Cyclic Lie-Rinehart algebras

Alina Dobrogowska
Faculty of Mathematics
University of Białystok

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The notion of Lie-Rinehart algebra is the algebraic counterpart of the differential geometric notion of Lie algebroid, which in turn is a generalization of Lie algebras.

Lie Algebroid $(A, [\cdot,\cdot]_A, a_A)$

Definition

A Lie algebroid $(A, [\cdot,\cdot]_A, a_A)$ is a vector bundle $A \longrightarrow M$ over a manifold M, together with a vector bundle map $a_A : A \longrightarrow TM$, called the anchor map, and a Lie bracket $[\cdot,\cdot]_A : \Gamma(A) \times \Gamma(A) \longrightarrow \Gamma(A)$, such that the following Leibniz rule is satisfied

$$[\alpha, f\beta]_A = f[\alpha,\beta]_A + a_A(\alpha)(f)\beta,$$

for all $\alpha, \beta \in \Gamma(A)$, $f \in C^\infty(M)$.

The anchor map is a Lie algebra homomorphism

$$a_A\left([\alpha,\beta]_A\right) = [a_A(\alpha), a_A(\beta)].$$

Paradines, J., *Théorie de Lie pour les grupoïdes différentiables. Relations entre propriétés locales et globales*, C. R. Acad. Sc. Paris, Ser. A, 264, 245-248, 1967.

Examples $(A, [\cdot,\cdot]_A, a_A)$

$$[\alpha, f\beta]_A = f[\alpha,\beta]_A + a_A(\alpha)(f)\beta,$$

Example

Any tangent bundle $A = TM$ of a manifold M, with $a_A = id$ and the usual Lie bracket of vector fields, is a Lie algebroid.

Example

Any Lie algebra $A = \mathfrak{g}$, with trivial anchor $a_A = 0$, is a Lie algebroid.

Poisson manifold $(M, \{\cdot,\cdot\})$

Definition

A Poisson manifold $(M, \{\cdot,\cdot\})$ is a smooth manifold M (equipped with a Poisson structure) with a fixed bilinear and antisymmetric mapping $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$, which satisfies Jacobi identity and Leibniz rule

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0,$$

$$\{f,gh\} = \{f,g\}h + g\{f,h\},$$

where $f, g, h \in C^\infty(M)$.

Poisson bracket can be written in terms of **Poisson tensor** ($\pi \in \Gamma\left(\bigwedge^2 TM\right)$ such that $[\pi,\pi]_{SN} = 0$) as follows

$$\{f,g\} = \pi(df, dg).$$

Poisson tensor

In the local coordinates $x_1, x_2, \ldots, x_N$ on M

$$\{f, g\} = \sum_{i,j=1}^{N} \pi_{ij}(x) \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}.$$

Components of Poisson tensor are given by the formula

$$\pi_{ij}(x) = \{x_i, x_j\}$$

and satisfy

- $\pi_{ij} = -\pi_{ji}$,
- $\frac{\partial \pi_{ij}}{\partial x_s} \pi_{sk} + \frac{\partial \pi_{ki}}{\partial x_s} \pi_{sj} + \frac{\partial \pi_{jk}}{\partial x_s} \pi_{si} = 0.$

Lie–Poisson structure

- Lie, Kirillov, Kostant, Souriau

$(\mathfrak{g}, [\cdot,\cdot])$ – a Lie algebra is a vector space $\mathfrak{g}$ together with a bilinear, skew-symmetric bracket $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ (called the Lie bracket) which satisfies the Jacobi identity

$$[[X,Y],Z] + [[Z,X],Y] + [[Y,Z],X] = 0,$$

for all $X, Y, Z \in \mathfrak{g}$.
$\mathfrak{g}^*$ – dual space to Lie algebra $\mathfrak{g}$ — Poisson manifold.
Lie–Poisson bracket for $f, g \in C^\infty(\mathfrak{g}^*)$ is given by formula

$$\{f,g\}(\rho) = \langle \rho, [df(\rho), dg(\rho)] \rangle ,$$

where $\rho \in \mathfrak{g}^*$, $df(\rho)$, $dg(\rho)$ – linear forms on $\mathfrak{g}^*$. However due to the canonical isomorphism valid in the finite dimensional case $(\mathfrak{g}^*)^* = \mathfrak{g}$ we can identify them with elements of Lie algebra $\mathfrak{g}$.

Linear fiber-wise Poisson structure

If $(A, [\cdot,\cdot]_A, a_A)$ is a Lie algebroid then on the total space A^* of dual bundle $A^* \xrightarrow{q} M$ there exists a Poisson structure given by

$$\{f \circ q, g \circ q\} = 0,$$

$$\{l_X, g \circ q\} = a_A(X)(g) \circ q,$$

$$\{l_X, l_Y\} = l_{[X,Y]_A},$$

where $X, Y \in \Gamma(A)$, $\;l_X(v) = \langle v, X(q(v))\rangle$, $\;v \in A^*$ and $f, g \in C^\infty(M)$.

Example $A = T^*M$

Let $(M, \{.,.\})$ be a Poisson manifold, then its cotangent bundle $q^* : T^*M \to M$ possesses a Lie algebroid structure

$$\begin{tikzcd} A = T^*M \arrow[r, "a_{T^*M}"] \arrow[d, "q^*"] & TM \arrow[d, "q"] \\ M \arrow[r, "id"] & M \end{tikzcd}$$

given by

$$a_{T^*M}(df)(\cdot) = \{f, \cdot\},$$
$$[df, dg]_{T^*M} = d\{f, g\},$$

where $f, g \in C^\infty(M)$. In general form

$$[\alpha, \beta]_{T^*M} = \pounds_{\pi^\#(\alpha)}\beta - \pounds_{\pi^\#(\beta)}\alpha - d(\pi(\alpha, \beta))$$

for $\alpha, \beta \in \Gamma(T^*M)$, where $\pi^\# : \Gamma(T^*M) \to \Gamma(TM)$ is given by $\pi^\#(\alpha)(\cdot) = \pi(\alpha, \cdot)$ and $a_{T^*M} = \pi^\#$.

Lifting of a Poisson structure from M to TM

If $(M, \{,\}) = (M, \pi)$ is a Poisson manifold, then the manifold TM possesses a Poisson structure given by

$$\{f \circ q, g \circ q\}_{TM} = 0,$$

$$\{l_{df}, g \circ q\}_{TM} = \{f, g\} \circ q,$$

$$\{l_{df}, l_{dg}\}_{TM} = l_{d\{f,g\}},$$

where $l_{df}(v) = \langle v, df(q_M(v))\rangle$, $v \in TM$ and $f, g \in C^\infty(M)$. Let $\mathbf{x} = (x_1, \ldots, x_N)$ be a system of local coordinates on M. Then the Poisson tensor π^C on the manifold TM associated with π has the form

$$\pi^C(\mathbf{x}, \mathbf{y}) = \left(\begin{array}{c|c} 0 & \pi(\mathbf{x}) \\ \hline \pi(\mathbf{x}) & \sum_{s=1}^N \frac{\partial \pi}{\partial x_s}(\mathbf{x}) y_s \end{array}\right),$$

in the system of local coordinates $(\mathbf{x}, \mathbf{y}) = (x_1, \ldots, x_N, y_1 = l_{dx_1}, \ldots, y_N = l_{dx_N})$ on TM.

The complete lift, the vertical lift

$$\pi(\mathbf{x}) = \sum_{1\leq i<j}^{N} \pi^{ij}(\mathbf{x})\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j},$$

$$\Downarrow$$

$$\pi^C(\mathbf{x},\mathbf{y}) = \sum_{1\leq i<j}^{N} \left(\pi^{ij}(\mathbf{x})\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial y^j} + \pi^{ij}(\mathbf{x})\frac{\partial}{\partial y^i} \wedge \frac{\partial}{\partial x^j}\right.$$

$$\left. + \sum_{s=1}^{N} \frac{\partial \pi^{ij}}{\partial x^s}(\mathbf{x}) y^s \frac{\partial}{\partial y^i} \wedge \frac{\partial}{\partial y^j}\right) \Longrightarrow \left(\begin{array}{c|c} 0 & \pi(\mathbf{x}) \\ \hline \pi(\mathbf{x}) & \sum_{s=1}^N \dfrac{\partial \pi}{\partial x_s}(\mathbf{x}) y_s \end{array}\right)$$

$$\pi^V(\mathbf{x},\mathbf{y}) = \sum_{1\leq i<j}^{N} \pi^{ij}(\mathbf{x})\frac{\partial}{\partial y^i} \wedge \frac{\partial}{\partial y^j} \Longrightarrow \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \pi(\mathbf{x}) \end{array}\right)$$

Lifting of Casimir functions from M to TM

Theorem

Let $c_1, \ldots, c_r$, where $r = \dim M - \operatorname{rank} \pi$, be Casimir functions for the the Poisson structure π, then the functions

$$c_i \quad \text{and} \quad l_{dc_i} = \sum_{s=1}^{N} \frac{\partial c_i}{\partial x_s} y_s, \quad i = 1, \ldots r,$$

are the Casimir functions for the Poisson tensor π^C.

J. Grabowski, P. Urbański, *Tangent lifts of Poisson and related structures*, J. Phys. A: Math. Gen., 28, 6743-6777, 1995.

Example: Lifting of a Poisson structure from $\mathfrak{so}(3)$

Let us consider the Lie algebra $\mathfrak{so}(3)$ of skew-symmetric matrices $[X, Y] = XY - YX$. The Poisson tensors can be written:

$$\pi_1(X) = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}, \quad c_1(X) = x_1^2 + x_2^2 + x_3^2.$$

The Poisson structures on $T\left(\mathfrak{so}(3)\right)^*$ are given by tensors

$$\pi^C(X, Y) = \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & -x_3 & x_2 \\ 0 & 0 & 0 & x_3 & 0 & -x_1 \\ 0 & 0 & 0 & -x_2 & x_1 & 0 \\ \hline 0 & -x_3 & x_2 & 0 & -y_3 & y_2 \\ x_3 & 0 & -x_1 & y_3 & 0 & -y_1 \\ -x_2 & x_1 & 0 & -y_2 & y_1 & 0 \end{array}\right).$$

Moreover the Casimirs are given by

$$c_1(X) = x_1^2 + x_2^2 + x_3^2, \qquad \frac{1}{2} l_{dc_1} = x_1 y_1 + x_2 y_2 + x_3 y_3.$$

In this case we recognize the Lie-Poisson structure of $\mathfrak{e}(3)$.

Lie algebroids with a Poisson structure

Let $(A, [\cdot,\cdot]_A, a_A)$ be a Lie algebroid and assume that $\pi \in \Gamma\left(\bigwedge^2 A\right)$ satisfies $[\pi,\pi]_A = 0$. Then (A,π) is called a Lie algebroid with a Poisson structure.
Let us define

$$[\alpha,\beta]_\pi = \pounds_{\pi^\sharp\alpha}\beta - \pounds_{\pi^\sharp\beta}\alpha - d\left(\pi(\alpha,\beta)\right),$$

for $\alpha,\beta \in \Gamma\left(A^*\right)$, where $\pounds$ denotes the Lie derivation defined by

$$\pounds_X\alpha(Y) = a_A(X)\alpha(Y) - \alpha\left([X,Y]_A\right),$$

for $X, Y \in \Gamma(A)$ and $\pi^\sharp : A^* \longrightarrow A$ is defined by $\pi^\sharp\alpha(\cdot) = \pi(\alpha,\cdot)$, and set $a_{A^*} = a_A \circ \pi^\sharp$.
Then $(A^*, [\cdot,\cdot]_\pi, a_{A^*})$ is a Lie algebroid.

Substitution $\pi = X \wedge Y$

We rewrite

$$[\alpha, \beta]_\pi = \pounds_{\pi^\sharp \alpha}\beta - \pounds_{\pi^\sharp \beta}\alpha - d\left(\pi(\alpha, \beta)\right),$$

for π of the form $\pi = X \wedge Y$

$$\begin{aligned}[\alpha, \beta]_\pi =& \beta(Y)\pounds_X\alpha - \alpha(Y)\pounds_X\beta - \left(\beta(X)\pounds_Y\alpha - \alpha(X)\pounds_Y\beta\right)\\ =& [\alpha, \beta]_{X,Y} - [\alpha, \beta]_{Y,X}.\end{aligned}$$

General situation

$$[\alpha, \beta]_{X,Y} + \lambda\, [\alpha, \beta]_{Y,X}.$$

Substitution $\pi = X \wedge Y$

For π of the form $\pi = X \wedge Y$

$$\begin{aligned}[\alpha,\beta]_\pi =& \beta(Y)\pounds_X\alpha - \alpha(Y)\pounds_X\beta - (\beta(X)\pounds_Y\alpha - \alpha(X)\pounds_Y\beta) \\ =& [\alpha,\beta]_{X,Y} - [\alpha,\beta]_{Y,X}.\end{aligned}$$

General situation

$$[\alpha,\beta]_{X,Y} + \lambda\,[\alpha,\beta]_{Y,X}.$$

A. Dobrogowska, G. Jakimowicz, *Generalization of the concept of classical r-matrix to Lie algebroids*, J. Geom. Phys. 165, 1-15, 2021.

On some constructions of Lie algebroids on the cotangent bundle of a manifold

It is well known that if M is a manifold then TM is the tangent algebroid of M, with the identity map as the anchor map and the standard commutator of vector fields. However, we will use these fields and give the construction of another algebroid structures.

Theorem

Suppose that M is a manifold and $X, Y \in \Gamma(TM)$ are vector fields such that $[X, Y] = fY$, $f \in C^\infty(M)$. Then $(T^*M, [\cdot,\cdot]_{X,Y}, a_{X,Y})$ is a Lie algebroid, where the Lie bracket and the anchor map are given by

$$
\begin{aligned}
[\alpha, \beta]_{X,Y} &= \beta(Y)\pounds_X \alpha - \alpha(Y)\pounds_X \beta, \\
a_{X,Y}(\alpha) &= -\alpha(Y)X,
\end{aligned}
$$

where $\alpha, \beta \in \Gamma(T^*M)$.

The Poisson structure on the tangent bundle TM

In local coordinates $(\mathbf{x},\mathbf{y})$ when $X=\sum_{i=1}^N v^i(\mathbf{x})\frac{\partial}{\partial x^i}$ and $Y=\sum_{i=1}^N w^i(\mathbf{x})\frac{\partial}{\partial x^i}$ the Poisson tensor is given by formula

$$\pi_{X,Y}=\left(\begin{array}{c|c} 0 & v(\mathbf{x})w^\top(\mathbf{x}) \\ \hline -w(\mathbf{x})v^\top(\mathbf{x}) & \sum_{s=1}^N \left(\frac{\partial v}{\partial x^s}(\mathbf{x})w^\top(\mathbf{x}) - w(\mathbf{x})\left(\frac{\partial v}{\partial x^s}(\mathbf{x})\right)^\top\right) y^s \end{array}\right),$$

where $\mathbf{v}^\top=(v^1,\dots,v^N)$ and $\mathbf{w}^\top=(w^1,\dots,w^N)$.

Substitution $\pi = X \wedge Y$

General situation

$$[\alpha, \beta]_{X,Y} + \lambda\, [\alpha, \beta]_{Y,X}\,.$$

On some constructions of Lie algebroids on the cotangent bundle of a manifold

In addition, we will get a similar structure by swapping vector fields X, Y. Moreover, if we take a linear combination of these structures, we will again obtain a Poisson structure. The same thing also happens on the level of the Lie algebroid.

Theorem

Let $X, Y \in \Gamma(TM)$ be such that $[X, Y] = 0$, then a structure $\left(T^*M, [\cdot,\cdot]^{\lambda}_{X,Y}, a^{\lambda}_{X,Y}\right)$ is a Lie algebroid, where the Lie bracket and the anchor map are given by

$$\begin{aligned}
[\alpha, \beta]^{\lambda}_{X,Y} &= [\alpha, \beta]_{X,Y} + \lambda[\alpha, \beta]_{Y,X} \\
&= \beta(Y)\pounds_X\alpha - \alpha(Y)\pounds_X\beta + \lambda\left(\beta(X)\pounds_Y\alpha - \alpha(X)\pounds_Y\beta\right),
\end{aligned}$$

$$a^{\lambda}_{X,Y}(\alpha) = a_{X,Y}(\alpha) + \lambda a_{Y,X}(\alpha) = -\alpha(Y)X - \lambda\alpha(X)Y$$

and λ is a real parameter.

Remark

In the case when $\lambda = -1$, the assumption of $[X, Y] = 0$ can be weakened. It is sufficient to assume that $[X, Y] = fX + gY$, where $f, g \in C^\infty(M)$.

The Poisson structure on the tangent bundle TM

This structure also leads to the Poisson bracket. In the local coordinates expression of the Poisson structure is the following tensor

$$\pi^{\lambda}_{X,Y}(\mathbf{x},\mathbf{y}) =$$

$$\left(\begin{array}{c|c} 0 & v(\mathbf{x})w^\top(\mathbf{x}) + \lambda w(\mathbf{x})v^\top(\mathbf{x}) \\ \hline -w(\mathbf{x})v^\top(\mathbf{x}) & \sum_{s=1}^{N} \left(\frac{\partial v}{\partial x^s}(\mathbf{x})w^\top(\mathbf{x}) - w(\mathbf{x}) \left(\frac{\partial v}{\partial x^s}(\mathbf{x})\right)^\top \right. \\ -\lambda v(\mathbf{x})w^\top(\mathbf{x}) & \left. + \ \lambda \ \left(\frac{\partial w}{\partial x^s}(\mathbf{x})v^\top(\mathbf{x}) - v(\mathbf{x}) \left(\frac{\partial w}{\partial x^s}(\mathbf{x})\right)^\top \right) \right) y^s \end{array}\right).$$

In this construction, the block $vw^\top + \lambda wv^\top$ is symmetric for $\lambda = 1$ in contrast to the construction of the Poisson bracket from the algebroid bracket of differential forms. Moreover, this block is antisymmetric for $\lambda = -1$ and it is also a Poisson tensor on manifolds M. In this case it is a complete lift of $\pi = X \wedge Y$.

Example

Let us consider again the Lie algebra $\mathfrak{so}(3)$ of skew-symmetric matrices. Thus on $\mathfrak{so}(3)$ we have the linear Poisson structure

$$\pi(X) = -x^3 \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} + x^2 \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^3} - x^1 \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^3}.$$

Observe that defining the vector fields

$$X = x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2}, \quad Y = \frac{\partial}{\partial x^3}, \quad U = -x^3 \frac{\partial}{\partial x^1}, \quad W = \frac{\partial}{\partial x^2}.$$

we can split the above Poisson tensor into two terms
$\pi(X) = X \wedge Y + U \wedge W$.

Example

We obtain the following splitting

$$\begin{tikzcd}
 & & & X \wedge Y + U \wedge W \arrow[dlll] \arrow[dll] \arrow[drr] \arrow[drrr] & & & \\
X^C \wedge Y^V \arrow[dr] & -Y^C \wedge X^V \arrow[d] & & & & U^C \wedge W^V \arrow[d] & -W^C \wedge U^V \arrow[dl] \\
 & (X \wedge Y)^C \arrow[drr] & & & & (U \wedge W)^C \arrow[dll] & \\
 & & & (X \wedge Y + U \wedge W)^C & & &
\end{tikzcd}$$

Example

We obtain the following splitting

$$
\begin{tikzcd}
 & & \mathfrak{so}(3) \arrow[dll] \arrow[dl] \arrow[dr] \arrow[drr] & & \\
\mathfrak{g}_{5,17}^{100} \oplus < x^3 > \arrow[dr] & \mathfrak{g}_{5,1} \oplus < y^3 > \arrow[d] & & \mathfrak{g}_{5,1} \oplus < x^2 > \arrow[d] & \mathfrak{g}_{3,1} \oplus < x^1, y^2, y^3 > \arrow[dl] \\
 & \mathfrak{e}(2) = \mathfrak{g}_{3,6} \arrow[dr] & & \mathfrak{g}_{5,1} \oplus < x^1 + x^2 > \arrow[dl] & \\
 & & \mathfrak{e}(3) & &
\end{tikzcd}
$$

The particular case of above construction $A = \mathfrak{g}$

Because a Lie algebra $\mathfrak{g}$ can be thought of as a Lie algebroid over a point, so we have the opportunity to construct a Lie bracket on the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$.

Corollary

If $(\mathfrak{g}, [\cdot,\cdot])$ is a Lie algebra and $X, Y \in \mathfrak{g}$ such that $[X,Y] = cY$ (or $[X,Y] = 0$) are fixed, then $(\mathfrak{g}^*, [\cdot,\cdot]_{X,Y})$ is a Lie algebra, where

$$[\alpha, \beta]_{X,Y} = \alpha(Y) ad^*_X \beta - \beta(Y) ad^*_X \alpha,$$

(or $\left(\mathfrak{g}_*, [\cdot,\cdot]^\lambda_{X,Y}\right)$ is a Lie algebra, where the commutator is constructed as follows

$$[\alpha, \beta]^\lambda_{X,Y} = \alpha(Y) ad^*_X \beta - \beta(Y) ad^*_X \alpha + \lambda \left(\alpha(X) ad^*_Y \beta - \beta(X) ad^*_Y \alpha\right),$$

for $\alpha, \beta \in \mathfrak{g}^*$

Note that when $\lambda = -1$ the bracket can be rewritten as

$$[\alpha, \beta]^{-1}_{X,Y} = -\alpha(X) ad^*_Y \beta + \alpha(Y)\rangle ad^*_X \beta$$

$$+\beta(X) ad^*_Y \alpha - \beta(Y) ad^*_X \alpha = [\alpha, \beta]_r.$$

It is a formula for the r-bracket or classical r-matrix. If $r = Y \wedge X$ the assumptions of corollary can be weakened. In this case we obtain a Lie bracket if r satisfies the Yang-Baxter equation or some of its modifications (modified Yang-Baxter equation). It means that $r^\sharp : \mathfrak{g}^* \longrightarrow \mathfrak{g}$ given by $r^\sharp(\alpha)(\beta) = r(\alpha, \beta)$ satisfies the condition

$$\langle \alpha | [r^\sharp(\beta), r^\sharp(ad^*_Z \gamma)] \rangle + \langle \beta | [r^\sharp(ad^*_Z \gamma), r^\sharp(\alpha)] \rangle + \langle ad^*_Z \gamma | [r^\sharp(\alpha), r^\sharp(\beta)] \rangle = 0$$

for all $\alpha, \beta, \gamma \in \mathfrak{g}^*$ and $Z \in \mathfrak{g}$. Then we can think about the formula as a generalization of the notion of classical r-matrices by introducing a parameter $\lambda \in \mathbb{R}$.

Classical R-matrix

One of the important tools of the integrable systems theory is the so-called classical R-matrix. Given a Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$, a linear operator $R : \mathfrak{g} \longrightarrow \mathfrak{g}$ is called a classical R-matrix if the R–bracket

$$[X, Y]_R = \frac{1}{2}\left([R(X), Y] + [X, R(Y)]\right)$$

is a Lie bracket. The Lie algebra $\mathfrak{g}$ equipped with two Lie brackets: $[\cdot, \cdot]$ and R-bracket $[\cdot, \cdot]_R$ is called a double Lie algebra. A certain class of R-matrices can be obtained from the modified Yang-Baxter equation

$$R([R(X), Y] + [X, R(Y)]) - [R(X), R(Y)] = c[X, Y].$$

Generalization of the concept of classical r-matrix

Ultimately this concept can be extended to the level of arbitrary $r \in \mathfrak{g} \otimes \mathfrak{g}$. If we define mappings $\underline{r}, \overline{r} : \mathfrak{g}^* \longrightarrow \mathfrak{g}$ such that $\underline{r}(\alpha) = r(\alpha, \cdot)$, $\overline{r}(\alpha) = r(\cdot, \alpha)$ then we obtain the following generalization:

Theorem

Assume that the map r satisfies the condition

$$\langle \alpha | [\overline{r}(\gamma), \underline{r}(\beta)] \rangle + \langle \beta | [\underline{r}(\alpha), \overline{r}(\gamma)] \rangle$$

$$+ \langle \gamma | [\underline{r}(\alpha), \underline{r}(\beta)] \rangle = 0,$$

for all $\alpha, \beta, \gamma \in \mathfrak{g}^*$. Then

$$[\alpha, \beta]_{\underline{r}} = ad^*_{\underline{r}(\alpha)} \beta - ad^*_{\underline{r}(\beta)} \alpha$$

is a Lie bracket on $\mathfrak{g}^*$.

A new look at Lie algebras

Lie bracket has a form

$$[\alpha, \beta]_{X,Y} = \langle \beta, Y \rangle ad_X^* \alpha - \langle \alpha, Y \rangle ad_X^* \beta,$$

for $\alpha, \beta \in \mathfrak{g}^*$, where $X, Y \in \mathfrak{g}$ fulfill $[X, Y] = fY$. It means that Lie algebra structure of $\mathfrak{g}$ gives Lie algebra structure on $\mathfrak{g}^*$. However, as it was shown the bracket can be generalized to the linear space V^* to the form

$$[\alpha, \beta]_{(F,v)} = \beta(v)F^*(\alpha) - \alpha(v)F^*(\beta),$$

where $\alpha, \beta \in V^*, F \in End(V)$ and v is an eigenvector of F.

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Eigenvalue problem

We present some constructions of a Lie bracket on a space V^* having a pair: linear mapping and its eigenvector. A pair (F, v) gives a Lie bracket on a dual space V^*:

Theorem

If V is a vector space, $F : V \longrightarrow V$ is a linear map and $v \in V$ is an eigenvector of the map F, then $(V^*, [\cdot,\cdot]_{(F,v)})$, is a Lie algebra, where the Lie bracket is given by

$$[\psi, \phi]_{(F,v)} = \phi(v)F^*(\psi) - \psi(v)F^*(\phi)$$

for $\psi, \phi \in V^*$.

We can identify V and V^* with $\mathbb{R}^N$ with the canonical basis $\{e_1, e_2, \ldots, e_N\}$ (i.e. $V \simeq V^* \simeq \mathbb{R}^N$), so that the pairing between V and V^* is given by the scalar product. Then the Lie bracket can be rewritten in the form

$$[u, w]_{(F,v)} = \langle w|v\rangle F^T u - \langle u|v\rangle F^T w \quad \text{for } u, w \in \mathbb{R}^N,$$

where $\langle\cdot|\cdot\rangle$ is the scalar product in $\mathbb{R}^N$.

Cyclic Lie-Rinehart algebras

- $\langle\cdot,\cdot\rangle\colon L\times N\to R$ is a duality pairing of two R-modules L, N;
- $E_0\in End(L)$, $D_0\in End(N)$, $X_0\in Der(R)$ and $\ell\in L$ and $n, y_0\in N$ satisfying the conditions that the mapping

$$L\times L\to R,\quad (\ell_1,\ell_2)\mapsto\langle\ell_1,y_0\rangle\langle\ell_2,D_0(y_0)\rangle$$

is symmetric and we have

$$X_0(\langle\ell,n\rangle)=\langle E_0(\ell),n\rangle+\langle\ell,D_0(n)\rangle.$$

If we define the anchor

$$a\colon L\to T_R,\quad a(\ell):=\langle\ell,y_0\rangle X_0$$

then the mapping

$$L\times L\to L,\quad (\ell_1,\ell_2)\mapsto\ell_1\cdot\ell_2:=\nabla_{\ell_1}\ell_2=\langle\ell_1,y_0\rangle E_0(\ell_2)$$

defines the structure of a pre-Lie-Rinehart algebra on L ($[\ell_1,\ell_1]=\ell_1\cdot\ell_2-\ell_2\cdot\ell_1$).

field of real numbers: $\mathbb{R}$	$\mathbb{K}$
algebra of smooth functions: $C^\infty(M, \mathbb{R})$	R
Lie algebra of tangent vector fields: $\Gamma(TM)$	$Der(R)$
space of differential 1-forms: $\Omega^1(M)$	$Hom_R(Der(R), R)$

$$D_0 = ad_X = [X, \cdot],$$

$$E_0 = \pounds_X,$$

$$X_0 = X,$$

$$y_0 = Y,$$

$$[\alpha, \beta]_{X,Y} = \beta(Y)\pounds_X \alpha - \alpha(Y)\pounds_X \beta,$$
$$a_{X,Y}(\alpha) = -\alpha(Y)X.$$

$$D_0 = F,$$

$$E_0 = F^*,$$

$$X_0 = 0,$$

$$y_0 = v$$

$$[\psi, \phi]_{(F,v)} = \phi(v)F^*(\psi) - \psi(v)F^*(\phi).$$

Cyclic Lie-Rinehart algebras

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- $E_0\in End(L)$, $D_0\in End(N)$, $X_0\in Der(R)$ and $\ell\in L$ and $n,y_0\in N$ satisfying the conditions that the mapping

$$L\times L\to R,\quad (\ell_1,\ell_2)\mapsto\langle\ell_1,y_0\rangle\langle\ell_2,(D_0)^2(y_0)\rangle$$

is symmetric and we have

$$X_0(\langle\ell,n\rangle)=\langle E_0(\ell),n\rangle+\langle\ell,D_0(n)\rangle.$$

If we define the anchor

$$a\colon L\to T_R,\quad a(\ell):=\langle\ell,y_0\rangle X_0$$

then the mapping

$$L\times L\to L,\quad (\ell_1,\ell_2)\mapsto\ell_1\cdot\ell_2:=\nabla_{\ell_1}\ell_2=\langle\ell_1,y_0\rangle E_0(\ell_2)-\langle\ell_1,D_0(y_0)\rangle\ell_2$$

defines the structure of a pre-Lie-Rinehart algebra on L $([\ell_1,\ell_1]=\ell_1\cdot\ell_2-\ell_2\cdot\ell_1)$.

Example

In the special case $\mathbb{K} = \mathbb{R}$ and $R = C^\infty(\mathbb{R})$ we have $[X, [X, Y]] = cY$ for $X = \frac{d}{dt}$ and $Y = g\frac{d}{dt}$, $c(t) = t$ for all $t \in \mathbb{R}$.
For the R-module Ω^1_R the Lie bracket has the form

$$[f dt, h dt] = -g(t)\mathcal{D}_H(f, h)(t) dt$$

for all $f, h \in C^\infty(\mathbb{R})$, where $\mathcal{D}_H$ denotes Hirota's operator which acts on the pair of functions f and h in the following way

$$\mathcal{D}_H(f, h)(t) := \left(\frac{d}{dt} - \frac{d}{d\tilde{t}}\right) f(t)h(\tilde{t})\bigg|_{t=\tilde{t}} = f'(t)h(t) - f(t)h'(t).$$

This operator is used in the method of finding soliton solutions for non-linear equations, as for the example KdV.
Lie-Rinehart algebra structures on the space of differential forms Ω^1_R have been constructed before in the algebraic theory of Dirac structures, motivated by the study of integrability of certain nonlinear differential equations.

Thank you for your attention