Infinite Grassmannians and operator theory

Eduardo Chiumiento

Centro de Matemática La Plata- UNLP

Instituto Argentino de Matemática
"Alberto P. Calderón"

School on Geometry and Physics

Białystok (2026)

Second lecture: Examples

Shift-invariant subspaces

$$\mathbb{D} = \{ z \in \mathbb{C} \, : \, |z| < 1 \}, \quad \mathbb{T} = \partial \mathbb{D}$$

Hardy space:

$$H^2 = H^2(\mathbb{D}) = \left\{ f \in H(\mathbb{D}) \, : \, \|f\|_{H^2} := \left(\sup_{0 \leqslant r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 dt \right)^{1/2} < \infty \right\}$$

$$\simeq \left\{ f \in L^2(\mathbb{T}) \, : \, f = \sum_{n \geqslant 0} f_n e^{int} \right\}$$

Shift operator:

$$S : L^2(\mathbb{T}) \to L^2(\mathbb{T}), \quad (Sf)(e^{it}) = e^{it} f(e^{it})$$

Theorem (Beurling-Helson)

E closed subspace of $L^2(\mathbb{T})$, $S(E) \subsetneq E \Longrightarrow E = \varphi H^2$, $|\varphi| = 1$ a.e.

¿Geodesics joining shift-invariant subspaces of $\mathcal{P}(L^2(\mathbb{T}))$? Given unimodular functions φ, ψ, we have to understand if

$$dim(\varphi H^2 \cap (\psi H^2)^{\perp}) = dim(\psi H^2 \cap (\varphi H^2)^{\perp})$$

$L^2 = L^2(\mathbb{T})$, $L^\infty = L^\infty(\mathbb{T})$, $P_+ =$ projection onto H^2

Toeplitz operator

Given $\varphi \in L^\infty$, $T_\varphi : H^2 \to H^2$, $T_\varphi(f) = P_+(\varphi f)$

Infinite matrix

Properties: $\|T_\varphi\| = \|\varphi\|_\infty$; $T_\varphi^* = T_{\overline{\varphi}}$; $ker(T_{\varphi\overline{\psi}}) \simeq \psi H^2 \cap (\varphi H^2)^\perp$

Theorem (Coburn's lemma)

$0 \neq \varphi \in L^\infty \implies ker(T_\varphi) = \{0\}$ *or* $ker(T_\varphi^*) = \{0\}$

Theorem (E. Andruchow, E.C., G. Larotonda)

Let φ, ψ be unimodular functions. The following are equivalent:

1. $ker(T_{\varphi\overline{\psi}}) = ker(T_{\overline{\varphi}\psi}) = \{\,0\,\}$
2. *There exists a **unique** minimal geodesic in $\mathcal{P}(L^2)$ joining φH^2 and ψH^2*

Injectivitiy of Toeplitz operators

- M. Lee, D. Sarason (J. Math. Anal. Appl. 33, 529-523 (1971))
- N. Makarov, A. Poltoratski (Invent. Math. 180, 443-480 (2010))

Sufficient conditions:

$$T_{\varphi\overline{\psi}} \text{ invertible} \Longrightarrow ker(T_{\varphi\overline{\psi}}) = ker(T_{\overline{\varphi}\psi}) = \{0\}$$

The converse " $\Longleftarrow$ " does not hold (example later)

Notation: $C = C(\mathbb{T})$, $\quad H^\infty := L^\infty \cap H^2$

Sarason algebra:

$$H^\infty + C = \{ f + g \, : \, f \in H^\infty, \, g \in C \}$$

- $H^\infty \subseteq H^\infty + C \subseteq L^\infty$ are subalgebras
- Well-defined notion of index for an invertible $f \in H^\infty + C$

Theorem (Invertible Toeplitz operators)

Given $f \in L^\infty$, then

1. T_f *is invertible* $\iff$ T_f *is Fredholm of zero index*
2. *If* $f \in H^\infty + C$, T_f *invertible* $\iff$ f *invertible in* $H^\infty + C$, $ind(f) = 0$

$$\begin{aligned} ind(\varphi\overline{\psi}) &= -ind(T_{\varphi\overline{\psi}}) = dim\,ker(T_{\psi\overline{\varphi}}) - dim\,ker(T_{\varphi\overline{\psi}}) \\ &= dim(\varphi H^2 \cap (\psi H^2)^\perp) - dim(\psi H^2 \cap (\varphi H^2)^\perp) \end{aligned}$$

Corollary (E. Andruchow, E.C., G. Larotonda)

Let φ, ψ be invertible functions in $H^\infty + C$. Then, we have the following:

$ind(\varphi) = ind(\psi) \iff \exists!$ *minimal geodesic in* $\mathcal{P}(L^2)$ *joining* φH^2 *and* ψH^2

Examples:

- $z\left(\frac{(z-1/2)}{1-z/2}\right)H^2$ and $z^{-1}\left(\frac{(z-1/3)}{1-z/3}\right)\left(\frac{(z-1/4)}{1-z/4}\right)\left(\frac{(z-1/5)}{1-z/5}\right)H^2 \rightsquigarrow$ $\exists$ geodesic
- zH^2 and $z^{-1}H^2 \rightsquigarrow$ $\nexists$ geodesic

$f \in H^2$ is *inner* if $|f| = 1$ a.e.

$f \in H^2$ is *outer* if $\overline{\text{span}}\{\, f\chi_n \,:\, n \geqslant 0\,\} = H^2$, where $\chi_n(e^{it}) = e^{int}$

Theorem (Inner-outer factorization)

For $f \in H^2$, $f = f_{int} f_{out}$ (unique up to a constant)

Inner functions can still be factorized...

Blaschke products: $a_1, \ldots, a_n \in \mathbb{D}\backslash\{\,0\,\}$

$$b(z) = z^k \prod_{j=1}^{n} \frac{\overline{a_j}}{|a_j|} \frac{a_j - z}{1 - \overline{a_j} z}, \quad z \in \mathbb{D}, \quad k \geqslant 0.$$

For infinite points $a_1, a_2, \ldots$, we have the *Blaschke condition*

$$\text{Convergent product} \iff \sum_{j=1}^{\infty} (1 - |a_j|) < \infty$$

Remark: Blaschke products are inner functions with zeros $\{a_j\}_{j \geqslant 1}$

Singular functions: Let μ be a finite positive measure on $\mathbb{T}$ and μ singular with respect to the Lebesgue measure

$$s_\mu(z) = \exp\left(-\int_{\mathbb{T}} \frac{w+z}{w-z} d\mu(w)\right)$$

Remark: These are inner functions and $s_\mu \neq 0$ en $\mathbb{D}$

Example: If μ is supported on $\{1\}$, then $s_\mu(z) = e^{\frac{z+1}{z-1}}$

Theorem (Canonical factorization)

Every $f \in H^2$ is factorized as

$$f = \lambda\, b\, s_\mu f_{out}\,,$$

where $\lambda \in \mathbb{T}$, b Blaschke product, s_μ singular and f_{out} outer part of f

For φ inner function, $z \in sup(\varphi) \subseteq \mathbb{T}$ if z is a limit point of zeros of φ or z is in the support of μ

Theorem (M. Lee - D. Sarason '71)

Let φ, ψ be inner functions

1. $sup(\varphi)\backslash sup(\psi) \neq \emptyset \implies ker(T_{\varphi\overline{\psi}}) = \{0\}$
2. $sup(\varphi) \neq sup(\psi) \implies \sigma(T_{\varphi\overline{\psi}}) = \overline{\mathbb{D}}$

Example: there exists geodesic between

$$\varphi H^2 = \prod_{j=1}^{\infty}(-1)\frac{-1 + 1/j^2 - z}{1 - (-1 + 1/j^2)z}H^2 \qquad \text{and} \qquad \psi H^2 = e^{\frac{z+1}{z-1}}H^2$$

Remark: $T_{\varphi\overline{\psi}}$ injective, dense range and non invertible

Zero sets in RKHS

Now we take $\mathcal{H}$ *reproducing kernel Hilbert space (RKHS)* consisting of complex-valued functions on a set X. That is, for each $w \in X$, there exists $k_w \in \mathcal{H}$ such that

$$f(w) = \langle f, k_w \rangle, \quad \forall f \in \mathcal{H}.$$

The function k_w is the *reproducing kernel for* $w \in \mathcal{H}$, and the two-variables function $K(z,w) = k_w(z)$ is the *reproducing kernel for* $\mathcal{H}$

Examples:

a) **Hardy space of the disk:** $\mathcal{H} = H^2 = H^2(\mathbb{D}), \quad X = \mathbb{D}$

$$H^2 = \left\{ f \in H(\mathbb{D}) : \|f\|_{H^2} := \left(\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 dt \right)^{1/2} < \infty \right\}$$
$$= \left\{ f \in H(\mathbb{D}) : \ f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ satisfying } \sum_{n=0}^{\infty} |a_n|^2 < \infty \right\}$$

The reproducing kernel for H^2 is given by

$$K(z,w) = \frac{1}{1 - \bar{w}z}, \quad z, w \in \mathbb{D}, \qquad \text{'Szegő kernel'}$$

b) **Bergman space of the disk:** $\mathcal{H} = A^2 = A^2(\mathbb{D}), \quad X = \mathbb{D}$

$$A^2 = \left\{ f \in H(\mathbb{D}) : \|f\|_{A_2} := \left(\int_{\mathbb{D}} |f(z)|^2 dxdy \right)^{1/2} < \infty \right\}, \quad z = x + iy$$

$$= \left\{ f \in H(\mathbb{D}) : \; f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ satisfying } \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1} < \infty \right\}$$

The reproducing kernel for A^2 is given by $K(z,w) = \frac{1}{(1-\bar{w}z)^2}, \quad z, w \in \mathbb{D}$

c) **Segal-Bargmann space of the plane:** $\mathcal{H} = \mathcal{F}^1 = \mathcal{F}^1(\mathbb{C}), \quad X = \mathbb{C}$

$$\mathcal{F}^1 = \left\{ f \in H(\mathbb{C}) : \|f\|_{\mathcal{F}^1} := \left(\int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2} dxdy \right)^{1/2} < \infty \right\}$$

$$= \left\{ f \in H(\mathbb{C}) : \; f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ satisfying } \sum_{n=0}^{\infty} n! \, |a_n|^2 < \infty \right\}$$

The reproducing kernel for $\mathcal{F}^1$ is given by $K(z,w) = e^{z\bar{w}}, \quad z, w \in \mathbb{C}$

Geodesics joining zero sets in $\mathcal{P}(\mathcal{H})$? Given $a = \{a_j\}_{j=1}^N$ and $b = \{b_j\}_{j=1}^M$ points in X, we study if there is a geodesic in $\mathcal{P}(\mathcal{H})$ joining the following *zero sets in* $\mathcal{H}$

$$Z_a = \{f \in \mathcal{H} : f(a_j) = 0,\ j = 1, \ldots, N\}$$

and

$$Z_b = \{f \in \mathcal{H} : f(b_j) = 0,\ j = 1, \ldots, M\}$$

Proposition (E. Andruchow, E.C., A. Varela '21)

The following assertions hold:

1. *There exists a minimal geodesic joining Z_a and Z_b if and only if $N = M$*
2. *If $N = M$, then the minimal geodesic is unique if and only if*

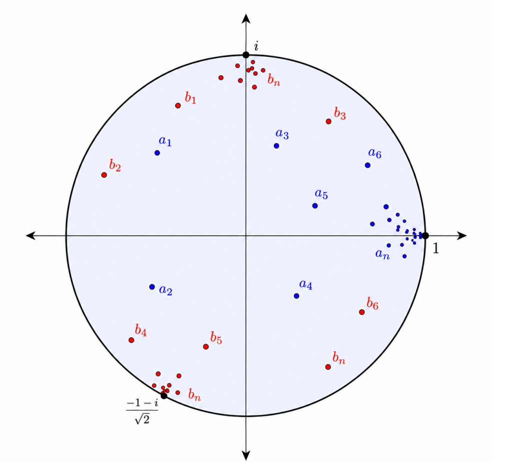
$$\det \left(K(a_j, b_i)\right)_{i,j=1,\ldots,N} \neq 0.$$

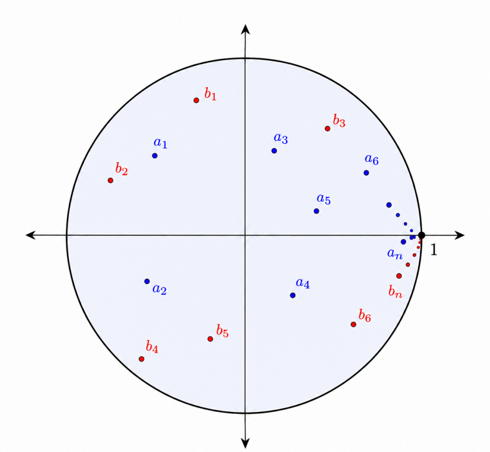
Otherwise, there are infinitely many minimal geodesics joining the subspaces.

Case of infinitely many points? Take two infinite sequences $a = \{a_j\}_{j \geqslant 1}$ and $b = \{b_j\}_{j \geqslant 1}$ in X. We consider only the RKHS space $\mathcal{H} = H^2(\mathbb{D})$ (Hardy space)

Remark: $a = \{a_j\}_{j \geqslant 1}$ and $b = \{b_j\}_{j \geqslant 1}$ in $\mathbb{D}$ must satisfy Blaschke condition

$$\sum_{j=1}^{\infty}(1 - |a_j|) < \infty \quad \text{and} \quad \sum_{j=1}^{\infty}(1 - |b_j|) < \infty$$





Then we rewrite the zero sets as follows

$$Z_a = B_a H^2 \quad \text{and} \quad Z_b = B_b H^2,$$

where B_a and B_b are the Blaschke products associated to a and b, respectively.

Remark

$\{z \in \mathbb{T} : z \textit{ is limit point of } a\} \cap \{z \in \mathbb{T} : z \textit{ is limit point of } b\} = \varnothing \implies$
$\exists!$ *minimal geodesic joining Z_a and Z_b (Sarason Lee)*

Theorem (E. Andruchow, E.C., A. Varela '21)

Given $n \in \{0, \ldots, \infty\}$, there exist two disjoint sequences $a = \{a_j\}_{j \geqslant 1}$ and $b = \{b_j\}_{j \geqslant 1}$ in $\mathbb{D}$ with the following properties:

- *a and b satisfy Blaschke condition.*
- $\lim a_j = \lim b_j = 1$
- *$\dim(Z_a \cap Z_b^{\perp}) = 0$ and $\dim(Z_a^{\perp} \cap Z_b) = n$*

In particular, there exists a (unique) geodesic joining Z_a and Z_b if and only if $n = 0$.

Uncertainty principle

Theorem (D. Donoho, P. Stark'89)

Let $I, J \subseteq \mathbb{R}^n$ be Lebesgue-measurable sets of finite measure and let $f \in L^2(\mathbb{R}^n)$, $\|f\|_2 = 1$, be such that

$$\int_{\mathbb{R}^n \setminus I} |f(x)|^2 dx \leqslant \epsilon_I, \qquad \int_{\mathbb{R}^n \setminus J} |\hat{f}(\xi)|^2 d\xi \leqslant \epsilon_J.$$

Then,

$$|I|\,|J| \geqslant (1 - \epsilon_I - \epsilon_J)^2$$

We then consider P_I and Q_J the projections on $L^2(\mathbb{R}^n)$ defined by

$$P_I f = \chi_I f, \qquad Q_J f = (\chi_J \hat{f})^{\check{}}$$

where χ_L is the characteristic function of the set L

Remark: The proof of the above theorem has two main stages

1. $\sqrt{|I|\,|J|} = \|P_I Q_J\|_{HS}$ (Hilbert-Schmidt norm)

2. $\|P_I Q_J\| \geqslant 1 - \epsilon_I - \epsilon_J$

Theorem (E. Andruchow, G. Corach)

Let I, J be measurable subsets of $\mathbb{R}^n$ with finite measure. Then there exists a unique self-adjoint operator $X_{I,J}$ satisfying the following:

1. $\|X_{I,J}\| = \frac{\pi}{2}$
2. $X_{I,J}$ *is* P_I*-codiagonal*
3. $e^{iX_{I,J}} P_I e^{-iX_{I,J}} = Q_J$
4. *If* $\gamma : [0,1] \to \mathcal{P}(L^2(\mathbb{R}^n))$ *is a* C^1 *piecewise curve joining* P_I *and* Q_J*, then*

$$L(\gamma) = \int_0^1 \|\dot{\gamma}(t)\| dt \geqslant \frac{\pi}{2}$$

<u>Ideas:</u> Apply E. Andruchow's characterization together with

$$ran(P_I) \cap ker(Q_J) = ker(P_I) \cap ran(Q_J) = \{0\}, \quad \|P_I - Q_J\| = 1.$$

These conditions were proved by A. Lenard '72

Corollary (E. Andruchow, G. Corach)

With the previous notations, we have

- $\|P_I Q_J\| = \cos(\gamma(X_{I,J}))$
- $\sqrt{|I|\,|J|} = \|P_I Q_J\|_{HS} = \|\cos(X)\|_{HS}$, *where* $X^* = X$ *is defined by*

$$X_{I,J}|_{ker(X_{I,J})^{\perp}} = \begin{pmatrix} 0 & iX \\ -iX & 0 \end{pmatrix}$$

- *Furthermore, the spectrum of* $X_{I,J}$ *has the form*

$$\sigma(X_{I,J}) = \left\{-\frac{\pi}{2}, 0, \frac{\pi}{2}\right\} \cup \{\mu_n : n \geqslant 1\} \cup \{-\mu_n : n \geqslant 1\},$$

where 0 *is an infinite eigenvalue,* $\mu_n > 0$ *are finite eigenvalues and* $\mu_n \nearrow \frac{\pi}{2}$.

Thanks for your attention

Note that $\left\{\frac{e^{int}}{\sqrt{2\pi}}\right\}_{n\in\mathbb{Z}}$ is an orthonormal basis of L^2 and $\left\{\frac{e^{int}}{\sqrt{2\pi}}\right\}_{n\geqslant 0}$ is an orthonormal basis of H^2.

For $\varphi \in L^\infty$,

$$\varphi = \sum_{n\in\mathbb{Z}} \varphi_n \frac{e^{int}}{\sqrt{2\pi}}$$

the Toeplitz operator is given by

$$T_\varphi = \begin{pmatrix} \varphi_0 & \varphi_{-1} & \varphi_{-2} & \varphi_{-3} & \cdots \\ \varphi_1 & \varphi_0 & \varphi_{-1} & \varphi_{-2} & \cdots \\ \varphi_2 & \varphi_1 & \varphi_0 & \varphi_{-1} & \cdots \\ \varphi_3 & \varphi_2 & \varphi_1 & \varphi_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \end{pmatrix}$$

Back