Infinite Grassmannians and operator theory

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General framework: Geometric aspects of Grassmannians in operator theory

- Operator theory: projection, idempotent, partial isometry, Toeplitz operator
- Manifolds: Banach-Lie group, homogeneous space, Finsler metric, geodesic

These lectures are based on a book in preparation with

Esteban Andruchow

Gustavo Corach

Topics:

First lecture: The Grasmannian of a Hilbert space

Second lecture: Examples

Third lecture: Homogeneous spaces related to Grassmannians

First lecture: The Grassmannian of a Hilbert space

Miscellaneous facts

Let $\mathcal{H}$ be a separable infinite-dimensional complex Hilbert space, and $\mathcal{B}(\mathcal{H})$ the algebra of bounded linear operators acting on $\mathcal{H}$

Definition (Grassmannian of a Hilbert space)

$$Gr(\mathcal{H}) = \{\mathcal{S} \subseteq \mathcal{H} : \mathcal{S} \textit{ is a closed subspace of } \mathcal{H}\}.$$

Remark: From now on we say '*subspaces*' to refer to '*closed subspaces*' of $\mathcal{H}$

Definition (Projections on a Hilbert space)

$$\mathcal{P}(\mathcal{H}) = \{P \in \mathcal{B}(\mathcal{H}) : P = P^2 = P^*\}$$

Notation: $P_{\mathcal{S}}$ is the orthogonal projection onto $\mathcal{S}$

Identification: $Gr(\mathcal{H}) \simeq \mathcal{P}(\mathcal{H})$ via the map

$$Gr(\mathcal{H}) \to \mathcal{P}(\mathcal{H}), \quad \mathcal{S} \to P_{\mathcal{S}}$$

Distance: If $\mathcal{S}, \mathcal{T} \in Gr(\mathcal{H})$, then

$$d(\mathcal{S}, \mathcal{T}) := \|P_{\mathcal{S}} - P_{\mathcal{T}}\|,$$

where $\|\cdot\|$ is the operator norm.

- $Gr(\mathcal{H})$ is a smooth manifold, whose tangent space at $\mathcal{S}$ is

$$T_{\mathcal{S}} Gr(\mathcal{H}) \simeq \mathcal{B}(\mathcal{S}, \mathcal{S}^{\perp}), \;\; \mathcal{S} \in Gr(\mathcal{H})$$

- $\mathcal{P}(\mathcal{H})$ is a smooth submanifold of $\mathcal{B}(\mathcal{H})_{sa}$, whose tangent space at $P = P_{\mathcal{S}}$ is

$$\begin{aligned} T_P \mathcal{P}(\mathcal{H}) &\simeq \{XP - PX : X = -X^*\} \\ &= \left\{ \begin{pmatrix} 0 & X_{12} \\ X_{12}^* & 0 \end{pmatrix} : X_{12} \in \mathcal{B}(\mathcal{S}^{\perp}, \mathcal{S}) \right\} \end{aligned}$$

- Actions of the unitary group $\mathcal{U}(\mathcal{H})$ of the Hilbert space $\mathcal{H}$

$$\mathcal{U}(\mathcal{H}) \curvearrowright Gr(\mathcal{H}) \quad U \cdot \mathcal{S} = U(\mathcal{S}), \, U \in \mathcal{U}(\mathcal{H}), \, \mathcal{S} \in Gr(\mathcal{H})$$

$$\mathcal{U}(\mathcal{H}) \curvearrowright \mathcal{P}(\mathcal{H}) \quad U \cdot P = UPU^*, \, U \in \mathcal{U}(\mathcal{H}), \, P \in \mathcal{P}(\mathcal{H})$$

- The map

$$Gr(\mathcal{H}) \to \mathcal{P}(\mathcal{H}), \ \ \mathcal{S} \to P_{\mathcal{S}},$$

is a diffeomorphism. Furthermore, the actions are compatible, one has $P_{U(\mathcal{S})} = UP_{\mathcal{S}}U^*$, for $U \in \mathcal{U}(\mathcal{H})$ and $\mathcal{S} \in Gr(\mathcal{H})$.

- $\|P - Q\| \leq 1$, for all $P, Q \in \mathcal{P}(\mathcal{H})$

- If $\|P - Q\| < 1$, then there exists $U \in \mathcal{U}(\mathcal{H})$ such that $Q = UPU^*$.

- For $m, n \in \{0, 1, \ldots, \infty\}$ such that $m + n = \infty$, the sets

$$\mathcal{P}_{m,n} = \{P \in \mathcal{P}(\mathcal{H}) : dim\, ran(P) = m, \ dim\, ker(P) = n\}$$

are the connected components of $\mathcal{P}(\mathcal{H})$, which coincide with the orbits of the above action

Geodesics of minimal length joining two projections

- Projection onto tangent spaces:

$$E_P : \mathcal{B}(\mathcal{H})_{sa} \to T_P\mathcal{P}(\mathcal{H}), \quad E_P\left(\begin{pmatrix} x_{11} & x_{12} \\ x_{12}^* & x_{22} \end{pmatrix}\right) = \begin{pmatrix} 0 & x_{12} \\ x_{12}^* & 0 \end{pmatrix}$$

- Linear connection: $X(t)$ tangent field along $\delta(t) \subseteq \mathcal{P}(\mathcal{H})$

$$\frac{DX}{dt} = E_{\delta(t)}(\dot{X}(t))$$

Proposition (Geodesics in $\mathcal{P}(\mathcal{H})$)

*The **geodesic** δ such that $\delta(0) = P$, $\dot{\delta}(0) = XP - PX = \begin{pmatrix} 0 & x_{12} \\ x_{12}^* & 0 \end{pmatrix}$ is given by*

$$\delta(t) = e^{t\begin{pmatrix} 0 & -x_{12} \\ x_{12}^* & 0 \end{pmatrix}} P \, e^{-t\begin{pmatrix} 0 & -x_{12} \\ x_{12}^* & 0 \end{pmatrix}}$$

- Length functional: for a piecewise C^1 curve $\gamma : [0,1] \to \mathcal{P}(\mathcal{H})$, set

$$L(\gamma) = \int_0^1 \|\dot\gamma(t)\| \, dt. \qquad (\| \cdot \| \text{ is the operator norm})$$

- Geodesic distance:

$$d_g(P,Q) := \inf\{L(\gamma) : \gamma \text{ piecewise } C^1 \text{ joining } P \text{ and } Q\}$$

Remark

$(\mathcal{P}(\mathcal{H}), d_g)$ is a complete metric space.

Some questions we will consider:

1)Existence of geodesics of minimal length joining two projections?

2) Local existence?

4) Uniqueness?

4) How many geodesic segments are needed to join two given projections?

Remark: Recall that $\|P - Q\| \leq 1, \forall\, P, Q \in \mathcal{P}(\mathcal{H})$

Theorem (H. Porta, L. Recht '87)

$\|P - Q\| < 1 \implies$ *There is a unique minimal geodesic joining P and Q* Thm 1 proof

Theorem (E. Andruchow '14)

Given $P, Q \in \mathcal{P}(\mathcal{H})$, the following are equivalent:

1. *There is a minimal geodesic joining P and Q*
2. $dim(ran(P) \cap ker(Q)) = dim(ker(P) \cap ran(Q))$

Moreover, there is unique minimal geodesic iff the above dimension is zero. If this condition does not hold, then there are infinitely many geodesics joining the two given projections Thm 2 proof

Remark: Set $\mathcal{S} = ran(P)$ and $\mathcal{T} = ran(Q)$. Then, the third condition is rewritten as

$$dim\,(\mathcal{S} \cap \mathcal{T}^{\perp}) = dim\,(\mathcal{S}^{\perp} \cap \mathcal{T})$$

Subspaces with or without common complement

Notation: $\dot{+}$ means direct sum (not necessarily orthogonal)

Definition (Common complemented subspaces of a Hilbert space)

Let $\mathcal{H}$ be a Hilbert space. Two subspaces $\mathcal{S}$ and $\mathcal{T}$ of $\mathcal{H}$ have a ***common complement*** *in $\mathcal{H}$ if there exists a subspace $\mathcal{Z}$ of $\mathcal{H}$ such that*

$$\mathcal{S} \dot{+} \mathcal{Z} = \mathcal{T} \dot{+} \mathcal{Z} = \mathcal{H}$$

Definition (Idempotents on a Hilbert space)

$$\mathcal{Q}(\mathcal{H}) = \{Q \in \mathcal{B}(\mathcal{H}) : Q = Q^2\}$$

- $\mathcal{S} \dot{+} \mathcal{Z} = \mathcal{H} \iff \exists Q \in \mathcal{Q}(\mathcal{H})$ such that $ran(Q) = \mathcal{S}$ and $ker(Q) = \mathcal{Z}$

Notation: $Q = P_{\mathcal{S}||\mathcal{Z}}$ is the (unique) idempotent above described

- $\mathcal{Q}_{m,n}$, where $m + n = \infty$, are the connected components of $\mathcal{Q}(\mathcal{H})$

Notation: $P \sim Q$ whenever P, Q lie in the same connected component

Proposition (M. Lauzon, S. Treil '04)

$\mathcal{S}$ and $\mathcal{T}$ have a common complement if and only if there exists an idempotent E onto one of the subspaces, say for instance $\mathcal{T}$, such that

$$E\big|_{\mathcal{S}} : \mathcal{S} \to \mathcal{T}$$

is an isomorphism.

- For finite-dimensional subspaces $\mathcal{S}, \mathcal{T}$, we have that $\mathcal{S}$ and $\mathcal{T}$ have a common complement if and only if $\mathcal{S}$, $\mathcal{T}$ belong to the same connected component
- Analogous statement for finite-codimensional subspaces
- The interesting component is $\mathcal{P}_\infty = \mathcal{P}_{\infty,\infty}$

Let $\mathcal{S}$, $\mathcal{T}$ be two closed subspaces of $\mathcal{H}$. The *Gramian operator* $G = G_{\mathcal{S},\mathcal{T}} : \mathcal{S} \to \mathcal{T}$ associated to these subspaces is defined by

$$G := P_{\mathcal{T}}|_{\mathcal{S}}.$$

Notice that $G^* : \mathcal{T} \to \mathcal{S}$ is given by $G^* = P_{\mathcal{S}}|_{\mathcal{T}}$.

Theorem (M. Lauzon, S. Treil '04)

Let $\mathcal{H}$ be a (separable) Hilbert space. Let $\mathcal{S}$ and $\mathcal{T}$ be two closed subspaces of $\mathcal{H}$, and let $G = G_{\mathcal{S},\mathcal{T}}$ be the Gramian operator. The following are equivalent.

i) *The operator $(I - G^*G)|_{ker(G)^\perp} : ker(G)^\perp \to ker(G)^\perp$ is not compact, or $dim\,(\mathcal{S}^\perp \cap \mathcal{T}) = dim\,(\mathcal{S} \cap \mathcal{T}^\perp)$.*

ii) *Let E_{G^*G} be the spectral measure of G^*G. Then*

$$dim\,(\mathcal{S}^\perp \cap \mathcal{T}) + dim\, ran(E_{G^*G}((0,a))) = dim\,(\mathcal{S} \cap \mathcal{T}^\perp) + dim\, ran(E_{G^*G}((0,a)))$$

for some $a \in (0,1)$.

iii) *$\mathcal{S}$ and $\mathcal{T}$ have a common complement in $\mathcal{H}$.*

Next we use the following conventions:

$$\begin{array}{ccc} & & Q \\ & \diagup & \\ P & & \end{array} \qquad \text{if } \mathit{ran}(P) = \mathit{ran}(Q)$$

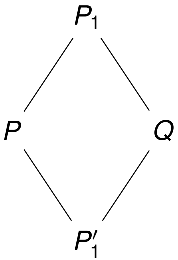
and

$$\begin{array}{ccc} P & & \\ & \diagdown & \\ & & Q \end{array} \qquad \text{if } \mathit{ker}(P) = \mathit{ker}(Q)$$

Let P and Q be idempotents, $P \sim Q$.

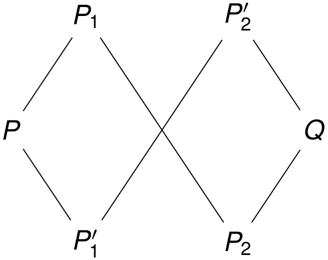
The **counting function** $s(P, Q)$ is defined in the following three cases.

(i) $s(P, Q) \leq 2$ if there exist two idempotents P_1, P_1' such that the following diagram holds:



$s(P, Q) \leq 2$

(ii) $s(P, Q) \leq 3$ if there exist four idempotents P_1, P_1', P_2, P_2' such that the following diagram holds:



$s(P, Q) \leq 3$

(iii) $s(P, Q) \leq 4$ if there exist six idempotents $P_1, P_1', P_2, P_2', P_3, P_3'$ such that the following diagram holds:

$$\begin{tikzcd}[column sep=small]
 & P_1 \arrow[ddrr, dash] & & P_2' \arrow[ddrr, dash] & & P_3 \arrow[dr, dash] & & \\
P \arrow[ur, dash] \arrow[dr, dash] & & & & & & Q & s(P, Q) \leq 4 \\
 & P_1' \arrow[uurr, dash] & & P_2 \arrow[uurr, dash] & & P_3' \arrow[ur, dash] & &
\end{tikzcd}$$

Theorem (J. Giol '05)

Let $\mathcal{H}$ be a (separable) Hilbert space. Let $\mathcal{S}$ and $\mathcal{T}$ be two subspaces of $\mathcal{H}$. Then the following are equivalent.

(i) $\|P_{\mathcal{S}} - P_{\mathcal{T}}\| < 1$

(ii) $s(P_{\mathcal{S}}, P_{\mathcal{T}}) \leq 2$

(iii) $\mathcal{S} \dot{+} \mathcal{T}^{\perp} = \mathcal{H}$

Theorem (J. Giol '05)

Let $\mathcal{H}$ be a (separable) Hilbert space. Let $\mathcal{S}$ and $\mathcal{T}$ be two closed subspaces of $\mathcal{H}$. Then the following are equivalent.

(i) *There exists $E \in \mathcal{P}(\mathcal{H})$ such that $\|P_{\mathcal{S}} - E\| < 1$ and $\|E - P_{\mathcal{T}}\| < 1$*

(ii) $s(P_{\mathcal{S}}, P_{\mathcal{T}}) \leq 3$

(iii) *$\mathcal{S}$ and $\mathcal{T}$ have a common complement*

Theorem (J. Giol '05)

$s(P, Q) \leq 4$, for every $P, Q \in \mathcal{Q}(\mathcal{H})$

Corollary (J. Giol '05)

Let $\mathcal{S}, \mathcal{T} \in \mathcal{P}_\infty$. Then we have the following equivalence:

$\mathcal{S}$ and $\mathcal{T}$ do not have a common complement $\iff s(P_{\mathcal{S}}, P_{\mathcal{T}}) = 4$.

<u>Remark:</u> In the last case, there exist $E_1, E_2 \in \mathcal{P}(\mathcal{H})$ such that

$$\|P_{\mathcal{S}} - E_1\| < 1, \quad \|E_1 - E_2\| < 1, \quad \|E_2 - P_{\mathcal{T}}\| < 1.$$

Corollary

Minimal number of geodesic segments joining P and $Q \leq s(P,Q) - 1$

Corollary

If $\mathcal{S},\mathcal{T}$ do not have a common complement, then the open neighbourhoods

$$\{E \in \mathcal{P}(\mathcal{H}) : \|E - P_{\mathcal{S}}\| < 1\} \ \text{and} \ \{F \in \mathcal{P}(\mathcal{H}) : \|F - P_{\mathcal{T}}\| < 1\}$$

are disjoint.

Corollary

Consider two projections $P_{\mathcal{S}},\ P_{\mathcal{T}} \in \mathcal{P}_{\infty}$ such that $\mathcal{S}$ and $\mathcal{T}$ do not have a common complement. Then the following hold:

1. *$P_{\mathcal{S}}$ and $P_{\mathcal{T}}$ cannot be joined by a geodesic*
2. *Moreover, the set*

   $$\{E \in \mathcal{P}(\mathcal{H}) : \text{there exists a geodesic joining } E \text{ and } P\}$$

   is not dense in $\mathcal{P}(\mathcal{H})$.

Thanks for your attention

Ideas of proofs

Ideas: Change to symmetries $\epsilon_P = 2P - I \iff P = P^2 = P^*$

- $\|\epsilon_P - \epsilon_Q\| < 2 \Rightarrow \exists\, X = 2\epsilon_P \log(A|A|^{-1})$, where $A = (I + \epsilon_P \epsilon_Q)/2$

$$Q = e^{X\epsilon_P/2} P e^{-X\epsilon_P/2}$$

The geodesic is given by $\delta(t) = e^{tX\epsilon_P/2} P e^{-tX\epsilon_P/2}$, $t \in [0,1]$

- Existence of a unit norming vector: $\xi \in \mathcal{H}$, $X^2\xi = \|X\|^2\xi$

Set $F(\gamma) = \gamma\xi$ for a curve γ joining ϵ_P and ϵ_Q. Then,

$$L(\gamma) \underset{\textbf{(A)}}{\geq} L(F\gamma) \underset{\textbf{(B)}}{\geq} L(F\delta) \underset{\textbf{(C)}}{=} L(\delta)$$

(A) F is length-reducing

(B) $F\delta$ is a geodesic in the sphere of $\mathcal{H}$ of length $\leq \pi$

(C) Computation with the Finsler metric Back

Ideas: (3) $\Longrightarrow$ (2) is based on the decomposition in the two projections theorem:
$\mathcal{S} = ran(P)$, $\mathcal{T} = ran(Q)$, $\mathcal{S}^\perp = ker(P)$, $\mathcal{T}^\perp = ker(Q)$

$$\mathcal{H} = \underbrace{(\mathcal{S} \cap \mathcal{T}) \oplus (\mathcal{S}^\perp \cap \mathcal{T}^\perp) \oplus (\mathcal{S} \cap \mathcal{T}^\perp) \oplus (\mathcal{S}^\perp \cap \mathcal{T})}_{\mathcal{H}_0^\perp} \oplus \mathcal{H}_0$$

The condition $\dim \mathcal{S} \cap \mathcal{T}^\perp = \dim \mathcal{S}^\perp \cap \mathcal{T}$ allows to construct an operator $X = -X^*$, X is P-codiagonal and $\|X\| \leq \pi/2$, such that

$$e^X P e^{-X} = Q.$$

Two projections theorem: (J. Dixmier '48/P. Halmos '69/W. Amrein K. Sinha'94) After a unitary conjugation, the projections P and Q in the decomposition

$$\mathcal{H} = (\mathcal{S} \cap \mathcal{T}) \oplus (\mathcal{S}^\perp \cap \mathcal{T}^\perp) \oplus (\mathcal{S} \cap \mathcal{T}^\perp) \oplus (\mathcal{S}^\perp \cap \mathcal{T}) \oplus \mathcal{H}_0$$

have the following form

$$P = \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & C^2 & CS \\ 0 & 0 & 0 & 0 & CS & S^2 \end{pmatrix},$$

where $\mathcal{H}_0 = \mathcal{L} \times \mathcal{L}$, and $C = \cos(\Gamma)$, $S = \sin(\Gamma)$ and $\Gamma = \Gamma^*$ injective acting on $\mathcal{L}$ with $\sigma(\Gamma) \subseteq [0, \frac{\pi}{2}]$

Then, the operator X on $\mathcal{H}$ such that $e^X P e^{-X} = Q$ is constructed as a block diagonal operator $X = X_2 \oplus X_1 \oplus X_0$

Using the decomposition $\mathcal{H} = (\mathcal{S} \cap \mathcal{T}) \oplus (\mathcal{S}^\perp \cap \mathcal{T}^\perp) \oplus (\mathcal{S} \cap \mathcal{T}^\perp) \oplus (\mathcal{S}^\perp \cap \mathcal{T}) \oplus \mathcal{H}_0$

$$P = \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & C^2 & CS \\ 0 & 0 & 0 & 0 & CS & S^2 \end{pmatrix}$$

- Take $X_2 = 0$ on $\mathcal{H}_2 := (\mathcal{S} \cap \mathcal{T}) \oplus (\mathcal{S}^\perp \cap \mathcal{T}^\perp)$ since $P|_{\mathcal{H}_2} = Q|_{\mathcal{H}_2}$
- By hypothesis, $\exists\, X_1$ on $\mathcal{H}_1 := (\mathcal{S} \cap \mathcal{T}^\perp) \oplus (\mathcal{S}^\perp \cap \mathcal{T})$ such that $e^{X_1} P|_{\mathcal{H}_1} e^{-X_1} = Q|_{\mathcal{H}_1}$
- Take $X_0 = \begin{pmatrix} 0 & \Gamma \\ -\Gamma & 0 \end{pmatrix}$ on $\mathcal{H}_0$ which satisfies $e^{X_0} P|_{\mathcal{H}_0} e^{-X_0} = Q|_{\mathcal{H}_0}$ because

$$\exp\left(\begin{pmatrix} 0 & \Gamma \\ -\Gamma & 0 \end{pmatrix}\right) \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \exp\left(-\begin{pmatrix} 0 & \Gamma \\ -\Gamma & 0 \end{pmatrix}\right) = \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix}$$

Back