Infinite Grassmannians and operator theory

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Third lecture: Homogeneous spaces related to Grassmannians

Metric structure of homogeneous spaces of unitary groups

Let $\mathcal{A}$ be a unital C^*-algebra, $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$.

Unitary group of $\mathcal{A}$:

$$\mathcal{U}_{\mathcal{A}} = \{U \in \mathcal{A} : UU^* = U^*U = I\}$$

Lie algebra:

$$\mathcal{A}^{ah} = \{X \in \mathcal{A} : X = -X^*\}$$

Theorem (Homogeneous spaces)

Let $\mathcal{B} \subseteq \mathcal{A}$ be a C^*- subalgebra of $\mathcal{A}$ such that $\mathcal{U}_{\mathcal{B}}$ is a Lie subgroup of $\mathcal{U}_{\mathcal{A}}$. Then the following assertions hold:

1. *$\mathcal{O} := \mathcal{U}_{\mathcal{A}}/\mathcal{U}_{\mathcal{B}}$ is a manifold endowed with the quotient topology*
2. *$\mathcal{U}_{\mathcal{A}}$ acts smoothly and transitively on $\mathcal{O}$ as follows*

$$\mathcal{U}_{\mathcal{A}} \times \mathcal{O} \to \mathcal{O}, \quad (W, [U]) \mapsto [WU]$$

3. *For every $[U] \in \mathcal{O}$ the following maps are submersions*

$$\pi_{[U]} : \mathcal{U}_{\mathcal{A}} \to \mathcal{O}, \quad \pi_{[U]}(W) = [WU]$$

Definition (Finsler metric on homogeneous spaces)

We consider homogeneous spaces $\mathcal{O} = \mathcal{U}_{\mathcal{A}}/\mathcal{U}_{\mathcal{B}}$ as above endowed with the following Finsler metric: for $[X] \in T_{[U]}\mathcal{O}$,

$$\|[X]\|_{[U]} := \inf\{\|Z\| : (d\pi_{[U]})_I(Z) = [X]\}$$

Remark: $(d\pi_{[U]})_I : \mathcal{A}^{ah} \to T_{[U]}\mathcal{O}$ is surjective, and $ker(d\pi_{[U]})_I = U\mathcal{B}^{ah}U^*$ is the Lie algebra of the isotropy group at $[U]$ given by

$$\mathcal{I}_{[U]} = \{W \in \mathcal{U}_{\mathcal{A}} : [UW] = [U]\} = U\mathcal{U}_{\mathcal{B}}U^*$$

Then,

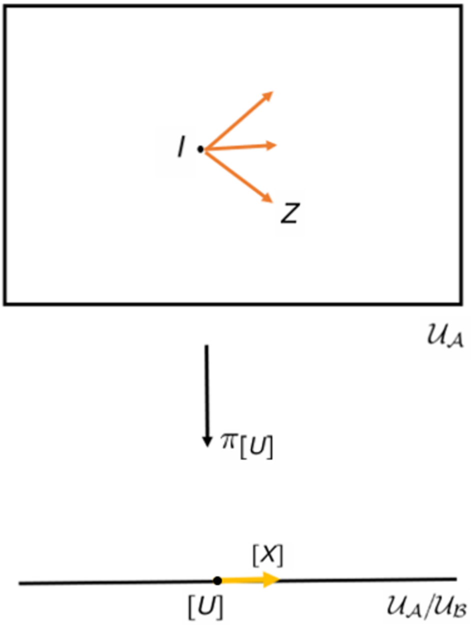
$$[X] \in T_{[U]}\mathcal{O} \simeq \mathcal{A}^{ah}/U\mathcal{B}^{ah}U^*, \quad X \in \mathcal{A}^{ah}$$

and

$$(d\pi_{[U]})_I(Z) = [X] \Longleftrightarrow Z = X + Y, \quad Y \in U\mathcal{U}_{\mathcal{B}}U^*$$

Alternative expression for the Finsler metric

$$\|[X]\|_{[U]} := \inf\{\|X + Y\| : Y \in U\mathcal{B}^{ah}U^*\}$$


I
Z
$\mathcal{U}_{\mathcal{A}}$
$\pi_{[U]}$
$[X]$
$[U]$
$\mathcal{U}_{\mathcal{A}}/\mathcal{U}_{\mathcal{B}}$

- Any $Z \in \mathcal{A}^{ah}$ is called a *lift* of $[X]$ if $(d\pi_{[U]})_I(Z) = [X]$.
- If, in addition, Z satisfies $\|Z\| = \|[X]\|_{[U]}$, then Z is called a *minimal lift* of $[X]$
- Length functional: let $\gamma : [0,1] \to \mathcal{O}$ be a piecewise C^1 curve

$$L(\gamma) = \int_0^1 \|\dot{\gamma}(t)\|_{\gamma(t)} dt$$

Theorem (Durán, Mata-Lorenzo and Recht '04)

Take $[U] \in \mathcal{O}$ and $[X] \in T_{[U]}\mathcal{O}$. If there exists $Z \in \mathcal{A}^{ah}$ minimal lift of $[X]$, then the curve $\delta(t) = [e^{tZ}U]$ has minimal length in the class of all the curves joining $\delta(0)$ and $\delta(t)$ for $|t| \leqslant \frac{\pi}{2\|Z\|}$

Theorem (Durán, Mata-Lorenzo and Recht '04)

If $\mathcal{B} \subseteq \mathcal{A}$ are von Neumann algebras, then minimal lifts do exist.

Remark: Minimal lifts may not exist in arbitrary C^*-algebras. When they exist, it is in general a difficult task to compute them explicitly.

Projections with fixed difference

$\mathcal{H}$ Hilbert space, $\mathcal{B}(\mathcal{H})$ algebra of bounded operators on $\mathcal{H}$

$\mathcal{P}(\mathcal{H})$ (orthogonal) projections on $\mathcal{H}$

Differences of projections

$$\mathcal{D} = \{P - Q : P, Q \in \mathcal{P}(\mathcal{H})\}$$

For $A \in \mathcal{D}$, set

$$\mathcal{D}_A = \{(P, Q) \in \mathcal{P}(\mathcal{H}) \times \mathcal{P}(\mathcal{H}) : P - Q = A\}$$

<u>Remark:</u>

$$\begin{aligned} ker(A^2 - I) &= ker(A - I) \oplus ker(A + I) \\ &= (ran(P) \cap ker(Q)) \oplus (ker(P) \cap ran(Q)) \end{aligned}$$

Furthermore,

$$\mathcal{H} = ker(A) \oplus ker(A^2 - I) \oplus \mathcal{H}_0$$

where $ker(A) = (ran(P) \cap ran(Q)) \oplus (ker(P) \cap ker(Q))$, and $\mathcal{H}_0$ is the *generic part*.

Remark (Factorization of $\mathcal{D}_A$)

Using the decomposition $\mathcal{H} = ker(A) \oplus ker(A^2 - I) \oplus \mathcal{H}_0$, the set $\mathcal{D}_A$ is factorized as

$$\mathcal{D}_A = \mathcal{P}(ker(A)) \oplus \{A_\pm\} \oplus D_{A_0}$$

where $\mathcal{P}(ker(A))$ are the projections on $ker(A)$, $A_\pm = P_{ker(A-I)} - P_{ker(A+I)}$ is symmetry on $ker(A^2 - I)$ and $A_0 = A|_{\mathcal{H}_0}$

Remark: That is, $(P, Q) \in \mathcal{D}_A$ is given by

$$P = E \oplus P_{ker(A-I)} \oplus P_0$$
$$Q = E \oplus P_{ker(A+I)} \oplus Q_0$$

where $E \in \mathcal{P}(ker(A))$, $P_0 = P|_{\mathcal{H}_0}$ and $Q_0 = Q|_{\mathcal{H}_0}$

Theorem (Davis '58)

There is a bijection between the following sets:

1. $\mathcal{D}_{A_0} = \{(P_0, Q_0) \in \mathcal{P}(\mathcal{H}_0) \times \mathcal{P}(\mathcal{H}_0) : P_0 - Q_0 = A_0\}$
2. $\{V = V^* = V^{-1} \in \mathcal{B}(\mathcal{H}_0) : A_0 V = -VA_0\}$ Davis thm

Corollary

Given $A \in \mathcal{B}(\mathcal{H})$, then $A = P - Q$ for some $P, Q \in \mathcal{P}(\mathcal{H}) \iff \exists\ V$ symmetry on $\mathcal{H}' = \ker(A^2 - I)^\perp$ such that $VA = -AV$ on $\mathcal{H}'$

Action on $\mathcal{D}_A$: Take $\mathcal{A} = \{X \in \mathcal{B}(\mathcal{H}) : XA = AX\} = \{A\}'$ the commutant of A, and the unitary group

$$\mathcal{U}_\mathcal{A} = \{U \in \mathcal{U}(\mathcal{H}) : UA = AU\}$$

Then, $\mathcal{U}_A$ acts on $\mathcal{D}_A$ as follows:

$$U \cdot (P, Q) = (UPU^*, UQU^*), \quad U \in \mathcal{U}_\mathcal{A}, (P, Q) \in \mathcal{D}_\mathcal{A}$$

Indeed, note

$$UPU^* - UQU^* = U(P - Q)U^* = UAU^* = A.$$

Put $\mathcal{A}_0 := \{X \in \mathcal{B}(\mathcal{H}_0) : XA_0 = A_0X\} = \{A_0\}'$.

Theorem (Andruchow, Corach, Recht '19)

$\mathcal{U}_{\mathcal{A}_0}$ acts transitively on $\mathcal{D}_{A_0}$

Corollary (Andruchow, Corach, Recht '19)

$\mathcal{D}_{A_0}$ is connected. The connected components of $\mathcal{D}_A$ are parametrized by the rank and corank of projections in $\mathcal{P}(\ker(A))$.

Geometric structure of the components of $\mathcal{D}_A$ can be reduced to that of $\mathcal{D}_{A_0}$

In the Hilbert space $\mathcal{H}_0 \simeq \mathcal{L} \times \mathcal{L}$,

$$P_0 \simeq \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad Q_0 \simeq \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix}$$

where $C = \cos(\Gamma)$ and $S = \sin(\Gamma)$, where $\Gamma^* = \Gamma$ and $\sigma(\Gamma) \subseteq [0, \frac{\pi}{2}]$

$$\Longrightarrow A_0 = P_0 - Q_0 \simeq \begin{pmatrix} S^2 & -CS \\ -CS & -S^2 \end{pmatrix}$$

Then, we obtain

$$\mathcal{A}_0 = \{A_0\}' = \left\{ \begin{pmatrix} X & Y \\ Y & Z \end{pmatrix} : X, Y, Z \in \{\Gamma\}',\ C(X - Z) + 2SY = 0 \right\}$$

We write $\mathcal{U}_{\mathcal{A}_0}$ for its unitary group

Isotropy group:

$$\mathcal{I}_{(P_0,Q_0)} = \{U \in \mathcal{U}_{\mathcal{A}_0} : UP_0U^* = P_0,\ UQ_0U^* = Q_0\}$$
$$= \left\{ \begin{pmatrix} W & 0 \\ 0 & W \end{pmatrix} : W \in \{\Gamma\}' \right\} =: \mathcal{U}_{\mathcal{B}}$$

is the unitary group of the von Neumann algebra

$$\mathcal{B} := \left\{ \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix} : X \in \{\Gamma\}' \right\}$$

Important remark: By the above charecterizations, it follows that the Lie algebra of $\mathcal{U}_{\mathcal{B}}$ (i.e. $\mathcal{B}^{ah}$) is closed and complemented in the Lie algebra of $\mathcal{U}_{\mathcal{A}_0}$ (i.e. $\mathcal{A}_0^{ah}$). Indeed, a continuous projection is given by

$$\mathcal{E}_{(P_0,Q_0)} : \mathcal{A}_0^{ah} \to \mathcal{A}_0^{ah}$$
$$\mathcal{E}_{(P_0,Q_0)}\left(\begin{pmatrix} X & Y \\ Y & Z \end{pmatrix} \right) = \begin{pmatrix} \frac{1}{2}(X+Z) & 0 \\ 0 & \frac{1}{2}(X+Z) \end{pmatrix}$$

Theorem (Andruchow, Corach, Recht '19)

The following assertions hold:

1. $\mathcal{D}_{A_0} \simeq \mathcal{U}_{\mathcal{A}_0}/\mathcal{U}_{\mathcal{B}}$ *is a homogeneous space*
2. *If $A_0 - I$ has closed range, then $\mathcal{D}_{A_0}$ is a submanifold of $\mathcal{B}(\mathcal{H}_0) \times \mathcal{B}(\mathcal{H}_0)$*

Remark: For the second item we construct a continuous local cross sections for the map induced by the action

$$\pi : \mathcal{U}_{\mathcal{A}_0} \to D_{A_0}, \quad \pi(U) = (UP_0U^*, UQ_0U^*).$$

The construction needs the partial isometry in the polar decomposition of the operators of the form $I - P - Q = V_{P,Q}|I - P - Q|$. The map associated to the polar decomposition is not continuous in general.

Curves of minimal length with respect to the quotient Finsler metric in $\mathcal{D}_{A_0}$?

Remarks:

1) $\mathcal{A}_0$ and $\mathcal{B}$ are both von Neumann algebras.

2) The following inequality holds:

$$\left\| \begin{pmatrix} \frac{1}{2}(X-Z) & Y \\ Y & \frac{1}{2}(Z-X) \end{pmatrix} \right\| \leqslant \left\| \begin{pmatrix} X & Y \\ Y & Z \end{pmatrix} + \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix} \right\|$$

for all D.

Minimal lifts have the form: $R = \begin{pmatrix} L & Y \\ Y & -L \end{pmatrix}$

Curve (locally) minimal along its path: $\delta(t) = (e^{tR}P_0e^{-tR}, e^{tR}Q_0e^{-tR})$

Furthermore, in this case we have the following

Theorem (Andruchow, Corach, Recht '19)

*Every pair of points in $\mathcal{D}_{A_0}$ can be joined by a **geodesic** of minimal length.*

Projections with fixed commutator

Commutators of projections

$$\mathcal{C} = \{[P, Q] : P, Q \in \mathcal{P}(\mathcal{H})\}$$

For $A \in \mathcal{C}$, set

$$\mathcal{C}_A = \{(P, Q) \in \mathcal{P}(\mathcal{H}) \times \mathcal{P}(\mathcal{H}) : [P, Q] = A\}$$

Theorem (Li '04)

A is a commutator of two orthogonal projections if and only if A is an antihermitian operator such that $\|A\| \leqslant \frac{1}{2}$ and A is unitarily equivalent to its adjoint.

Set $\mathcal{A} = \{A\}'$. The unitary group $\mathcal{U}_{\mathcal{A}}$ acts on $\mathcal{C}_A$ as follows

$$W \cdot (P, Q) = \big(WPW^*, WQW^*\big), \quad W \in \mathcal{U}_{\mathcal{A}}, (P, Q) \in \mathcal{C}_A.$$

Note

$$\big[UPU^*, UQU^*\big] = UPQU^* - UQP\big)U^* = UAU^* = A.$$

We consider orbits as follows:

$$\mathcal{O}_{(P,Q)} = \{\big(WPW^*, WQW^*\big) : W \in \mathcal{U}_{\mathcal{A}}\}$$

If $A = [P, Q]$, then

$$ker(A) = (ker(P) \cap ker(Q)) \oplus (ker(P) \cap ran(Q)) \oplus (ran(P) \cap ker(Q)) \oplus (ran(P) \cap ran(Q))$$

Assumption: From now on, we suppose $ker(A) = \{0\}$

Recall that $\mathcal{H} = ker(A) \oplus \mathcal{H}_0$ ('Two projections thm')

Fix $(P_0, Q_0) \in \mathcal{C}_A$. Then, after a unitary conjugation by the Two projections thm,

$$P_0 \simeq \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad Q_0 \simeq \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix},$$

where $C = \cos(\Gamma)$ and $S = \sin(\Gamma)$, where $\Gamma^* = \Gamma$ and $\sigma(\Gamma) \subseteq [0, \frac{\pi}{2}]$

After another conjugation,

$$P_0 \simeq \frac{1}{2}\begin{pmatrix} I & I \\ I & I \end{pmatrix}, \quad Q_0 \simeq \frac{1}{2}\begin{pmatrix} I & e^{-2i\Gamma} \\ e^{2i\Gamma} & I \end{pmatrix}$$

$$\Longrightarrow A = [P_0, Q_0] = \begin{pmatrix} iCS & 0 \\ 0 & -iCS \end{pmatrix}$$

$$\mathcal{U}_{\mathcal{A}} = \{W \in \mathcal{U}(\mathcal{H}) : UA = UA\} = \left\{ \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix} : V_1, V_2 \in \mathcal{U}_{\{CS\}'} \right\}$$

Isotropy group:

$$\mathcal{I}_{(P_0,Q_0)} = \{U \in \mathcal{U}_{\mathcal{A}} : UP_0U^* = P_0,\ UQ_0U^* = Q_0\}$$
$$= \left\{ \begin{pmatrix} W & 0 \\ 0 & W \end{pmatrix} : W \in \mathcal{U}_{\{\Gamma\}'} \right\} =: \mathcal{U}_{\mathcal{B}}$$

Theorem (E.C., M. Chaile '26)

$\mathcal{O}_{(P_0,Q_0)} \simeq \mathcal{U}_{\mathcal{A}}/\mathcal{U}_{\mathcal{B}}$ is a homogeneous space.

Remark:

1) $\{\Gamma\}' \subseteq \{CS\}'$. The reversed inclusion may not hold

2) The key point in the above thm is to give a (continuous) projection from $\{CS\}'$ onto $\{\Gamma\}'$

Theorem (E.C., M. Chaile '26)

The following hold:

1. *If $\{CS\}' = \{\Gamma\}'$, then the exists a minimal* ***geodesic*** *joining any pair of points.*
2. *If $\{CS\}' \neq \{\Gamma\}'$, then there exists (locally) minimal curves along their path.*

Remark: There is a characterization of when $\{CS\}' = \{\Gamma\}'$ in terms of the spectral measure of Γ

Thanks for your attention

- Given $P_0 - Q_0 = A_0$, take $S = P_0 + Q_0 - I = P_0 - (I - Q_0)$, then

$$ker(S) = (ran(P_0) \cap ker(Q_0)) \oplus (ker(P_0) \cap ran(Q_0)) = \{0\}.$$

The polar decomposition $S = V|S| = |S|V$ gives a symmetry $V = \mathrm{sign}(S)$ such that $VP_0 = Q_0V$ and $VQ_0 = P_0V$ because $SP_0 = Q_0S$ and $SQ_0 = P_0S$

$$VA_0 = V(P_0 - Q_0) = (Q_0 - P_0)V = -A_0V.$$

- Conversely, take V symmetry on $\mathcal{H}_0$ such that $VA_0 = -A_0V$. Define

$$P_V = \frac{1}{2}(I + A_0 + (I - A_0)^{1/2}V), \quad Q_V = \frac{1}{2}(I - A_0 + (I - A_0)^{1/2}V)$$

It can be proved that $P_V, Q_V \in \mathcal{P}(\mathcal{H}_0)$ and $A_0 = P_V - Q_V$

- The correspondence is indeed a bijection Back