Hidden symmetries and their algebra structures on Courant algebroids

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Contents

Leibniz algebroid

- Let E be a vector bundle over a manifold M
- $\pi \in Hom(E, TM)$ a smooth vector bundle morphism called the **anchor**
- $[\,,\,] : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ be a $\mathbb{R}$-bilinear map

- **Definition:** $(E, \pi, [\,,\,])$ is called a **Leibniz algebroid**, if

$$[\phi, f\phi'] = f[\phi, \phi'] + (\pi(\phi)f)\phi'$$

and $(E, \Gamma(E))$ satisfies the Leibniz identity

$$[\phi, [\phi', \phi'']] = [[\phi, \phi'], \phi''] + [\phi', [\phi, \phi'']]$$

for all $\phi, \phi', \phi'' \in \Gamma(E)$ and $f \in C^\infty(M)$.

Lie algebroid

- The **anchor** preserves the brackets

$$\pi([\phi, \phi']) = [\pi(\phi), \pi(\phi')]$$

where the RHS is the vector field commutator.

- **Definition:** A Leibniz algebroid $(E, \pi, [\,,\,])$ with the bracket $[\,,\,]$ is skew-symmetric is called a **Lie algebroid**.
The Leibniz identity corresponds to the **Jacobi identity** and $(\Gamma(E), [\,,\,])$ is a Lie algebra.

- A **fibrewise metric** $< , >: \Gamma(E) \times \Gamma(E) \to C^\infty(M)$ on E can be defined if
  $< \phi, \phi' >= 0$ for all $\phi' \in \Gamma(E)$ implies $\phi = 0$.

- A map $\mathcal{D} : C^\infty(M) \to \Gamma(E)$ is also defined as

$$< \mathcal{D}f, \phi >= \pi(\phi)f$$

for all $\phi \in \Gamma(E)$ and $f \in C^\infty(M)$.

Courant algebroid

- **Definition:** Let $(E, \pi, [\,,\,])$ be a Leibniz algebroid and $<\,,\,>$ is a fiberwise metric. Then $(E, \pi, [\,,\,], <\,,\,>)$ is called a **Courant algebroid** if the following identities are satisfied (Liu, Weinstein, Xu 1997)

$$\pi(\phi) < \phi', \phi'' > = < [\phi, \phi'], \phi'' > + < \phi', [\phi, \phi''] >$$

$$[\phi, \phi'] + [\phi', \phi] = \mathcal{D} < \phi, \phi' >$$

$$[f\phi, \phi'] = f[\phi, \phi'] - (\pi(\phi')f)\phi + < \phi, \phi' > \mathcal{D}f$$

for all $\phi, \phi' \in \Gamma(E)$ and $f \in C^\infty(M)$.

Exact Courant algebroid

- A Courant algebroid is **exact** if E is a direct sum of tangent and cotangent bundels which is called generalized tangent bundle (Courant 1990)

$$E = TM \oplus T^*M$$

- Exact Courant algebroids are classified in terms of the de Rham cohomology $H^3_{dR}(M)$ (Severa 1998).

Generalized geometry

- Every exact Courant algebroid is isomorphic to the one of which is presented in the following example
- Let $E = TM \oplus T^*M$ be the generalized tangent bundle. Set $\pi = pr_{TM}$ and $[H] \in \Omega^3(M)$ be represented by a closed 3-form H.
- The fiberwise metric is defined as

$$< (X, \xi), (Y, \eta) >= i_X \eta + i_Y \xi$$

for all $X, Y \in \mathcal{X}(M)$ and $\xi, \eta \in \Omega^1(M)$.

- and $[\,,\,]$ is defined as the H-twisted Dorfman bracket $[\,,\,]_D^H$

$$[(X, \xi), (Y, \eta)]_D^H = ([X, Y], \mathcal{L}_X \eta - i_Y d\xi - i_X i_Y H)$$

for all $(X, \xi), (Y, \eta) \in \Gamma(E)$.

Generalized geometry

- A **generalized metric** can be defined by using the metric $g: TM \to T^*M$ which is invertible and a 2-form $B \in \Omega^2(M)$ satisfying $H = dB$ (Gualtieri 2004).

$$\mathcal{G}_B = \begin{pmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{pmatrix}$$

Since $\mathcal{G}^2 = I$, ± 1 eigenspaces of $\mathcal{G}$ which are denoted by $V_\pm$ give a metric splitting of E

$$TM \oplus T^*M = V_+ \oplus V_-$$

and the generalized metric can be written in terms of the fiberwise metrics on V_+ and V_- as

$$\mathcal{G}(\, , \,) = < \, , \, >_+ - < \, , \, >_-$$

Generalized geometry

- Dorfman bracket corresponds to the generalized Lie derivative
- By antisymmetrizing the Dorfman bracket we can also define the **Courant bracket** as follows

$$[(X,\xi),(Y,\eta)]_C^H = ([X,Y],\mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d(i_X\eta - i_Y\xi) - i_Xi_YH)$$

which also satisfies the identity

$$\pi([\phi,\phi']_C^H) = [\pi(\phi),\pi(\phi')]$$

for all $\phi,\phi' \in \Gamma(E)$.

- But it does not satisfy the Jacobi identity.

Courant algebroid connection

- Let $(E, \pi, <,>, [,])$ be a Courant algebroid and

$$\mathbb{D} : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$$

is a $\mathbb{R}$-bilinear map.

- $\mathbb{D}$ is a Courant algebroid connection if it satisfies

$$\mathbb{D}_{f\phi}\phi' = f\mathbb{D}_{\phi}\phi'$$

$$\mathbb{D}_{\phi}f\phi' = f\mathbb{D}_{\phi}\phi' + (\pi(\phi)f)\phi'$$

for all $\phi, \phi' \in \Gamma(E)$ and $f \in C^{\infty}(M)$

- and compatible with the fiberwise metric $<,>$

$$\pi(\phi) < \phi', \phi'' >' = < \mathbb{D}_{\phi}\phi', \phi'' > + < (\phi', \mathbb{D}_{\phi}\phi'' >$$

for all ϕ, ϕ', ϕ''.

Levi-Civita connection

- The torsion 3-form on a Courant algebriod is defined as (Alekseev, Xu + Gualtieri 2007)

$$\mathbb{T}^{\mathbb{D}}(\phi,\phi',\phi'') =< \mathbb{D}_{\phi}\phi' - \mathbb{D}_{\phi'}\phi - [\phi,\phi'],\phi'' > + < \mathbb{D}_{\phi''}\phi,\phi' >$$

- If $\mathbb{D}$ is a Courant algebroid connection and it is compatible with the generalized metric

$$\mathbb{D}\mathcal{G} = 0$$

and torsion-free

$$\mathbb{T}^{\mathbb{D}} = 0$$

Generalized connection

- By fixing a divergence operator

$$div_{\mathbb{D}}(\phi) =< \mathbb{D}_{\mathcal{X}_A}\phi, \mathcal{X}^A >$$

for a basis of generalized vectors $\mathcal{X}_A$,

- one can define a metric compatible torsion-free generalized connection $\mathbb{D}^{\pm H}$ in terms of the 3-form field H as (Gualtieri 2010 + Garcia-Fernandez 2019)

$$\mathbb{D}^{\pm H}_{\mathcal{X}}\mathcal{Y} = \nabla_X \begin{pmatrix} Y \\ \eta \end{pmatrix} \pm \frac{1}{6} \begin{pmatrix} \widetilde{i_Y i_X H} \\ i_{\widetilde{\eta}} i_X H \end{pmatrix}$$

where $\mathbb{D}^{\pm H}$ acts on subbundles $V_{\pm}$ respectively, ∇ is the Levi-Civita connection on M and $\mathcal{X} = (X, \xi), \mathcal{Y} = (Y, \eta) \in \Gamma(E)$.

Torsionful connection

- Indeed, the defined generalized connection corresponds to the connection ∇^H with respect to a skew-symmetric torsion 3-form H on M

$$\mathbb{D}^H = \nabla^H \oplus \nabla^{*H}$$

- where

$$\nabla^H = \nabla_X Y + \frac{1}{6}\widetilde{i_X i_Y H}$$

Killing and conformal Killing vectors

- On a n-dimensional manifold M, **Killing vectors** are defined as vector fields that the metric remains constant along their flows

$$\mathcal{L}_K g = 0$$

- and **conformal Killing vectors** are defined as vector fields that the metric remains constant along their flows up to a conformal factor

$$\mathcal{L}_C g = 2\lambda g$$

Hidden symmetries

- The antisymmetric generalization of conformal Killing vectors to higher degree differential forms are called **conformal Killing-Yano (CKY) forms** and they satisfy the following equation

$$\nabla_X \alpha = \frac{1}{p+1} i_X d\alpha - \frac{1}{n-p+1} \widetilde{X} \wedge \delta\alpha$$

  where X is any vector field, $\widetilde{X}$ is its metric dual and α is a CKY p-form.

- A subset of CKY forms satisfying $\delta\alpha = 0$ are called **Killing-Yano (KY) forms** and correspond to the antisymmetric generalizations of Killing vectors

$$\nabla_X \alpha = \frac{1}{p+1} i_X d\alpha$$

- These are called **hidden symmetries** of the manifold.

Hidden symmetries

- In the presence of a torsionful connection ∇^H, we can define the torsionful CKY equation as

$$\nabla^H_X \alpha = \frac{1}{p+1} i_X d^H \alpha - \frac{1}{n-p+1} \widetilde{X} \wedge \delta^H \alpha$$

- Here d^H and δ^H can be written in terms of the exterior and co-derivatives as follows

$$d^H \alpha = d\alpha + H \underset{1}{\wedge} \alpha$$

$$\delta^H \alpha = \delta\alpha + \frac{1}{2} H \underset{2}{\wedge} \alpha$$

- and the contracted wedge product is defined by

$$\alpha \underset{k}{\wedge} \beta = i_{X_{a_k}} ... i_{X_{a_1}} \alpha \wedge i_{X^{a_k}} ... i_{X^{a_1}} \beta$$

Integrability condition

- By using the curvature operator

$$R^H(X,Y) = \nabla^H_X \nabla^H_Y - \nabla^H_Y \nabla^H_X - \nabla^H_{[X,Y]}$$

- and the torsion operator

$$T^H(X,Y) = \nabla^H_X Y - \nabla^H_Y X - [X,Y]$$

- we can write the integrability condition of torsionful CKY equation as

$$\frac{p}{p+1}\delta^H d^H \alpha + \frac{n-p}{n-p+1} d^H \delta^H \alpha = -e^a \wedge i_{X^b} \widehat{R}(X_a, X_b)\alpha$$

where $\widehat{R}(X_a, X_b)\alpha = R^H(X_a, X_b)\alpha + \nabla^H_{T^H(X_a,X_b)}\alpha$.

Hidden symmetries on Courant algebroids

- By using the generalized Lie derivative $\mathbb{L}_{\mathcal{X}}$ and generalized connection $\mathbb{D}^H_{\mathcal{X}}$, one can write the generalized CKY equation on Courant algebroids as

$$\mathbb{D}^H_{\mathcal{X}}\alpha = \frac{1}{p+1} i_{\mathcal{X}} d^H \alpha - \frac{1}{n-p+1} \widetilde{\mathcal{X}} \wedge \delta^H \alpha$$

  Here d^H and δ^H are defined from the generalized connection $\mathbb{D}^H$, $\mathcal{X}$ is an arbitrary generalized vector and α is a generalized p-form

- and similarly generalized KY equation is written as

$$\mathbb{D}^H_{\mathcal{X}}\alpha = \frac{1}{p+1} i_{\mathcal{X}} d^H \alpha$$

Algebra structure

- In the absence of torsion, one can prove by using the integrability condition that CKY forms satisfy a graded Lie algebra structure on constant curvature manifolds w.r.t. the following bracket (Ertem 2016)

$$[\alpha_1, \alpha_2]_{CKY} = \frac{1}{q+1} i_{X^a}\alpha_1 \wedge i_{X_a} d\alpha_2 + \frac{(-1)^p}{p+1} i_{X^a} d\alpha_1 \wedge i_{X_a}\alpha_2 + \frac{(-1)^p}{n-q+1}\alpha_1 \wedge \delta\alpha_2 + \frac{1}{n-p+1}\delta\alpha_1 \wedge \alpha_2$$

  where α_1 is a CKY p-form and α_2 is a CKY q-form.
- A subset of CKY forms called normal CKY forms and satisfy a special kind of integrability condition

$$\frac{p}{p+1}\delta d\alpha + \frac{n-p}{n-p+1} d\delta\alpha = -2(n-p) K_a \wedge i_{X^a}\alpha$$

  where K_a is defined as $K_a = \frac{1}{n-2}\left(\frac{\mathcal{R}}{2(n-1)} e_a - P_a\right)$ in terms of curvature 2-forms R_{ab} and Ricci 1-forms P_a
  also satisfy a graded Lie algebra structure on Einstein manifolds.

Algebra structure

- Moreover, KY forms satisfy a graded Lie algebra structure on constant curvature manifolds w.r.t. the Schouten-Nijenhuis bracket (Kastor, Ray, Traschen 2007)

$$[\alpha_1, \alpha_2]_{SN} = i_{X^a}\alpha_1 \wedge \nabla_{X_a} d\alpha_2 + (-1)^p \nabla_{X^a}\alpha_1 \wedge i_{X_a}\alpha_2$$

where α_1 is a KY p-form and α_2 is a KY q-form.

- Another subset of CKY forms called closed conformal CKY (CCKY) forms that satisy the following equation

$$\nabla_X \alpha = -\frac{1}{n-p+1}\widetilde{X} \wedge \delta\alpha$$

are subject to a graded Lie algebra structure that is a combination of odd KY-even CCKY forms and even KY-odd CCKY forms satisfy a graded Lie algebra structure w.r.t. CKY bracket (Açık, Ertem 2021)

Algebra structure

- For the torsionful case, the bracket of torsionful CKY forms is modified as follows (Ertem, Kelekçi, Açık 2016)

$$[\alpha_1, \alpha_2]_{HCKY} = \frac{1}{q+1} i_{X^a}\alpha_1 \wedge i_{X_a} d^H \alpha_2 + \frac{(-1)^p}{p+1} i_{X^a} d^H \alpha_1 \wedge i_{X_a}\alpha_2 + \frac{(-1)^p}{n-q+1}\alpha_1 \wedge \delta^H \alpha_2 + \frac{1}{n-p+1}\delta^H \alpha_1 \wedge \alpha_2$$

- And two constraints appear to satisfy a graded Lie algebra structure for a closed torsion $dH = 0$

$$\nabla^H_X H = 0$$

corresponding to parallel torsion and

$$\nabla^H_{T(X,Y)}\alpha = 0$$

corresponding to a subset of torsionful CKY forms satisfying the parralelity condition w.r.t. torsion operator.

Algebra structure

- If these two constraints are satisfied then the conditions to satisfy an algebra structure reduce to the torsionles case and we have a graded Lie algebra structure for a subset of torsionful CKY forms on constant curvature or Einstein manifolds.
- It is proved in (Agricola, Ferreira 2014) that any Einstein manifold with parallel skew-symmetric torsion H has constant scalar curvature.
- Moreover, any Einstein-Sasaki manifold has deformations into Einstein manifolds with skew-symmetric torsion. So, the manifolds that have parallel skew-symmetric torsion and satisfy the constant curvature or Einstein condition can be found in the literature.

Algebra structure

- The algebra structure for the torsionful case can easily be transferred to the generalized hidden symmetries on Courant algebroids.
- Since the equalities have the same structure in both cases, the conditions to obtain an algebra structure for generalized hidden symmetries is the same as before.
- So, we define a bracket $[\,,\,]_{HCKY}$ on Courant algebroids and have constraints on torsion 3-form H and a subset of CKY forms defined by the generalized torsion operator have to be considered.

Summary and Applications

- Courant algebroids, especially generalized geometry, is the natural geometrical language for supergravity theories (especially for 10d).
- Extension of the Lie algebra structure of isometries for higher degree generalizations can give way to obtain the full symmetry structure of a background. By using these Lie algebra structures one can construct extended symmetry superalgebra structures by including special types of spinors.

Summary and Applications

- We find that when the skew-symmetric torsion 3-form H is closed and parallel then a special subset of CKY forms constitute a graded Lie algebra structure with respect to HCKY bracket on constant curvature or Einstein manifolds. The special subset consists of CKY forms satisfying a special parallelity condition with respect to torsion operator.
- By extending these graded Lie algebra structures to superalgebra structures by including generalized spinors, one can find full symmetry structures of special Courant algebroids corresponding to supergravity theories and obtain hints about the classification problem of supergravity backgrounds.

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