text{ad}}, \dots, H_\ell^{\text{ad}}) : Y \to \mathbb{R}^\ell \quad \text{with} \quad H_j^{\text{ad}}(C_1, C_2, C_3) := \Xi_j(C_1 C_2),$$

where $\Xi_j := \sum_{1 \le k \le \ell} Q_{j,k} \chi_k$ using the transposed inverse Q of the Cartan matrix of $\mathfrak{k}^{\mathbb{C}}$.

Recall in passing that $\mathbb{T}^{\text{ad}}$ is the quotient of $\mathfrak{t}$ by 2πi-times the coweight lattice, and $\mathbb{T}$ is the quotient of $\mathfrak{t}$ by 2πi-times the coroot lattice. The fundamental coweights $\omega_j^\vee$ satisfy $\alpha_k(\omega_j^\vee) = \delta_{j,k}$ and are related to the simple coroots h_{α_k} by $\omega_j^\vee = \sum_k Q_{j,k} h_{\alpha_k}$. One has the diffeomorphism

$$T_{\omega^\vee} : \mathbb{R}^\ell/(2\pi\mathbb{Z})^\ell \to \mathbb{T}^{\text{ad}} \text{ with } T_{\omega^\vee}(\underline{\tau}) := \exp\Big(-\mathrm{i} \sum_{j=1}^{\ell} \tau_j \mathrm{ad}_{\omega_j^\vee}\Big) \text{ for } \underline{\tau} = (\tau_1, \dots, \tau_\ell) \in \mathbb{R}^\ell.$$

It turns out that the principal isotropy group for the $K \times \mathbb{T}^{\rm ad}$ action on Y is $\mathcal{Z}(K) \times \{e\}$ and our Scenario is again applicable. Thus, we obtain an integrable system of rank ℓ on $(M_{0,3})_*/K$ as well as on every symplectic leaf of Y_0/K and on all such symplectic leaves of Y_0^1/K that intersect Y_0. In short, Proposition 5 is word for word valid if we replace $\mathbb{D}(K)$ by $M_{0,3}$, and also replace $\mathbb{T}$ by $\mathbb{T}^{\rm ad}$ and $(H_1, \dots, H_\ell)$ by $(H_1^{\rm ad}, \dots, H_\ell^{\rm ad})$.

An alternative approach to reduced integrability proceeds through an explicit description of the commutant of $\mathfrak{H}$ in $C^\infty(M)^K$, denoted $\mathfrak{F}^K$. In our case, $\mathfrak{F}^K$ can be obtained using the maps

$$\Psi_{1,2} : M_{0,3} \to K \times K, \qquad \Psi_{1,2}(C_1, C_2, C_3) := (C_1, C_2),$$

$$\Psi_{(12),3} : M_{0,3} \to K \times K, \qquad \Psi_{(12),3}(C_1, C_2, C_3) := (C_1 C_2, C_3),$$

which induce the constants of motion $\Psi_{1,2}^*(C^\infty(K \times K)^K)$ and $\Psi_{(12),3}^*(C^\infty(K \times K)^K)$. The intersection of these two sets of constants of motion is $\mathfrak{H}$. One has

$$\mathrm{ddim}\left(\Psi_{1,2}^*(C^\infty(K \times K)^K)\right) = \mathrm{ddim}\left(\Psi_{(12),3}^*(C^\infty(K \times K)^K)\right) = \dim(K)$$

$$\text{and} \quad \mathrm{ddim}(\mathfrak{F}^K) + \mathrm{ddim}(\mathfrak{H}) = 2\dim(K) = \dim((M_{0,3})_*/K).$$

Moreover, one easily calculates that the dimension of the generic symplectic leaves of $(M_{0,3})_*/K$ is $2\dim(K) - 4\ell$. Except for SU(2), we have

$$\mathrm{ddim}(\mathfrak{H}) = \ell < \dim(K) - 2\ell,$$

and this shows superintegrability on $(M_{0,3})_*/K$ and on its generic symplectic leaves.

Our method based on action variables gave us stronger results.

A conjecture motivated by [Chalykh and Ryan, arXiv:2410.23456]

Chalykh and Ryan identified the celebrated trigonometric, complex van Diejen system with 5 independent parameters as a Liouville integrable system on a symplectic leaf of dimension $2(n-1)$ of the character variety

$$\mathrm{Hom}(\pi_1(\Sigma_{0,4}), \mathrm{SL}(2n,\mathbb{C}))/\,\mathrm{SL}(2n,\mathbb{C}).$$

Their work combined DAHA techniques with the Fock–Rosly approach to moduli spaces. The relevant small symplectic leaf arises from fixing the holonomies along the 4 boundary componenst to special semisimple conjugacy classes.

A similar construction should work also in the compact setting, using $\mathrm{SU}(2n)$ and fixing C_1, C_2, C_3 and $C_4 = (C_1C_2C_3)^{-1}$ to conjugacy classes having representatives of the form

$$\lambda_1 = \mathrm{diag}(\kappa_0^{-1},\dots,\kappa_0^{-1},\kappa_0,\dots,\kappa_0),$$

$$\lambda_2 = \mathrm{diag}(v_0^{-1},\dots,v_0^{-1},v_0,\dots,v_0),$$

$$\lambda_3 = \mathrm{diag}(v_n^{-1},\dots,v_n^{-1},v_n,\dots,v_n),$$

$$\lambda_4 = \mathrm{diag}(\kappa_n^{-1},\dots,\kappa_n^{-1},\kappa_n T^{-2},\dots,\dots,\kappa_n T^{-2},\kappa_n T^{2n-2}),$$

where $\kappa_0, v_0, v_n, \kappa_n, T$ are generic parameters, all taken from $\mathrm{U}(1)$.

The resulting 'compactified classical van Diejen systems' should be related to the quantum mechanical systems studied in [van Diejen and Stokman, arXiv:q-alg/9607003] and in [van Diejen and Görbe, arXiv:2108.00499].

The double torus with one hole

We take the quotient space of $M := \mathbb{D}(K) \circledast \mathbb{D}(K) = \{X \mid X = (A_1, B_1, A_2, B_2) \in K^4\}$.
We have five straightforward possibilities for defining the Abelian Poisson algebra $\mathfrak{H}$:

1. $\mathfrak{H}_1$ consists of the functions
$$H(X) = \chi(A_1), \qquad \forall \chi \in C^\infty(K)^K.$$
2. $\mathfrak{H}_2$ consists of the functions
$$H(X) = \chi([A_1, B_1]), \qquad \forall \chi \in C^\infty(K)^K.$$
3. $\mathfrak{H}_3$ contains all products and sums of the functions
$$H(X) = \chi(A_1) \quad \text{and} \quad H(X) = \chi([A_2, B_2]), \qquad \forall \chi \in C^\infty(K)^K.$$
4. $\mathfrak{H}_4$ consists of all products and sums of the functions
$$H(X) = \chi(A_1) \quad \text{and} \quad H(X) = \chi(A_2), \qquad \forall \chi \in C^\infty(K)^K.$$
5. $\mathfrak{H}_5$ is obtained from the functions
$$H(X) = \chi([A_1, B_1]) \quad \text{and} \quad H(X) = \chi([A_2, B_2]), \qquad \forall \chi \in C^\infty(K)^K.$$

In all cases, $Y \subset M$ is the submanifold, where the arguments of χ above belong to K^{reg}. Torus actions are generated by using the functions $(\chi_1, \ldots, \chi_\ell)$ and the functions $(\Xi_1, \ldots, \Xi_\ell)$ when the argument of χ is A_1, A_2 or a group commutator, respectively.

Our general Scenario is applicable and leads to many superintegrable systems. These are real forms of systems studied in [`Arthamonov and Reshetikhin, arXiv:1909.08682`].

Remark: There exist further possibilities, too. For example, $\mathfrak{H} = \langle \mathfrak{H}_4, \mathfrak{H}_5 \rangle$ is also an Abelian Poisson subalgebra of $C^\infty(M)^K$. In this case, we have not yet investigated the connectedness of Y and the principal isotropy group for the $K \times \mathbb{T} \times \mathbb{T} \times \mathbb{T}^{\text{ad}} \times \mathbb{T}^{\text{ad}}$-action.

Construction of a general class of integrable systems

Consider $M_{m,n}$ with generic non-negative integers m, n. We write the elements of $M_{m,n}$ as $X = (A_1, B_1, \dots, A_m, B_m, C_1, \dots, C_n)$. Below, we specify an Abelian Poisson subalgebra $\mathfrak{H}(I, J) \subset C^\infty(M_{m,n})^K$.

First, we select a subset

$$I := \{i_1, i_2, \dots, i_p\} \subset \{1, 2, \dots, m\}, \quad \text{where} \quad i_1 < i_2 < \cdots < i_p.$$

We associate with the set I the Hamiltonians of the form

$$H^\chi_{i_\alpha}(X) := \chi(A_{i_\alpha}) \quad \text{or} \quad H^\chi_{i_\alpha}(X) := \chi([A_{i_\alpha}, B_{i_\alpha}]) \quad \forall \chi \in C^\infty(K)^K, \forall \alpha = 1, \dots, p.$$

Second, we also fix a set of non-intersecting closed intervals

$$J := \{[\lambda_1, \rho_1], [\lambda_2, \rho_2], \dots, [\lambda_q, \rho_q]\},$$

such that the boundaries of the intervals are integers satisfying

$$1 \leq \lambda_1 < \rho_1 < \lambda_2 < \rho_2 < \cdots < \lambda_q < \rho_q \leq n.$$

We associate with J the Hamiltonians

$$H^\chi_{[\lambda_\beta, \rho_\beta]}(X) := \chi(C_{\lambda_\beta} C_{\lambda_\beta + 1} \cdots C_{\rho_\beta}), \quad \forall \chi \in C^\infty(K)^K, \ \forall \beta = 1, \dots, q.$$

By definition, $\mathfrak{H}(I, J)$ contains the arbitrary finite products and sums of these Hamiltonians. Then, we let Y be the *dense open connected subspace* of $M_{m,n}$ where all arguments of χ above belong to K^{reg}. We can generate an action of $\mathbb{T}^{p-r} \times (\mathbb{T}^{\text{ad}})^{q+r}$ on Y by using the sets of standard functions χ_j and Ξ_j ($j = 1, \dots, \ell$) similarly as before (r is the number of group commutators used). We proved that the Scenario works in all these cases. For a few further possibilities, see our paper.

Conclusion

We developed a general method for showing the superintegrability of Hamiltonian systems constructed by Poisson reduction. The point was that we utilized 'torus' actions whose invariants gave the required constants of motion. We applied our approach to several interesting examples, and there could be many more applications.

Regarding the moduli spaces, our method covers a subset of the class of superintegrable systems constructed in [`Arthamonov and Reshetikhin, arXiv:1909.08682`]. To develop a concrete description of the systems (flows and phase spaces), they used the "r-matrix method" of [`Fock and Rosly, arXiv:math/9802054`], which is *not applicable to compact Lie groups*. Another difference is that we obtain action variables, too.

Recently, in [`Chalykh and Ryan, arXiv:2410.23456`], the complex trigonometric van Diejen system (with 5 coupling constants) was identified as a Liouville integrable system on a symplectic leaf of $\mathrm{Hom}(\pi_1(\Sigma_{0,4}), \mathrm{SL}(2n,\mathbb{C}))/\mathrm{SL}(2n,\mathbb{C})$. Compactified van Diejen sytems should result from a similar construction based on $\mathrm{SU}(2n)$.

A strength of our method is that it delivers integrability not only on generic symplectic leaves, but also on arbitrary symplectic leaves of a dense open subset of the smooth component M_*/K of the full reduced phase space M/K. The key is that we utilized action variables.

What happens outside those dense open subsets, and outside the principal orbit type?

Before engaging oneself is speculations, examples should be explored in depth.